ON THE FRACTIONAL DIRAC SYSTEMS WITH NON-SINGULAR OPERATORS

by

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Original scientific paper
https://doi.org/10.2298/TSCI190810405E

In this manuscript, we consider the fractional Dirac system with exponential and Mittag-Leffler kernels in Riemann-Liouville and Caputo sense. We obtain the representations of the solutions for Dirac systems by means of Laplace transforms.

Key words: *Dirac operator, fractional, exponential kernel, Mittag-Leffler kernel, Laplace transform*

Introduction

The aim of fractional derivatives has been used in many real world problems with great success. This led to new fractional definitions. Recently, fractional derivative with exponential kernel was defined by Caputo and Fabrizio [1]. It was followed by some related theoretical and applied results in [2, 3]. Definition of fractional derivative having Mittag-Leffler kernel was given by Atangana and Baleanu [4]. These new definitions give more precise results in some modelling problems because of having non-singularity in its kernels.

Some modelling problems and its solution methods having fractional operator with exponential kernel was considered by Baleanu *et al.* in [2, 5-7], Al-Refai and Abdeljawad [8] Owolabi and Atangana [9]. Baleanu *et al.* studied operator with Mittag-Leffler kernel in [10-12]. The comparisons of these two new definitions were studied by Abro *et al.* [13], Sheikh *et al.* [14], Saad *et al.* [15], and Bas *et al*. [16-18].

Dirac equation has a big importance in the modern field of atomic physics and its emergence began in 1928 while searched for a relativistic covariant wave equation of the Schrödinger form. Dirac equation expresses a wave equation for spin-1/2 particles [19]. A relativistic particle of spin-1/2 at high velocities corresponds to Dirac equation. Dirac equation expresses the position of electrons in a sense convenient with special relativity, needs that electrons have spin 1/2, and presuming the existence of an antiparticle partner to the electron (the positron). It is very important in the physics and mathematics, such that it is a first-order matrix linear differential equation whose solution is a 4-component wave function (a spinor) [19-21].

Let L denote a matrix operator:

$$\begin{bmatrix} V(x)+m & 0 \\ 0 & V(x)-m \end{bmatrix}$$

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where m is the mass of a particle and $V(x)$is a potential function. Then consider the equation:

$$\left(B\frac{\mathrm{d}}{\mathrm{d}x}+L-\lambda I\right)y=0$$

where λ is a parameter and:

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ y(x)=\begin{bmatrix}y_1(x)\\ y_2(x)\end{bmatrix}$$

is equivalent to a system of two simultaneous first-order ordinary differential equations:

$$\frac{\mathrm{d}y_2(x,\lambda)}{\mathrm{d}x}+\left[V(x)+m\right]y_1(x,\lambda)=\lambda y_1(x,\lambda),$$
$$-\frac{\mathrm{d}y_1(x,\lambda)}{\mathrm{d}x}+\left[V(x)-m\right]y_2(x,\lambda)=\lambda y_2(x,\lambda)$$

Major results for the classical spectral problem for Dirac operator are included the representations of the solutions, asymptotic behaviours of the eigenvalues, the eigen-vector-functions and the norming constants, the ortogonality of the eigenfunctions, the reality of the eigenvalues and this kind of approach is called as direct problem. This type of problems were studied in [22-24].

In this study, we consider fractional Dirac systems within non-singular operators. We derive the representations of solutions for Dirac systems by using Laplace transforms. Our this approachment will give rise to a lot of open problems in direct and inverse problems for Dirac system in the spectral theory.

Preliminaries

*Definition 1***.** [1] The left and right sided fractional derivatives in Caputo (C) sense with exponential kernel are defined:

$$\begin{aligned}{}^{CFC}_{\ a}\mathrm{D}^{\alpha}f(t)&=\frac{M(\alpha)}{1-\alpha}\int_a^t f'(s)\exp\left[\frac{-\alpha}{1-\alpha}(t-s)\right]\mathrm{d}s\\ {}^{CFC}\mathrm{D}_b^{\alpha}f(t)&=\frac{-M(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b f(s)\exp\left[\frac{-\alpha}{1-\alpha}(s-t)\right]\mathrm{d}s\end{aligned}\tag{1}$$

left and right derivatives in the Riemann-Liouville (RL) sense:

$${}^{CFR}_{\ a}\mathrm{D}^{\alpha}f(t)=\frac{M(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(s)\exp\left[\frac{-\alpha}{1-\alpha}(t-s)\right]\mathrm{d}s\tag{2}$$

where $f\in H^1(a,b)$, $a<b$, $\alpha\in[0,1]$, and:

$${}^{CFC}\mathrm{D}_b^{\alpha}f(t)=\frac{-M(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b f(s)\exp\left[\frac{-\alpha}{1-\alpha}(s-t)\right]\mathrm{d}s$$

where $M(\alpha)>0$ is a normalization function with $M(0)=M(1)=1$.

Definition 2. [1] Left and right fractional integrals for fractional derivatives with exponential kernel are defined, respectively by:

$${}^{CF}_{\ a}I^{\alpha}f(t)=\frac{1-\alpha}{M(\alpha)}f(t)+\frac{\alpha}{M(\alpha)}\int_a^t f(s)\mathrm{d}s$$

$$^{CF}I_b^\alpha f(t)=\frac{1-\alpha}{M(\alpha)}f(t)+\frac{\alpha}{M(\alpha)}\int_t^b f(s)\mathrm{d}s$$

Definition 3. [4] Left and right sided fractional derivatives having Mittag-Leffler kernel in C sense are defined, respectively:

$$\begin{aligned}{}^{ABC}_{\ a}\mathrm{D}^\alpha f(t)&=\frac{B(\alpha)}{1-\alpha}\int_a^t f'(s)E_\alpha\left[\frac{-\alpha}{1-\alpha}(t-s)^\alpha\right]\mathrm{d}s\\ {}^{ABC}\mathrm{D}_b^\alpha f(t)&=\frac{-B(\alpha)}{1-\alpha}\int_t^b f'(s)E_\alpha\left[\frac{-\alpha}{1-\alpha}(s-t)^\alpha\right]\mathrm{d}s\end{aligned}\tag{3}$$

left and right derivatives in RL sense:

$${}^{ABR}_{\ a}\mathrm{D}^\alpha f(t)=\frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(s)E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-s)^\alpha\right)\mathrm{d}s\tag{4}$$

where $f\in H^1(a,b)$, $a<b$, $\alpha\in[0,1]$, and:

$${}^{ABR}\mathrm{D}_b^\alpha f(t)=\frac{-B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b f(s)E_\alpha\left[\frac{-\alpha}{1-\alpha}(s-t)^\alpha\right]\mathrm{d}s$$

where $B(\alpha)>0$ is a normalizatio n function with $B(0)=B(1)=1$.

Definition 4. [12] Left and right fractional integrals for fractional derivative with Mittag-Leffler kernel are defined, respectively:

$$\begin{aligned}{}^{AB}_{\ a}I^\alpha f(t)&=\frac{1-\alpha}{B(\alpha)}f(t)+\frac{\alpha}{B(\alpha)}\ {}_aI^\alpha f(t)\\ {}^{AB}I_b^\alpha f(t)&=\frac{1-\alpha}{B(\alpha)}f(t)+\frac{\alpha}{B(\alpha)}\ I_b^\alpha f(t)\end{aligned}$$

Theorem 5. [1, 9] The fractional Laplace transforms of with exponential kernel (1) and (2) are defined:

$$\mathcal{L}\left\{{}^{CFR}_{\ a}D^\alpha f(t)\right\}(s)=\frac{M(\alpha)}{1-\alpha}\frac{s\mathcal{L}\{f(t)\}(s)}{s+\frac{\alpha}{1-\alpha}}\tag{5}$$

$$\mathcal{L}\left\{\left({}^{CFC}_{\ a}D^\alpha f\right)(t)\right\}(s)=\frac{M(\alpha)}{1-\alpha}\frac{s\mathcal{L}\{f(t)\}(s)}{s+\frac{\alpha}{1-\alpha}}-\frac{M(\alpha)}{1-\alpha}f(a)\mathrm{e}^{-as}\frac{1}{s+\frac{\alpha}{1-\alpha}}\tag{6}$$

Theorem 6. [2] The fractional Laplace transforms with Mittag-Leffler kernel (3) and (4) are defined:

$$\mathcal{L}\left\{{}^{ABR}_{\ a}D^\alpha f(t)\right\}(s)=\frac{B(\alpha)}{1-\alpha}\frac{s^\alpha\mathcal{L}\{f(t)\}(s)}{s^\alpha+\frac{\alpha}{1-\alpha}}\tag{7}$$

$$\mathcal{L}\left\{{}_{a}^{ABC}\mathrm{D}^{\alpha} f(t)\right\}(s)=\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\{f(t)\}(s)-s^{\alpha-1}f(a)}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}} \tag{8}$$

Definition 7. [17] The convolution of $f(t)$ and $g(t)$ is defined:

$$(f*g)(t)=\int_{0}^{t}f(s)g(t-s)\mathrm{d}s,\ f,g\ :\ [0,\infty)\to\mathrm{R} \tag{9}$$

Definition 8. [25] Mittag-Leffler function $E_{\delta}(z)$ is defined:

$$E_{\delta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\delta k+1)},\ \left[z\in\mathrm{C},\ \mathrm{Re}(\delta)>0\right] \tag{10}$$

and Mittag-Leffler function with two parameters is defined:

$$E_{\delta,\theta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\delta k+\theta)},\ \left[z,\theta\in\mathrm{C},\ \mathrm{Re}(\delta)>0\right] \tag{11}$$

Property 9. [17] Inverse Laplace transforms of some special functions are given:

$$\mathcal{L}^{-1}\left\{\frac{a}{s\left(s^{\delta}+a\right)}\right\}=1-E_{\delta}\left(-at^{\delta}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\delta}+a}\right\}=t^{\delta-1}E_{\delta,\delta}\left(-at^{\delta}\right)$$

Property 10. [17] The following equality is hold:

$$\mathcal{L}\{(f*g)(t)\}=\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

Main results

In this section, firstly we deal with fractional Dirac systems involving exponential kernel in RL and C sense. Secondly, we consider fractional Dirac systems having Mittag-Leffler kernel in RL and C sense. Then, we obtain the representations of solutions for Dirac systems by Laplace transforms. Assume that $h(x) = p(x)y_1(x)$ and $g(x) = r(x)y_2(x)$.

Theorem 11. Let's consider fractional Dirac system involving exponential kernel in RL sense:

$${}^{CF}L_1 f=\begin{cases}{}_{0}^{CFR}\mathrm{D}^{\alpha}y_2(x)+p(x)y_1(x)=\lambda y_1(x),\ x\in[0,n],n\in\mathrm{R}^{+}\\ -{}_{0}^{CFR}\mathrm{D}^{\alpha}y_1(x)+r(x)y_2(x)=\lambda y_2(x)\end{cases} \tag{12}$$

where $0 < \alpha < 1$, $p(x)$, and $r(x)$ are real valued continuous functions on $[0, n]$. Then, the representations of solutions of the system (12) are given by:

$$y_1(x,\lambda)=\left[\frac{M(\alpha)(1-\alpha)^{2}\delta(x)}{M^{2}(\alpha)+\lambda^{2}(1-\alpha)^{2}}+\right.$$

$$+\frac{\mathrm{e}^{\tilde{a}_1 x}\left[\left(-\alpha+2(\alpha-1)\tilde{a}_1\right)\lambda^{2}+(\alpha-1)^{2}\lambda^{2}\tilde{a}_1 M(\alpha)+\tilde{a}_1 M^{3}(\alpha)\right]}{2M(\alpha)\lambda i}-$$

$$-\frac{e^{\tilde{a}_2 x}\left[\left(-\alpha+2(\alpha-1)\tilde{a}_2\right)\lambda^2+(\alpha-1)^2\lambda^2\tilde{a}_2 M(\alpha)+\tilde{a}_2 M^3(\alpha)\right]}{2M(\alpha)\lambda i}\Bigg]r(x)y_2(x)+$$

$$+\left[\frac{\lambda(1-\alpha)^2\delta(x)}{M^2(\alpha)+\lambda^2(1-\alpha)^2}-\frac{e^{\tilde{a}_1 x}\left[(\alpha-1)^3\lambda^2-\left(1-2(\alpha-1)\tilde{a}_1\right)M^2(\alpha)\right]}{2M(\alpha)i}+\right.$$

$$\left.+\frac{e^{\tilde{a}_2 x}\left[(\alpha-1)^3\lambda^2-\left(1-2(\alpha-1)\tilde{a}_2\right)M^2(\alpha)\right]}{2M(\alpha)i}\right]p(x)y_1(x)$$

$$y_2(x,\lambda)=\left[-\frac{M(\alpha)(1-\alpha)^2\delta(x)}{M^2(\alpha)+\lambda^2(1-\alpha)^2}-\right.$$

$$-\frac{e^{\tilde{a}_1 x}\left[\left(-\alpha+2(\alpha-1)\tilde{a}_1\right)\lambda^2+(\alpha-1)^2\lambda^2\tilde{a}_1 M(\alpha)+\tilde{a}_1 M^3(\alpha)\right]}{2M(\alpha)\lambda i}+$$

$$\left.+\frac{e^{\tilde{a}_2 x}\left[\left(-\alpha+2(\alpha-1)\tilde{a}_2\right)\lambda^2+(\alpha-1)^2\lambda^2\tilde{a}_2 M(\alpha)+\tilde{a}_2 M^3(\alpha)\right]}{2M(\alpha)\lambda i}\right]p(x)y_1(x)+$$

$$+\left[\frac{\lambda(1-\alpha)^2\delta(x)}{M^2(\alpha)+\lambda^2(1-\alpha)^2}-\frac{e^{\tilde{a}_1 x}\left[(\alpha-1)^3\lambda^2-\left(1-2(\alpha-1)\tilde{a}_1\right)M^2(\alpha)\right]}{2M(\alpha)i}+\right.$$

$$\left.+\frac{e^{\tilde{a}_2 x}\left[(\alpha-1)^3\lambda^2-\left(1-2(\alpha-1)\tilde{a}_2\right)M^2(\alpha)\right]}{2M(\alpha)i}\right]r(x)y_2(x)$$

where

$$\tilde{a}_1=\frac{-\lambda^2\alpha(1-\alpha)+\alpha\lambda M(a)i}{M^2(\alpha)+\lambda^2(1-\alpha)^2},\quad \tilde{a}_2=\frac{-\lambda^2\alpha(1-\alpha)-\alpha\lambda M(a)i}{M^2(\alpha)+\lambda^2(1-\alpha)^2} \tag{13}$$

Proof. By applying Laplace transforms on both side of the eq. (12) and using *Theorem 5*, we find that:

$$\mathcal{L}\{(y_1)\}(s)=\frac{\frac{M(\alpha)}{1-\alpha}\frac{s}{s+\frac{\alpha}{1-\alpha}}}{\left[\frac{M(\alpha)}{1-\alpha}\frac{s}{s+\frac{\alpha}{1-\alpha}}\right]^2+\lambda^2}\mathcal{L}\{g\}(s)+\frac{\lambda}{\left[\frac{M(\alpha)}{1-\alpha}\frac{s}{s+\frac{\alpha}{1-\alpha}}\right]^2+\lambda^2}\mathcal{L}\{h\}(s) \tag{14}$$

$$\mathcal{L}\{(y_2)\}(s)=\frac{\lambda}{\left[\frac{M(\alpha)}{1-\alpha}\frac{s}{\frac{\alpha}{1-\alpha}}\right]^2+\lambda^2}\mathcal{L}\{g\}(s)-\frac{\frac{M(\alpha)}{1-\alpha}\frac{s}{s+\frac{\alpha}{1-\alpha}}}{\left[\frac{M(\alpha)}{1-\alpha}\frac{s}{s+\frac{\alpha}{1-\alpha}}\right]^2+\lambda^2}\mathcal{L}\{h\}(s) \tag{15}$$

Now, applying the inverse Laplace transform to the eqs. (14) and (15) and using convolution theorem, we can obtain the representations of solutions, of the system (12).

Theorem 12. Let's consider fractional Dirac system with exponential kernel in C sense:

$$ {}^{CF}L_2 f = \begin{cases} {}_0^{CFC}\mathrm{D}^{\alpha} y_2(x) + p(x) y_1(x) = \lambda y_1(x), \ x \in [0,n], n \in \mathrm{R}^+ \\ -{}_0^{CFC}\mathrm{D}^{\alpha} y_1(x) + r(x) y_2(x) = \lambda y_2(x) \end{cases} \tag{16} $$

where $0 < \alpha < 1$, $p(x)$, and $r(x)$ are real valued continuous functions on $[0, n]$. Then, the representations of solutions of system (16) are given by:

$$ \begin{aligned} y_1(x,\lambda) = {} & \frac{\left(\tilde{a}_1 e^{\tilde{a}_1 x} - \tilde{a}_2 e^{\tilde{a}_2 x}\right) M(\alpha)\left[(\alpha-1)^2\lambda^2 + M^2(\alpha)\right] c_1}{2\alpha\lambda i} + \\ & + \frac{\left[\alpha(\tilde{a}_1 - 1) - \tilde{a}_1\right] e^{\tilde{a}_1 x} + \left[\alpha(1-\tilde{a}_2) + \tilde{a}_2\right] e^{\tilde{a}_2 x} c_2}{2\alpha i} + \\ & + \left[\frac{M(\alpha)(1-\alpha)^2 \delta(x)}{M^2(\alpha) + \lambda^2(1-\alpha)^2} + \frac{e^{\tilde{a}_1 x}\left[\left(-\alpha + 2(\alpha-1)\tilde{a}_1\right)\lambda^2 + (\alpha-1)^2\lambda^2 \tilde{a}_1 M(\alpha) + \tilde{a}_1 M^3(\alpha)\right]}{2M(\alpha)\lambda i} - \right. \\ & \left. - \frac{e^{\tilde{a}_2 x}\left[\left(-\alpha + 2(\alpha-1)\tilde{a}_2\right)\lambda^2 + (\alpha-1)^2\lambda^2 \tilde{a}_2 M(\alpha) + \tilde{a}_2 M^3(\alpha)\right]}{2M(\alpha)\lambda i} \right] r(x) y_2(x) + \\ & + \left[\frac{\lambda(1-\alpha)^2 \delta(x)}{M^2(\alpha) + \lambda^2(1-\alpha)^2} - \frac{e^{\tilde{a}_1 x}\left[(\alpha-1)^3\lambda^2 - \left(1 - 2(\alpha-1)\tilde{a}_1\right) M^2(\alpha)\right]}{2M(\alpha) i} + \right. \\ & \left. + \frac{e^{\tilde{a}_2 x}\left[(\alpha-1)^3\lambda^2 - \left(1 - 2(\alpha-1)\tilde{a}_2\right) M^2(\alpha)\right]}{2M(\alpha) i} \right] p(x) y_1(x) \end{aligned} $$

$$ \begin{aligned} y_2(x,\lambda) = {} & \frac{\left(\tilde{a}_1 e^{\tilde{a}_1 x} - \tilde{a}_2 e^{\tilde{a}_2 x}\right) M(\alpha)\left[(\alpha-1)^2\lambda^2 + M^2(\alpha)\right] c_2}{2\alpha\lambda i} - \\ & - \frac{\left[\alpha(\tilde{a}_1 - 1) - \tilde{a}_1\right] e^{\tilde{a}_1 x} + \left[\alpha(1-\tilde{a}_2) + \tilde{a}_2\right] e^{\tilde{a}_2 x} c_1}{2\alpha i} + \\ & + \left[-\frac{M(\alpha)(1-\alpha)^2 \delta(x)}{M^2(\alpha) + \lambda^2(1-\alpha)^2} - \frac{e^{\tilde{a}_1 x}\left[\left(-\alpha + 2(\alpha-1)\tilde{a}_1\right)\lambda^2 + (\alpha-1)^2\lambda^2 \tilde{a}_1 M(\alpha) + \tilde{a}_1 M^3(\alpha)\right]}{2M(\alpha)\lambda i} + \right. \\ & \left. + \frac{e^{\tilde{a}_2 x}\left[\left(-\alpha + 2(\alpha-1)\tilde{a}_2\right)\lambda^2 + (\alpha-1)^2\lambda^2 \tilde{a}_2 M(\alpha) + \tilde{a}_2 M^3(\alpha)\right]}{2M(\alpha)\lambda i} \right] p(x) y_1(x) + \\ & + \left[\frac{\lambda(1-\alpha)^2 \delta(x)}{M^2(\alpha) + \lambda^2(1-\alpha)^2} - \frac{e^{\tilde{a}_1 x}\left[(\alpha-1)^3\lambda^2 - \left(1 - 2(\alpha-1)\tilde{a}_1\right) M^2(\alpha)\right]}{2M(\alpha) i} + \right. \\ & \left. + \frac{e^{\tilde{a}_2 x}\left[(\alpha-1)^3\lambda^2 - \left(1 - 2(\alpha-1)\tilde{a}_2\right) M^2(\alpha)\right]}{2M(\alpha) i} \right] r(x) y_2(x) \end{aligned} $$

where $c_1 = y_1(0)$, $c_2 = y_2(0)$

Proof. Proof can be done in a similar way to the proof of *Theorem 11*.

*Theorem 13***.** Consider the fractional Dirac system involving Mittag-Leffler kernel in C sense:

$$ {}^{CF}L_3 f = \begin{cases} {}_0^{ABC}\mathrm{D}^{\alpha} y_2(x) + p(x) y_1(x) = \lambda y_1(x) \\ -{}_0^{ABC}\mathrm{D}^{\alpha} y_1(x) + r(x) y_2(x) = \lambda y_2(x) \end{cases} \tag{17} $$

where $x[0, n]$, $0 < \alpha < 1$, $p(x)$, and $r(x)$ are real valued continuous functions on $[0, n]$. Then, representations of solutions of system (17) are given by:

$$ \begin{aligned} y_1(x,\lambda) = {} & \frac{B(\alpha)\left[B^2(\alpha)+(1-\alpha)^2\lambda^2\right]x^{\alpha}c_1}{2\lambda\alpha i}\left[E_{\alpha,1-\alpha}\left(a_1x^{\alpha}\right)-E_{\alpha,1-\alpha}\left(a_2x^{\alpha}\right)\right]- \\ & -\frac{(1-\alpha)\left[B^2(\alpha)+(1-\alpha)^2\lambda^2\right]x^{\alpha}c_2}{2\alpha i}\left[E_{\alpha,1-\alpha}\left(a_1x^{\alpha}\right)-E_{\alpha,1-\alpha}\left(a_2x^{\alpha}\right)\right]+ \\ & +\frac{\left[B^2(\alpha)+(1-\alpha)^2\lambda^2\right]c_2 i}{2}\left[E_{\alpha,1}\left(a_1x^{\alpha}\right)-E_{\alpha,1}\left(a_2x^{\alpha}\right)\right]+ \\ & +\left[\frac{(1-\alpha)B(\alpha)\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}(1-\alpha)\lambda i x^{\alpha-1}\left(a_1E_{\alpha,\alpha}\left(a_1x^{\alpha}\right)-a_2E_{\alpha,\alpha}\left(a_2x^{\alpha}\right)\right)+\right. \\ & \left.+\frac{x^{\alpha-1}\left[(1-\alpha)\lambda i+(1-\alpha)\lambda\alpha i+\alpha B(\alpha)\right]}{2}\left(E_{\alpha,\alpha}\left(a_1x^{\alpha}\right)-E_{\alpha,\alpha}\left(a_2x^{\alpha}\right)\right)\right]r(x)y_2(x)+ \\ & +\left[\frac{(1-\alpha)^2\lambda\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}-\frac{(1-\alpha)^3\lambda^2x^{\alpha-1}}{B(\alpha)i}\left(a_1E_{\alpha,\alpha}\left(a_1x^{\alpha}\right)-a_2E_{\alpha,\alpha}\left(a_2x^{\alpha}\right)\right)-\right. \\ & -\lambda\alpha i x^{\alpha-1}\left(E_{\alpha,\alpha}(a_1x^{\alpha})+E_{\alpha,\alpha}(a_2x^{\alpha})\right)+ \\ & \left.+\frac{\left[\alpha\left(B^2(\alpha)+(1-\alpha)^2\lambda^2\right)-(1-\alpha)^2(1+2\alpha)\lambda^2x^{\alpha-1}\right]}{2B(\alpha)i}\left(E_{\alpha,\alpha}\left(a_1x^{\alpha}\right)-E_{\alpha,\alpha}\left(a_2x^{\alpha}\right)\right)\right]p(x)y_1(x) \end{aligned} $$

$$ \begin{aligned} y_2(x) = {} & \frac{B(\alpha)\left(B^2(\alpha)+(1-\alpha)^2\lambda^2\right)x^{\alpha}c_2}{2\lambda\alpha i}\left(E_{\alpha,1-\alpha}\left(a_1x^{\alpha}\right)-E_{\alpha,1-\alpha}\left(a_2x^{\alpha}\right)\right)+ \\ & +\frac{(1-\alpha)\left(B^2(\alpha)+(1-\alpha)^2\lambda^2\right)x^{\alpha}c_1}{2\alpha i}\left(E_{\alpha,1-\alpha}\left(a_1x^{\alpha}\right)-E_{\alpha,1-\alpha}\left(a_2x^{\alpha}\right)\right)- \\ & -\frac{\left(B^2(\alpha)+(1-\alpha)^2\lambda^2\right)c_1 i}{2}\left(E_{\alpha,1}\left(a_1x^{\alpha}\right)-E_{\alpha,1}\left(a_2x^{\alpha}\right)\right)+ \\ & +\left[\frac{(1-\alpha)^2\lambda\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}-\frac{(1-\alpha)^3\lambda^2x^{\alpha-1}}{B(\alpha)i}\left(a_1E_{\alpha,\alpha}\left(a_1x^{\alpha}\right)-a_2E_{\alpha,\alpha}\left(a_2x^{\alpha}\right)\right)-\right. \end{aligned} $$

$$-\lambda\alpha i x^{\alpha-1}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)+E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)+$$

$$\left.+\frac{\left[\alpha\left(B^2(\alpha)+(1-\alpha)^2\lambda^2\right)-(1-\alpha)^2(1+2\alpha)\lambda^2x^{\alpha-1}\right]}{2B(\alpha)i}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)-E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)\right]r(x)y_2(x)-$$

$$-\left[\frac{(1-\alpha)B(\alpha)\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}+(1-\alpha)\lambda i x^{\alpha-1}\left(a_1E_{\alpha,\alpha}\left(a_1x^\alpha\right)-a_2E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)+\right.$$

$$\left.+\frac{x^{\alpha-1}\left[(1-\alpha)\lambda i+(1-\alpha)\lambda\alpha i+\alpha B(\alpha)\right]}{2}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)-E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)\right]p(x)y_1(x)$$

where

$$a_1=\frac{-\lambda^2\alpha(1-\alpha)+\lambda\alpha B(\alpha)i}{B^2(\alpha)+(1-\alpha)^2\lambda^2},\quad a_2=\frac{-\lambda^2\alpha(1-\alpha)-\lambda\alpha B(\alpha)i}{B^2(\alpha)+(1-\alpha)^2\lambda^2}$$

Proof. Applying Laplace transform on both side of the system (17) and by the help of *Theorem 6*, we find that:

$$\begin{aligned}\mathcal{L}\{(y_1)\}(s)=&\frac{\dfrac{B^2(\alpha)s^{2\alpha-1}}{\left[s^\alpha(1-\alpha)+\alpha\right]^2}c_1}{\left[\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]^2+\lambda^2}+\frac{\dfrac{B(\alpha)s^\alpha}{[s^\alpha(1-\alpha)+\alpha]}}{\left[\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]^2+\lambda^2}\mathcal{L}\{g\}(s)-\\&-\frac{\dfrac{\lambda B(\alpha)s^{\alpha-1}}{[s^\alpha(1-\alpha)+\alpha]}c_2}{\left[\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]^2+\lambda^2}+\frac{\lambda}{\left[\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]^2+\lambda^2}\mathcal{L}\{h\}(s)\end{aligned}\tag{18}$$

$$\begin{aligned}\mathcal{L}\{(y_2)\}(s)=&\frac{\dfrac{B^2(\alpha)s^{2\alpha-1}}{[s^\alpha(1-\alpha)+\alpha]^2}c_2}{\left[\dfrac{B^2(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]+\lambda^2}+\frac{\lambda}{\left[\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]+\lambda^2}\mathcal{L}\{g\}(s)+\\&+\frac{\dfrac{\lambda B(\alpha)s^{\alpha-1}}{s^\alpha(1-\alpha)+\alpha}c_1}{\left[\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]^2+\lambda^2}+\frac{\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}}{\left[\dfrac{B(\alpha)s^\alpha}{s^\alpha(1-\alpha)+\alpha}\right]^2}\mathcal{L}\{h\}(s)\end{aligned}\tag{19}$$

in here $c_1 = y_1(0)$ and $c_2 = y_2(0)$. Now, employing the inverse Laplace transform to the eqs. (18), and (19) and by means of convolution theorem, so we can derive the representations of solutions, of the system (17).

Theorem 14. Let consider fractional Dirac system with Mittag-Leffler kernel in RL sense:

$${}^{CF}L_4f=\begin{cases}{}_{0}^{ABR}D^\alpha y_2(x)+p(x)y_1(x)=\lambda y_1(x)\\-{}_{0}^{ABR}D^\alpha y_1(x)+r(x)y_2(x)=\lambda y_2(x)\end{cases}\tag{20}$$

where $x \in [0, n]$, $0 < \alpha < 1$, $p(x)$, and $r(x)$ are real valued continuous functions on $[0, n]$. Then, representations of solutions of system (20) are given by:

$$y_1(x,\lambda)=\left[\frac{(1-\alpha)^2\lambda\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}-\frac{(1-\alpha)^3\lambda^2x^{\alpha-1}}{B(\alpha)i}\left(a_1E_{\alpha,\alpha}\left(a_1x^\alpha\right)-a_2E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)-\right.$$

$$-\lambda\alpha ix^{\alpha-1}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)+E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)+$$

$$\left.+\frac{\left[\alpha\left(B^2(\alpha)+(1-\alpha)^2\lambda^2\right)-(1-\alpha)^2\lambda^2x^{\alpha-1}(1+2\alpha)\right]}{2B(\alpha)i}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)-E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)\right]p(x)y_1(x)+$$

$$+\left[\frac{(1-\alpha B)(\alpha)\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}+(1-\alpha)\lambda ix^{\alpha-1}\left(a_1E_{\alpha,\alpha}\left(a_1x^\alpha\right)-a_2E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)+\right.$$

$$\left.\frac{x^{\alpha-1}\left[(1-\alpha)\lambda i+(1-\alpha)\lambda\alpha i+\alpha B(\alpha)\right]}{2}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)-E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)\right]r(x)y_2(x)$$

$$y_2(x,\lambda)=\left[\frac{(1-\alpha)^2\lambda\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}-\frac{(1-\alpha)^3\lambda^2x^{\alpha-1}}{B(\alpha)i}\left(a_1E_{\alpha,\alpha}\left(a_1x^\alpha\right)-a_2E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)-\right.$$

$$-\lambda\alpha i-x^{\alpha-1}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)+E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)+$$

$$\left.+\frac{\left[\alpha\left(B^2(\alpha)+(1-\alpha)^2\lambda^2\right)-(1-\alpha)^2\lambda^2x^{\alpha-1}(1+2\alpha)\right]}{2B(\alpha)i}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)-E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)\right]r(x)y_2(x)-$$

$$-\left[\frac{(1-\alpha)B(\alpha)\delta(x)}{B^2(\alpha)+(1-\alpha)^2\lambda^2}+(1-\alpha)\lambda ix^{\alpha-1}\left(a_1E_{\alpha,\alpha}\left(a_1x^\alpha\right)-a_2E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)+\right.$$

$$\left.+\frac{x^{\alpha-1}\left[(1-\alpha)\lambda i+(1-\alpha)\lambda\alpha i+\alpha B(\alpha)\right]}{2}\left(E_{\alpha,\alpha}\left(a_1x^\alpha\right)-E_{\alpha,\alpha}\left(a_2x^\alpha\right)\right)\right]p(x)y_1(x)$$

Proof. Proof is similar to the proof of *Theorem 13*.

Conclusion

In this study, we have analyzed fractional Dirac systems involving exponential and Mittag-Leffler kernels in RL and C sense. The Laplace transforms has been employed to gain representations of solutions for Dirac systems.

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Paper Submitted: August 10, 2019
Paper revised: October 09, 2019
Paper accepted: October 14, 2019

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