ANTI-PERIODIC SOLUTIONS FOR FRACTIONAL-ORDER BIDIRECTIONAL ASSOCIATIVE MEMORY NEURAL NETWORKS WITH DELAYS

by

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Original scientific paper
https://doi.org/10.2298/TSCI190805406T

This paper concerns fractional-order bidirectional associative memory neural networks with distributed delays. Based on inequality technique and Lyapunov functional method, some novel sufficient conditions are obtained for the existence and exponential stability of anti-periodic solutions are established. An example is given to show the feasibility main results.

Key words: *bidirectional associative memory neural networks, existence, fractional differential equation, global exponential stability, delays, anti-periodic*

Introduction

The artificial neural networks (NN) enables the computing interaction of neurons. A neurons organized in couple layers is said to be a bidirectional associative memory (BAM) NN. The neurons in one layer interact with the neurons in the other layer. Thus, it shows the physical connection of the interconnecting network components.

The NN models (BAM) are widely used in scientific branches for instance signal and image processing, pattern recognition, combinatorial optimization and artificial intelligence. In studies conducted in these areas, it is important to ensure that the NN are stable.

So the time delay can significantly influence the efficiency, of recurrent NN. Thus, the presence, uniqueness, and stability of the balance point for BAM NN in the recent past have attracted the interest of researchers and have been discussed; see, for example, [1-8] and references therein. With the theory and practice of fractional differential equations developing recently [9-11], studies investigating the complex conditions of fractional NN have been intensified. First, a new class of cellular NN with a fractional derivative was created. The feature of this NN model is that the first-order cell previously used replaces a non-integer order. The inclusion of fractional-grade cells in chaotic cells in a two-cell system with less than three of the required parameters. At this point, the fractional theory has developed a new approach to this theory. In the studies carried out in this area [12], they examined the structure of chaos with harmonic equilibrium theory and prepared an algorithm for numerical solution. Thus, fractional differential equations and a fractional neuron network system were examined for chaos control and synchronization. In [13], they considered a Hopfield NN of fractional structure and observed their stability by using the energy function.

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Recently, studies on NN in fractional order are given in [14-16]. Kaslik and Sivasundaram [17, 18] also obtained interesting results in their studies on this subject. Wu *et al.* [19] examined the generalized Gronwall inequality and predictions of Mittag-Leffler functions. Alofi *et al.* [20] examined the distributed latency finite time stability of fractional sequential networks. Hopfield [21] is integer, bilateral relational memory model was first processed by Kosko [22]. This NN is of great importance for applications in the field of pattern recognition and automatic control. Recent studies of these networks have also been reported [23-26]. However, there are fewer studies on fractional networks in the literature.

The aim of this study is to investigate delayed BAM type fractional order in anti-periodic (AP) solutions for discrete time networks and to find global exponential stability. Our results prevent delays from being limited. Using techniques involving a new Lyapunov functionality and inequality technique, we present a new delay-dependent stability criterion for variable delay NN. Our results can be applied to more general networks with a wider time delay function. Our criterion is easy to control and implement in practice and therefore, has an important place in both application areas and in the design of NN.

Preliminaries

In this section, let us first recall some basic definitions of fractional calculation. We will use them to prove our main results in three chapters. Let N and R be the set of positive integers and the real numbers set, respectively.

Definition 1 [9] The fractional integral of order $\alpha > 0$ of a function $y : (a, b] \to R$:

$$I_a^\alpha + y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} y(\tau)\mathrm{d}\tau, \quad t\in(a,b]$$

Definition 2 [11] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (a, b] \to R$:

$$\mathrm{D}_a^\alpha + y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \frac{y(\tau)}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}s, \quad n=[\alpha]+1$$

Definition 3 [11] The Caputo fractional of order $\alpha > 0$ of function y on $(a, b]$ is explained by the Riemann-Liouville derivatives described previously:

$$\left({}^c\mathrm{D}_a^\alpha + y\right)(t) = \left(\mathrm{D}_{a^+}^\alpha + \left[y(t) - \sum_{k=0}^{n-1} \frac{(y^{(k)}(a)}{k!} :(t-a)^k\right]\right)t, \quad t\in(a,b]$$

In this study, we think that fractional-order BAM NN are delayed in leakage conditions:

$$\begin{aligned} \mathrm{D}_t^\alpha x_i(t) &= -a_i x_i(t) + \sum_{j=1}^m b_{ij} f\left[x_i(t)\right] + \sum_{j=1}^m c_{ij} h\left[x_j(t-\tau)\right] + I_i \\ \mathrm{D}_t^\alpha y_j(t) &= -d_j y_j(t) + \sum_{i=1}^n e_{ij} k\left[y_i(t)\right] - \sum_{i=1}^n \kappa_{ij} \int_{-\infty}^t \Upsilon_{ij}(t-\xi) g_{ij}\left[x_i(\xi)\right]\mathrm{d}\xi \end{aligned} \tag{1}$$

where $i = 1, 2,..., n, j = 1, 2,...,m$. The D_t^α, denotes Caputo fractional derivative of order α, $0 < \alpha < 1$.

Consider the following initial conditions with the system of eq. (1):

$$x_i(\Theta) = \delta_i(\Theta),\ \Theta \in [-\tau, 0],\ i = 1, 2, ..., n$$
$$y_j(\eta) = \phi_j(\eta),\ \eta \in (-\infty, 0],\ j = 1, 2..., m$$

where $\tau = \max\{\tau_{ij}\}$, δ_i, and ϕ_j are continuous and real valued functions.

Let $u_i(t) : R \to R$ be continuous in t, $u_i(t)$ is said to be T – AP on R:

$$u_i(t+T) = -u_i(t) \text{ for all } t \in R$$

Throughout this paper, for $i = 1, 2,..., n, j = 1, 2,...,m$, it will be assumed that ($T1$) a_i, d_j, I_i, b_{ij}, c_{ij}, e_{ij} : $R \to R$ and Y_{ij} : $[0, \infty) \to R$, and:

$$a_i(t+T) = a_i(t),\ d_j(t+T) = d_j(t)$$
$$b_{ij}(t+T) f_i(x) = -b_{ij}(t) f_i(-x),\ \forall t,\ x \in R$$
$$e_{ij}(t+T) k_i(y) = -e_{ij}(t) k_i(-y),\ \forall t, y \in R$$
$$c_{ij}(t+T) g_i(x) = -c_{ij}(t) g_i(-x),\ \forall t, x \in R$$
$$I_i(t+T) = -I_i(t),\ \forall t \in R$$

The ($T2$) b_{ij}, e_{ij} are locally Lipscitz continuos:

$$b_{ij}(x+u) - b_{ij}(x) \ge \delta_i x$$
$$e_{ij}(y+u) - b_{ij}(y) \ge \omega_i y$$

where $x, y \in R$, $i = 1, 2,...n, j = 1, 2,..., m$, and there existence positive constants δ_i and ω_i.

The ($T3$):

$$f_{ij}(0) = 0, \left|f_{ij}(x) - f_{ij}(y)\right| \le \sigma_{ij}\left|x-y\right|, |\, fi_j(x) | \le A_{ij}$$

$$h_{ij}(0) = 0, \left|h_{ij}(x) - h_{ij}(y)\right| \le \nu_{ij}\left|x-y\right|, |\, h_{ij}(x) | \le C_{ij}$$

$$k_{ij}(0) = 0, \left|k_{ij}(x) - k_{ij}(y)\right| \le \eta_{ij}\left|x-y\right|, |\, k_{ij}(x) \le B_{ij}$$

$$g_{ij}(0) = 0, |\, g_{ij}(x) - g_{ij}(y) | \le \upsilon_{ij}\left|x-y\right|, g_{ij}(x) \le L_{ij}$$

where existe costant $\beta_{ij} \ge 0$, $\nu_{ij} \ge 0$, $\eta_{ij} \ge 0$, $\upsilon_{ij} \ge 0$, $A_{ij} \ge 0$, $C_{ij} \ge 0$, $B_{ij} \ge 0$, and $L_{ij} \ge 0$ such that for all $x, y, \in R$, $i = 1, 2,..., n, j = 1, 2,..., m$.

The (T4) $\beta > 0$ constant is available. Thus:

$$0 \le \beta - a_i x_i(t) + \sum_{j=1}^{m} \sigma_{ij}\, \mathrm{e}^{\beta t} + \mathrm{e}^{\beta t} \sum_{j=1}^{m} \left|c_{ij}\right| \nu_{ij} + I_i$$

and

$$0 \le \beta - d_j y_j(t) + \sum_{i=1}^{n} \omega_i \mathrm{e}^{\beta t} - \sum_{i=1}^{n} |\, \kappa_{ij} \,| \int_{-\infty}^{t} Y_{ij}(t-\xi) \upsilon_{ij} \mathrm{e}^{\beta(t-\xi)} \mathrm{d}\xi$$

For $u^*(t) = \left[x_1(t), x_2(t),.., x_n(t), y_1(t), y_2(t),.., y_n(t)\right]^T \in R^{m+n}$, we define the norm:

$$\| u^* \| = sup \max\left\{ \max_{1\le j\le m}\left|x_j(t)\right|, \max_{1\le i\le n}\left|y_i(t)\right| \right\}$$

Definition 4 Let $\zeta^*(t) = [x_1^*(t), x_2^*(t),... x_m^*(t), y_1^*(t), y_2^*(t),... y_n^*(t)]^T$ be an AP solution of system (1) with initial value $[\delta_1^*(t), \delta_2^*(t),..., \delta_n^*(t), \phi_1^*(t), \phi_2^*(t),..., \phi_n^*(t),]^T$, and $\zeta(t) = [x_1(t), x_2(t),..., x_n(t), y_1(t), y_2(t),..., y_n(t)]^T$ be the solution of system (1) with initial value $[\delta_1(t), \delta_2(t),..., \delta_n(t), \phi_1(t), \phi_2(t),..., \phi_n(t),]^T$. Get $\beta > 0$ and $A = A(\beta) > 1$ such that:

$$|\, \zeta(t) - \zeta^*(t) | \le A_\delta \mathrm{e}^{-\beta t} \max\{\| \delta - \delta^* \|_\infty \| \phi - \phi^* \|_\infty,\ t > 0,\ i = 1,2,\ldots,n,\quad j = 1,2,\ldots,m$$

where:

$$\|\delta-\delta^*\|_\infty = sup \max_{1\le i\le n} |\delta_i(\xi)-\delta_i^*(\xi)|,\ \|\phi-\phi^*\|_\infty = sup \max_{1\le j\le m} |\phi(\xi)-\phi_i^*(\xi)|$$

Lemma 1 Let (T1)-(T4) be satisfied and $\tilde{\zeta}(t) = [\tilde{x}(t), \tilde{y}(t)]$, where:

$$\tilde{x}(t)=\left[\tilde{x}_1(t),\tilde{x}_2(t),..,\tilde{x}_n(t)\right]^T,\ \tilde{y}(t)=\left[\tilde{y}_1(t),\tilde{y}_2(t),..,\tilde{y}_n(t)\right]^T \tag{1}$$

is solution of system (1) with initial conditions:

$$\tilde{x}_i(\Theta)=\tilde{\delta}_i(\Theta),\ \left|\tilde{\delta}_i(\Theta)\right|<\frac{\sum_{i=1}^{n}|c_{ij}|A_{ij}}{\sigma_i},\ \Theta\in[-\tau,0],\ i=1,2,...,n \tag{2}$$

$$\tilde{y}(\eta)=\tilde{\phi}j(\eta),\ \left|\tilde{\phi}_j(\eta)\right|<\frac{\sum_{i=1}^{m}|\kappa_{ij}|L_{ij}\int_{-\infty}^{\alpha}|\Upsilon_{ij}(t-\xi)|\,\mathrm{d}\xi}{\varepsilon_i},\ \eta\in(-\infty,0],\ j=1,2,...,m \tag{3}$$

then

$$\left|\tilde{x}_i(t)\right|<\frac{\sum_{i=1}^{n}|c_{ij}|A_{ij}}{\sigma_i},\ \left|\tilde{y}_j(t)\right|<\frac{\sum_{i=1}^{m}|\kappa_{ij}|L_{ij}\int_{-\infty}^{\alpha}|\Upsilon_{ij}(t-\xi)|\,\mathrm{d}\xi}{\varepsilon_i} \tag{4}$$

where $t > 0$, $i = 1, 2,..., n$, $j = 1, 2,..., m$.

Lemma 2 Let hypotheses (T1)-(T4) be satisfied. Let $\zeta^*(t) = [x^*(t), y^*(t)]^T$, where $x^*(t) = x_1^*(t), x_2^*(t),... x_m^*(t)$, $y^*(t) = y_1^*(t), y_2^*(t),..., y_n^*(t)$ then, the solution of the system (1) having the condition (2) and (3). Let $\zeta(t) = [x_1(t, x_2(t),..., x_n(t), y_1(t), y_2(t),..., y_n(t)]^T$ is a solution of system (1) with the initial value $[\delta_1(t), \delta_2(t),..., \delta_n(t), \phi_1(t), \phi_2(t),..., \phi_n(t),]^T$. Then there exist constant $\beta > 0$ and $A = A(\beta) > 1$ such that:

$$|\zeta(t)-\zeta^*(t)|\le A_\delta \mathrm{e}^{-\beta t}\max\{\|\delta-\delta^*\|_\infty,\ \|\phi-\phi^*\|_\infty\},\ t>0$$

Remark. If $\zeta^*(t) = [x^*(t), y^*(t)]^T$, where $x^*(t) = x_1^*(t), x_2^*(t),... x_m^*(t)$, $y^*(t) = y_1^*(t), y_2^*(t),..., y_n^*(t)$ is the T-AP solution of system (1), with the use of *Lemma 2* and *Definition 4* that $\zeta^*(t)$ is globally exponentially stable.

Main results

In this section, our main results are:

Theorem 1. Under assumptions (T1)-(T4). The system (1), has only one T-AP solution $\zeta^*(t)$ which is globally exponentially stable.

Proof. Let $\zeta(t) = [x(t), y(t)]$ where $x(t) = [x_1(t), x_2(t),..., x_n(t)]^T$, $y(t) = [y_1(t), y_2(t),..., y_n(t)]^T$, is a solution of system (1) with initial conditions:

$$x_i(\Theta)=\delta_i^\varsigma(\Theta),\ \left|\delta_i^\varsigma(\Theta)\right|<\frac{\sum_{i=1}^{n}|c_{ij}|A_{ij}}{\sigma_i},\ \Theta\in[-\tau,0],\ i=1,2,...,n$$

$$y_j(\eta)=\phi_j^\varsigma(\eta),\ |\phi_j^\varsigma(\eta)|<\frac{\sum_{j=1}^{m}|\kappa_{ij}|L_{ij}\int_{-\infty}^{\alpha}|\Upsilon_{ij}(t-\xi)|\,\mathrm{d}(\xi)}{\varepsilon_i},\ \eta\in(-\infty,0],\ j=1,2,...,m$$

In this way with respect to *Lemma 2*, the solution $\zeta(t) = [x(t), y(t)]$ is limited:

$$\left|x_i(t)\right| < \frac{\sum_{i=1}^{n} | c_{ij} | Ai_j}{\sigma_i}, \quad \left|y_j(t)\right| < \frac{\sum_{j=1}^{m} | \kappa_{ij} | B_{ij} \int_{-\infty}^{\alpha} | \Upsilon_{ij}(t-\xi) | \mathrm{d}\xi}{\varepsilon_i}$$

where $t > 0$, $i = 1, 2,..., n, j = 1, 2,..., m$.

Using (1) and ($T1$)-($T4$), for an arbitrary natural number of κ, we obtain:

$$\left\{(-1)^{k+1} x_i\left[t+(k+1)\right]T\right\}' = (-1)^{k+1} x_i'\left[t+(k+1)\right]T = (-1)^{k+1} \{-a_i\left[x_i\left(t+(k+1)T\right)\right] +$$

$$+\sum_{j=1}^{m} b_{ij} f\left\{x_i\left[t+(k+1)\right]T\right\} + \sum_{j=1}^{m} c_{ij} h\left\{x_j\left[t+(k+1)T)-\tau\right]\right\} + I_i = a_i\left\{(-1)^{k+1} x_i\left[t+(k+1)T\right]\right\} +$$

$$+\sum_{j=1}^{m} b_{ij} f\left\{(-1)^{k+1} x_i\left[t+(k+1)T\right]\right\} + \sum_{j=1}^{m} c_{ij} h\left\{(-1)^{k+1} x_j\left[t+(k+1)T)-\tau\right]\right\} + I_i$$

and

$$\left\{(-1)^{k+1} y_j\left[t+(k+1)\right]T\right\}' = (-1)^{k+1} y_j'\left[t+(k+1)\right]T =$$

$$= (-1)^{k+1} \{-d_j y_j\left[t+(k+1)T)\right] + \sum_{i=1}^{n} e_{ij} k\left[y_j\left(t+(k+1)T\right)\right] -$$

$$-\sum_{i=1}^{n} \kappa_{ij} \int_{-\infty}^{t+(k+1)T} \Upsilon_{ij}\left(t+(k+1)T-\xi\right) g_{ij} x_i(\xi)\} =$$

$$= -d_j (-1)^{k+1} y_j\left[t+(k+1)T)\right] + \sum_{i=1}^{n} e_{ij} k\left\{(-1)^{k+1} y_j\left[t+(k+1)T\right]\right\} -$$

$$-\sum_{i=1}^{n} \kappa_{ij} \int_{-\infty}^{t+(k+1)T} \Upsilon_{ij} (-1)^{k+1}\left[t+(k+1)T-\xi\right] g_{ij} x_i(\xi)$$

Thus, for any natural number k, $(-1)^{k+1}\, \zeta[t+(k+1)T]$ are the solutions of system (1). Then, by *Lemma 2*, there exists a constant $A > 0$ such that:

$$\left|(-1)^{k+1} x_i\left[t+(k+1)T\right] - (-1)^k x_i(t+kT)\right| \leq A\mathrm{e}^{-\beta(t+kT)} sup \max_{1\leq i\leq n}\left|x_i(\xi+T)+x_i(\xi)\right| \leq$$

$$\leq 2A\mathrm{e}^{-\beta(t+kT)} \frac{\sum_{i=1}^{n} | c_{ij} | A_{ij}}{\sigma_i}, \text{ for } t+kT>0, \ i=1,2,...,n \tag{5}$$

and

$$\left|(-1)^{k+1} y_j\left[t+(k+1)T\right] - (-1)^k y_j(t+kT)\right| \leq A\mathrm{e}^{-\beta(t+kT)} sup \max_{1\leq j\leq m}\left|y_j(\xi+T)+y_j(\xi)\right| \leq$$

$$\leq 2A\mathrm{e}^{-\beta(t+kT)} \frac{\sum_{j=1}^{m} \left|\kappa_{ij}\right| L_{ij} \int_{-\infty}^{\alpha} \left|\Upsilon_{ij}(t-\xi)\right| \mathrm{d}\xi}{\varepsilon_i}, \text{ for } t+kT>0, \ j=1,2,...,m \tag{6}$$

So, for an arbitrary natural number of p we get:

$$(-1)^{p+1} x_i\left[t+(p+1)T\right]=x_i(t)+\sum_{k=0}^{p}\left\{(-1)^{k+1} x_i\left[t+(k+1)T\right]-(-1)^k x_i(t+kT)\right\}$$

and

$$\left|(-1)^{p+1} y_j\left[t+(p+1)T\right]\right|=y_j(t)+\sum_{k=0}^{p}\left\{(-1)^{k+1} y_j\left[t+(k+1)T\right]-(-1)^k y_j(t+kT)\right\}$$

where $i = 1, 2,..., n, j = 1, 2,..., m$. Then:

$$\left|(-1)^{p+1} x_i\left[t+(p+1)T\right]\right|=|x_i(t)|+\sum_{k=0}^{p}\left|(-1)^{k+1} x_i\left[t+(k+1)T\right]-(-1)^k x_i(t+kT)\right|$$

and

$$\left|(-1)^{p+1} y_j\left(t+(p+1)T\right)\right|=|y_i(t)|+\sum_{k=0}^{p}\left|(-1)^{k+1} y_j\left(t+(k+1)T\right)-(-1)^k y_j(t+kT)\right|$$

where $i = 1, 2,..., n, j = 1, 2,..., m$.

In view of eqs. (5) and (6), we can select enough big constants $L_1 > 0$, $L_2 > 0$, and positive constant ε_1, ε_2 such:

$$\left|(-1)^{k+1} x_i\left[t+(k+1)T\right]-(-1)^k x_i(t+kT)\right|\le\varepsilon_1(\mathrm{e}^{-\beta t})^k,\ \text{for}\ k>L_1,\ i=1,2,...,n$$

$$\left|(-1)^{k+1} y_j\left[t+(k+1)T\right]-(-1)^k y_j(t+kT)\right|\le\varepsilon_2(\mathrm{e}^{-\beta t})^k,\ \text{for}\ k>L_2,\ j=1,2,...,m$$

It follows from aforementioned that $\{(-1)^p\zeta(t + pT)\}$ uniformly approach to an incessant function $\zeta^*(t) = [x^*(t), y^*(t)]^T$, where:

$$x^*(t)=x_1^*(t),\ x_2^*(t),\dots x_m^*(t),\ \ y^*(t)=y_1^*(t),\ y_2^*(t),\dots y_n^*(t)$$

compact in any R set.

Presently, we aim demonstrate that $\zeta^*(t)$ is the T-AP solution of system (1). First $\zeta^*(t)$ is T-AP onward

$$\zeta^*(t+T)=\lim_{p\to\infty}(-1)^p\zeta(t+T+pT)=-\lim_{(p+1)\to\infty}(-1)^{p+1}\zeta\left[t+(p+1)T\right]=-\zeta^*(t)$$

Following we verify that $\zeta^*(t)$ is a solution of (1). Essentially, jointly with the continuity of the right side of (1), (12) mean that $\{((-1)^{p+1}\zeta(t + (p + 1)T))'\}$ regular approach to an incessant function on any compact set of R. In this way, allow $p \to \infty$, we get:

$$\mathrm{D}_t^\alpha x_i^*(t)=-a_i x_i^*(t)+\sum_{j=1}^{m} b_{ij} f\left[x_i^*(t)\right]+\sum_{j=1}^{m} c_{ij} h\left[x_j(t-\tau)\right]+I_i$$

and

$$\mathrm{D}_t^\alpha y_j^*(t)=-d_j y_j^*(t)+\sum_{i=1}^{n} e_{ij} k\left[y_j^*(t)\right]-\sum_{i=1}^{n}\kappa_{ij}\int_{-\infty}^{t}\Upsilon_{ij}(t-\xi)g_{ij}x_i(\xi)$$

Thence $\zeta^*(t)$ is a solution of system (1).

Consequently, by *Lemma 2* we can substantiate that $\zeta^*(t)$ has a global exponential stability. Thus the proof is completed.

An example

Take into account the following:

$$\mathrm{D}_t^\alpha x_1(t) = -a_1 x_1(t) + b_{11} f[x_1(t)] + b_{12} f[x_1(t)] + c_{11} h[x_1(t-\tau)] + c_{12} h[x_1(t-\tau)] + I_1$$

$$\mathrm{D}_t^\alpha x_2(t) = -a_2 x_2(t) + b_{21} f[x_2(t)] + b_{22} f[x_2(t)] + c_{21} h[x_2(t-\tau)] + c_{22} h[x_2(t-\tau)] + I_2$$

$$\mathrm{D}_t^\alpha y_1(t) = -d_1 y_1(t) + e_{11} k[y_1(t)] + e_{12} k[y_1(t)] +$$

$$+\kappa_{11} \int_{-\infty}^{t} \Upsilon_{11}(t-\xi) l_{11} x_1(\xi) \mathrm{d}s + \kappa_{12} \int_{-\infty}^{t} \Upsilon_{12}(t-\xi) l_{12} x_1(\xi) \mathrm{d}s \qquad (7)$$

$$\mathrm{D}_t^\alpha y_2(t) = -d_2 y_2(t) + e_{21} k[y_2(t)] + e_{22} k[y_2(t)] +$$

$$+\kappa_{21} \int_{-\infty}^{t} \Upsilon_{21}(t-\xi) g_{21} x_1(\xi) \mathrm{d}\xi + \kappa_{22} \int_{-\infty}^{t} \Upsilon_{22}(t-\xi) g_{22} x_1(\xi) \mathrm{d}\xi$$

where

$$f_{ij}(x) = g_{ij}(x) = k_{ij}(x) = \frac{|x+1| - |x-1|}{2}, \ i, j = 1,2$$

$$a_1 x_1(t) = 2 + \cos[x_1(t)], \ a_2 x_2(t) = 2 + \sin[x_2(t)], \ d_1 y_1(t) = 1 + \sin[y_1(t)]$$

$$d_2 y_2(t) = 2 + \cos[y_2(t)], \ c_{11} = c_{21} = 1, \ c_{21} = c_{22} = 1, \ \kappa_{11} = \kappa_{12} = 1, \ \kappa_{21} = \kappa_{22} = 2$$

$$\Upsilon_{ij}(\xi) = \mathrm{e}^{-\xi}, \ \int_0^\infty \Upsilon_{ij}(\xi) \mathrm{d}\xi = 1$$

$$b_{11} = e_{11} = \frac{1}{4}, \ b_{12} = e_{21} = \frac{1}{8}, b_{21} = e_{12} = \frac{1}{2}, \ b_{22} = e_{22} = \frac{1}{4}, \ g_{11} = g_{21} = \frac{1}{4}, \ g_{12} = \frac{1}{4}, \ g_{22} = \frac{1}{2}$$

$$\tau = 0.1 \alpha = 0.75 \ \text{and} \ I_1 = \frac{5}{8} \cos t + \frac{13}{8} \sin t, \ I_2 = \frac{1}{4} \cos t + \frac{3}{4} \sin t$$

We have $A_{ij} = C_{ij} = B_{ij} = L_{ij} = 1$, system (7) has availed all the axioms conditions in *Theorem 1*. Thus, the system (7) provides a *T*-AP solution. This results affirm the global exponential stability for AP solution with fractional order.

Conclusion

This manuscript we have investigated the AP solutions for delayed BAM type fractional NN. Lyapunov functional and Inequality techniques have been employed to solve the governing model. In chapter *Preliminaries*, some basic prefaces and preliminaries have been stated. The effectiveness of the obtained results are shown through some examples. A criterion for the AP problem of delayed fractional BAM NN is given in chapter *Main Results* to construct global exponential stability for the problem discussed. An example is given to illustrate the effectiveness of the new result. It is important to note that, the results of discrete time exponential stability of these NN have not been explored. The BAM NN need not be applied to discrete-time systems. Artificial NN require discretization for continuous time. Therefore, in this study, the dynamics of discrete-time NN play an important role in theory and practice. The criterion was applied in a fractional order equation. Global exponential stability, it is known that NN play an important role in many areas such as applications and synchronization in secure communication. Therefore, the global exponential stability analysis of the equation is important both in theory and in practice.

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Paper Submitted: August 5, 2019
Paper revised: October 11, 2019
Paper accepted: October 15, 2019