# A NUMERICAL SCHEME TO SOLVE VARIABLE ORDER DIFFUSION-WAVE EQUATIONS

by

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*In this work, we consider variable order diffusion-wave equations. We choose variable order derivative in the Caputo sense. First, we approximate the unknown functions and its derivatives using Bernstein basis. Then, we obtain operational matrices based on Bernstein polynomials. Finally, with the help of these operational matrices and collocation method, we can convert variable order diffusion-wave equations to an algebraic system. Few examples are given to demonstrate the accuracy and the competence of the presented technique.*

Key words*: variable order diffusion-wave equations, Bernstein polynomials, operational matrix, collocation method*

## Introduction

Fractional calculus (FC) can be used to simulate various real phenomena involving long memory accurately [1]. Many problems in various fiels such as physics, chemistry, biology and engineering such as viscoelasticity and damping, diffusion equations, electromagnetic waves can be modelling via systems of fractional ordinary/ partial/ integro-differential equations [2-7]. In most cases, It is difficult to obtain exact solution of most ordinary/ partial/ integro-differential equations. Therefore, we must use of the approximate and numerical methods. There are several numerical methods for solving such equations, see [8-16].

Variable order derivative is a new definition, Samko and Ross [17], in FC which means the order is a function of time, space or other variables. Since derivative operator has a kernel of the variable order, it is not simply task to find the solution of such equations. Recently, several numerical and approximate methods have been presented to solve variable order of differential equations (VODE). Yu and Erturk [18] applied a finite difference method to solve VO integro-differential equations. Jafari *et al.* [19] obtained the approximate solution of differential equations with variable order using operational matices. In [20, 21], the functional boundary value problems with variable order are solved by reproducing kernel method. Ganji and Jafari [22] applied Jacobi polynomials to obtain solution of multi VODE. Hassani and Naraghirad [23] solved variable-order time fractional Burgers equation via generalized polynomials. Heydari *et al.* [24] obtained a approximate solution for VO diffusion-wave equation by Chebyshev

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wavelets. Jiang and Guo [25] applied the reproducing kernel method to solve 2-D VO anomalous sub-diffusion equation and see [26, 27].

The aim of this work is obtainig a numerical scheme to solve variable order diffusion-wave equations (VODWE) by Bernstein polynomials (BP).

We investigate the following type of VODWE:

$$\frac{\partial^{\omega(x,t)}\chi(x,t)}{\partial t^{\omega(x,t)}}+\gamma(x,t)\frac{\partial^{\upsilon(x,t)}\chi(x,t)}{\partial x^{\upsilon(x,t)}}=\mathrm{H}\left(x,t,\chi\right),\quad 0\le x,t\le 1 \tag{1}$$

$$\chi(x,0)=f_0(x),\quad \chi(0,t)=f_1(t),\quad \chi(1,t)=f_2(t) \tag{2}$$

where $0<\omega(x,t)\le 1$ and $1<\upsilon(x,t)\le 2$. The $q$ and $q'$ are positive integer numbers. The $\gamma(x,t)\in L^2([0.1]^2)$ is a known function. The $\chi(x,t)$ is an unknown function. The $\partial^{\upsilon(x,t)}\chi(x,t)/\partial x^{\upsilon(x,t)}$ and $\partial^{\omega(x,t)}\chi(x,t)/\partial x^{\omega(x,t)}$ indicate the VO derivatives respect to space and time:

$$\frac{\partial^{\omega(x,t)}\chi(x,t)}{\partial t^{\omega(x,t)}}=\frac{1}{\Gamma[1-\omega(x,t)]}\int_0^t(t-s)^{-\omega(x,t)}\frac{\partial\chi(x,s)}{\partial s}\,\mathrm{d}s$$

$$\frac{\partial^{\upsilon(x,t)}\chi(x,t)}{\partial x^{\upsilon(x,t)}}=\frac{1}{\Gamma[2-\upsilon(x,t]}\int_0^x(x-s)^{1-\upsilon(x,t)}\frac{\partial^2\chi(s,t)}{\partial s^2}\,\mathrm{d}s$$

We can show easily that:

$$\frac{\partial^{\omega(x,t)}\ t^r}{\partial t^{\omega(x,t)}}=\begin{cases}\dfrac{\Gamma(r+1)\ t^{r-\omega(x,t)}}{\Gamma[r-\omega(x,t)+1]}, & r\in\mathbb{N},r\ge\lceil\omega(x,t)\rceil\text{ or }r\notin\mathbb{N},r>\lfloor\omega(x,t)\rfloor\\ 0, & r\in\mathbb{N}\cup\{0\},r<\lceil\omega(x,t)\rceil\end{cases} \tag{3}$$

Similarly:

$$\frac{\partial^{\omega(x,t)}\ x^r}{\partial x^{\omega(x,t)}}=\begin{cases}\dfrac{\Gamma(r+1)\ x^{r-\omega(x,t)}}{\Gamma[r-\omega(x,t)+1]}, & r\in\mathbb{N},\ r\ge\lceil\omega(x,t)\rceil\text{ or }r\notin\mathbb{N},\ r>\lfloor\omega(x,t)\rfloor\\ 0, & r\in\mathbb{N}\cup\{0\},\ r<\lceil\omega(x,t)\rceil\end{cases} \tag{4}$$

## Bernstein polynomials

The BP are important in numerous area of mathematics. These polynomails are positive and their sum is unit.

The $n^{\text{th}}$ degree BP:

$$\mathrm{B}_{k,n}(t)=\binom{n}{k}t^k(1-t)^{n-k},\qquad 0\le t\le 1,\qquad k=0,1,2,...,n \tag{5}$$

Applying the binomial expansion for:

$$\mathrm{B}_{k,n}(t)=\sum_{p=0}^{n-k}(-1)^p\binom{n}{k}\binom{n-k}{p}t^{k+p},\quad k=0,1,\ldots,n \tag{6}$$

We can write Bernstein basis polynomials in the matrix form:

$$\varphi(t)=[\mathrm{B}_{0,n}(t),\mathrm{B}_{1,n}(t),\ldots,\mathrm{B}_{n,n}(t)]^T=AT_n(t) \tag{7}$$

where

$$A=[a_{ij}]_{(n+1)\times(n+1)}=\begin{cases}(-1)^{j-i}\dbinom{n}{i}\dbinom{n-i}{j-i}, & i\le j\\ 0, & i>j\end{cases}\quad\text{and}\quad T_n(t)=\begin{bmatrix}1\\ t\\ \vdots\\ t^n\end{bmatrix}$$

## Function approximation

We can approximate $\chi(x, t) \in L^2([0.1]^2)$ by the first $n + 1$ terms of BP:

$$\chi(x,t) \approx \chi_n(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}\mathrm{B}_{i,n}(x)\mathrm{B}_{j,n}(t) = \varphi(x)^T \mathcal{C}\varphi(t) \tag{8}$$

where

$$\mathcal{C} = \mathcal{Q}^{-1}\left\langle \varphi(x), \left\langle \chi_n(x,t), \varphi(t)\right\rangle\right\rangle \mathcal{Q}^{-1}$$

where

$$\mathcal{Q} = [q_{ij}]_{(n+1)\times(n+1)}, \qquad q_{ij} = \int_0^1 \mathcal{B}_{i,n}(t)\mathcal{B}_{j,n}(t)\mathrm{d}t = \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}, \qquad i, j = 0,1,\ldots,n$$

where $\langle .,. \rangle$ is the inner product.

### *Convergence analysis*

Let $\Pi_n = span\{\mathrm{B}_{i,n}(x)\mathrm{B}_{j,n}(t),\ i, j = 0, 1,..., n\}$. Suppose that $\chi(x, t) \in I = ([0.1]^2)$ be a smooth function and $\overline{\chi}(x, t) \in \Pi_n$ is the best approximation of $\chi(x, t)$. We obtain an analytic expression for error.

In view of definition of the best approximation:

$$\forall \chi_n(x,t) \in \Pi_n, \| \chi(x,t) - \overline{\chi}(x,t) \|_\infty \le \| \chi(x,t) - \chi_n(x,t) \|_\infty$$

If $\chi_n(x, t)$ denotes the interpolating polynomial for $\chi_n(x, t)$ at points $(x_i, t_j)$, where $x_i$, $i = 0, 1..., n$ are the roots of $\mathrm{B}_{i,n}(x)$ while $t_j$, $j = 0, 1,...,n$ are the roots of $\mathrm{B}_{j,n}(t)$, then the previous inequality is true. Then by similar procedures as in [25, 28-30]:

$$\chi(x,t) - \chi_n(x,t) = \frac{\partial^{n+1}\chi(\xi,t)}{\partial x^{n+1}\,(n+1)!}\prod_{i=0}^{n}(x - x_i) + \frac{\partial^{n+1}\chi(x,\eta)}{\partial t^{n+1}\,(n+1)!}\prod_{j=0}^{n}(t - t_j) - \\ - \frac{\partial^{2n+2}\chi(\xi',\eta')}{\partial x^{n+1}\partial t^{n+1}[(n+1)!]^2}\prod_{i=0}^{n}(x - x_i)\prod_{j=0}^{n}(t - t_j)$$

where $\xi$, $\xi'$, $\eta$, $\eta' \in [0,1]$. Then we obtain:

$$\| \chi(x,t) - \chi_n(x,t) \|_\infty \le \max_{(x,t)\in I} \left| \frac{\partial^{n+1}\chi(\xi,t)}{\partial x^{n+1}} \right| \frac{\| \prod_{i=0}^{n}(x - x_i) \|_\infty}{(n+1)!} + \\ + \max_{(x,t)\in I} \left| \frac{\partial^{n+1}\chi(x,\eta)}{\partial t^{n+1}} \right| \frac{\| \prod_{j=0}^{n}(t - t_j) \|_\infty}{(n+1)!} + \\ + \max_{(x,t)\in I} \left| \frac{\partial^{2n+2}\chi(\xi',\eta')}{\partial x^{n+1}\,\partial t^{n+1}} \right| \frac{\| \prod_{j=0}^{n}(t - t_j) \|_\infty \| \prod_{i=0}^{n}(x - x_i) \|_\infty}{[(n+1)!]^2}$$

Since $\chi(x, t)$ is a smooth function on $I$, then there exist constants $s_1$, $s_2$, and $s_3$:

$$\max_{(x,t)\in I}\left|\frac{\partial^{n+1}\chi(x,t)}{\partial x^{n+1}}\right|\le s_1,\quad \max_{(x,t)\in I}\left|\frac{\partial^{n+1}\chi(x,t)}{\partial t^{n+1}}\right|\le s_2,\quad \max_{(x,t)\in I}\left|\frac{\partial^{2n+2}\chi(x,t)}{\partial x^{n+1}\,\partial t^{n+1}}\right|\le s_3$$

The factor $\| \Pi_{i=0}^{n} (x - x_i)\|_\infty$ by the mapping $x = (\tau + 1)/2$ between $[-1, 1]$ and $[0, 1]$:

$$\min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\prod_{i=0}^{n}(x-x_i)\right|=\min_{\tau_i\in[-1,1]}\max_{\tau\in[-1,1]}\left|\prod_{i=0}^{n}\frac{1}{2}(\tau-\tau_i)\right|=\frac{1}{2^{n+1}}\min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\prod_{i=0}^{n}(\tau-\tau_i)\right|=\frac{1}{2^{2n+1}}$$

where $\tau_i$ are the roots of Chebyshev polynomials. Then we obtain:

$$\|\chi(x,t)-\overline{\chi}(x,t)\|_\infty\le\|\chi(x,t)-\chi_n(x,t)\|_\infty\le\frac{s_1+s_2}{2^{2n+1}(n+1)!}+\frac{s_3}{2^{4n+2}[(n+1)!]^2} \tag{9}$$

By using of the approximate $\chi(x, t)$ as eq. (8), we obtain:

$$\|\chi(x,t)-\varphi(x)^T C\varphi(t)\|_2^2=\int_0^1\int_0^1|\chi(x,t)-\varphi(x)^T C\varphi(t)|^2\,\mathrm{d}x\mathrm{d}t\le\int_0^1\int_0^1|\chi(x,t)-\chi^*(x,t)|^2\,\mathrm{d}x\mathrm{d}t$$

where $\chi^*$ indicates the interpolating polynomial of degree $n$ on $I$. Using eq. (9):

$$\|\chi(x,t)-\chi^*(x,t)\|_\infty\le\frac{s_1+s_2}{2^{2n+1}(n+1)!}+\frac{s_3}{2^{4n+2}[(n+1)!]^2}$$

We know that $\|\cdot\|_2\le n^{1/2}\|\cdot\|_\infty$, then we obtain

$$\|\chi(x,t)-\varphi(x)^T C\varphi(t)\|_2\le\sqrt{n}\left(\frac{s_1+s_2}{2^{2n+1}(n+1)!}+\frac{s_3}{2^{4n+2}[(n+1)!]^2}\right)$$

Then, we have:

$$\|\chi(x,t)-\varphi(x)^T C\varphi(t)\|_2\to 0,\quad\text{as}\quad n\to\infty$$

## Operational matrix

### *Operational matrix of the integer order derivatives*

The differentiation of vectors $\varphi(t)$ and $\varphi(x)$ can be given:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)=D\varphi(t)\quad\text{and}\quad\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)=D\varphi(x)$$

where $D$ is the $(n + 1) \times (n + 1)$ operational matrix for derivative based on BP. For $l \ge 2$, where $l$ is the order of derivative:

$$\frac{\mathrm{d}^l}{\mathrm{d}t^l}\varphi(t)=D^l\varphi(t)\quad\text{and}\quad\frac{\mathrm{d}^l}{\mathrm{d}x^l}\varphi(x)=D^l\varphi(x) \tag{10}$$

The details of obtainning this matrix are given in [19].

### *Operational matrix of the VO derivative*

Here, we obtain the operational matrix of VO derivative for vector $\varphi(t)$:

$$\frac{\partial^{\omega(x,t)}}{\partial t^{\omega(x,t)}}\varphi(t)=\frac{\partial^{\omega(x,t)}}{\partial t^{\omega(x,t)}}[AT_n(t)]=A\frac{\partial^{\omega(x,t)}}{\partial t^{\omega(x,t)}}[1\quad t\ \cdots\ t^{q-1}\quad t^q\ \ldots\ t^n]^T$$

According to eq. (3), we take $q = \lceil\omega(x, t)\rceil$ and $q < n$, then:

$$\frac{\partial^{\omega(x,t)}}{\partial t^{\omega(x,t)}}\varphi(t) = A\left[0 \quad 0 \quad \cdots \quad 0 \quad \frac{\Gamma(q+1)\,t^{q-\omega(x,t)}}{\Gamma[q+1-\omega(x,t)]} \quad \cdots \quad \frac{\Gamma(n+1)\,t^{n-\omega(x,t)}}{\Gamma[n+1-\omega(x,t)]}\right]^T = A\Upsilon T_n(t)$$

where

$$\Upsilon = [\rho_{sj}]_{(n+1)\times(n+1)} = \begin{cases} \dfrac{\Gamma(s+1)\,t^{-\omega(x,t)}}{\Gamma[s+1-\omega(x,t)]}, & s = j \geq q \\ 0, & \text{otherwise} \end{cases}$$

From eq. (7), $T_n(t) = A^{-1}\,\varphi(t)$:

$$\frac{\partial^{\omega(x,t)}}{\partial t^{\omega(x,t)}}\varphi(t) = A\Upsilon A^{-1}\varphi(t)$$

We rewrite:

$$A\Upsilon A^{-1}\varphi(t) = \Psi\varphi(t) \tag{11}$$

Similarly, we can write:

$$\frac{\partial^{\upsilon(x,t)}}{\partial x^{\upsilon(x,t)}}\varphi(x) = \mathrm{Y}\varphi(x) \tag{12}$$

where $\mathrm{Y} = A\Phi A^{-1}$ and:

$$\Phi = [\sigma_{sj}]_{(n+1)\times(n+1)} = \begin{cases} \dfrac{\Gamma(s+1)\,x^{-\upsilon(x,t)}}{\Gamma[s+1-\upsilon(x,t)]}, & s = j \geq q' = \lceil \upsilon(x,t) \rceil \\ 0, & \text{otherwise} \end{cases}$$

where $\Psi$ and $\mathrm{Y}$ are the operational matrices for variable orders derivatives based on Bernstein polynomials.

## The method for solving eq. (1)

Substituting eqs. (8), (10)-(12) in eq. (1), we have:

$$\begin{gathered} \varphi(x)^T \mathcal{C}\Delta^{\omega(x,t)}\varphi(t) + \gamma(x,t)[\Pi^{\upsilon(x,t)}\varphi(x)]^T \mathcal{C}\varphi(t) = \mathrm{H}\left(x,t,\varphi(x)^T\mathcal{C}\varphi(t)\right) \\ \varphi(x)^T \mathcal{C}\varphi(0) = f_0(x) \end{gathered} \tag{13}$$

$$\varphi(0)^T \mathcal{C}\varphi(t) = f_1(t), \quad \varphi(1)^T \mathcal{C}\varphi(t) = f_2(t) \tag{14}$$

where

$$\Delta^{\omega(x,t)} = \begin{cases} \Psi, & 0 < \omega(x,t) < 1 \\ D, & \omega(x,t) = 1 \end{cases} \quad \text{and} \quad \Pi^{\upsilon(x,t)} = \begin{cases} \mathrm{Y}, & 1 < \upsilon(x,t) < 2 \\ D^2, & \upsilon(x,t) = 2 \end{cases}$$

We define the residual function:

$$R(x,t) = \varphi(x)^T \mathcal{C}\Delta^{\omega(x,t)}\varphi(t) + \gamma(x,t)[\Pi^{\upsilon(x,t)}\varphi(x)]^T \mathcal{C}\varphi(t) - \mathrm{H}\left(x,t,\varphi(x)^T\mathcal{C}\varphi(t)\right) \tag{15}$$

Substituting points $x_i = i\,/\,n$ and $t_j = j\,/\,n$ in eqs. (13)-(15), we obtain:

$$\begin{aligned} & R(x_i,t_j) = \varphi(x_i)^T \mathcal{C}\Delta^{\omega(x_i,t_j)}\varphi(t_j) + \gamma(x_i,t_j)[\Pi^{\upsilon(x_i,t_j)}\varphi(x_i)]^T \mathcal{C}\varphi(t_j) \\ & -\mathrm{H}\left(x_i,t_j,\varphi(x_i)^T\mathcal{C}\varphi(t_j)\right) = 0, \qquad i = 1,2,\ldots,n-1, \;\; j = 1,2,\ldots,n \\ & \varphi(x_i)^T \mathcal{C}\varphi(0) = f_0(x_i), \qquad i = 0,1,\ldots,n \\ & \varphi(0)^T \mathcal{C}\varphi(t_j) = f_1(t_j), \;\; \varphi(1)^T \mathcal{C}\varphi(t_j) = f_2(t_j), \quad j = 1,2,\ldots,n \end{aligned} \tag{16}$$

By solving system eq. (16), coefficients $c_{ij}$ can be calculated. Finally, we obtain the approximate solution for eq. (1).

## Test examples

We present three examples to show the efficiency of this method. We compare the exact and approximate solutions. Here the absolute errors are defined:

$$\text{Error} = |\chi(x,t) - \varphi(x)^T C\varphi(t)|, \qquad x,\, t \in [0,1] \tag{17}$$

*Example 1*. Consider the following VODWE:

$$\frac{\partial^{\omega(x,t)}\chi(x,t)}{\partial t^{\omega(x,t)}} = \frac{\partial^2\chi(x,t)}{\partial x^2} + \frac{2\,x\,t^{2-\omega(x,t)}}{\Gamma[3-\omega(x,t)]}, \qquad 0 \le x,t \le 1$$

$$\chi(x,0) = 0, \quad \chi(0,t) = 0, \quad \chi(1,t) = t^2$$

where $\omega(x,\, t) = \sin(x,\, t)$. The exact solution is $\chi(x,\, t) = xt^2$. The numerical results using the presented method are shown in fig. 1 and tab. 1 show the absolute error for various $\omega(x,\, t)$.

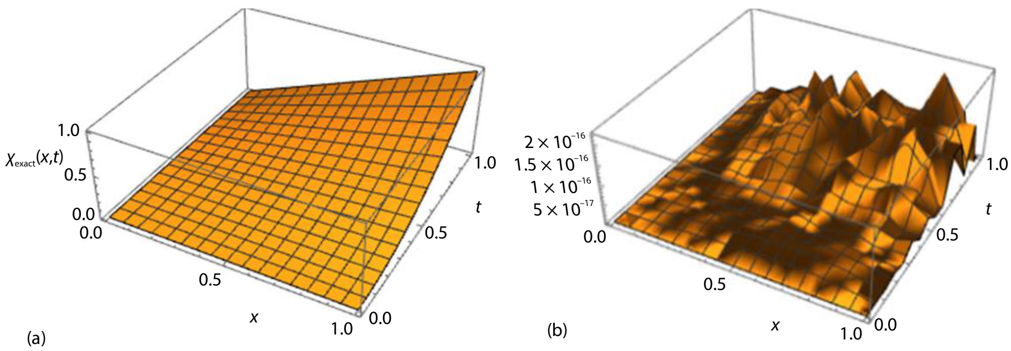


**Figure 1. (a) The exact solution, (b) the absolute errors, $n$ = 3**

**Table 1. Comparison absolute errors for various $\omega(x, t)$, $n$ = 3**

| $(x, t)$ | $\omega = [1-(xt)^4]/5$ | $\omega = x^2t^2$ | $\omega = \cos xt$ |
|---|---|---|---|
| (0.1, 0.1) | 1.51788e$^{-18}$ | 1.30104e$^{-18}$ | 8.67362e$^{-19}$ |
| (0.2, 0.2) | 6.93889e$^{-18}$ | 6.93889e$^{-18}$ | 5.20417e$^{-18}$ |
| (0.3, 0.3) | 0.00000 | 6.93889e$^{-18}$ | 6.93887e$^{-18}$ |
| (0.3, 0.4) | 1.38779e$^{-17}$ | 1.38779e$^{-17}$ | 1.38778e$^{-17}$ |
| (0.5, 0.5) | 0.00000 | 1.38779e$^{-17}$ | 0.00000 |
| (0.6, 0.6) | 2.77556e$^{-17}$ | 0.00000 | 8.32667e$^{-17}$ |
| (0.7, 0.7) | 0.00000 | 5.55112e$^{-17}$ | 0.00000 |
| (0.8, 0.8) | 1.11022e$^{-16}$ | 0.00000 | 1.11022e$^{-16}$ |
| (0.9, 0.9) | 0.00000 | 0.00000 | 0.00000 |

*Example 2***.** Consider the following VODWE:

$$\frac{\partial^{\omega(x,t)}\chi(x,t)}{\partial t^{\omega(x,t)}}+\frac{\partial^{\upsilon(x,t)}\chi(x,t)}{\partial x^{\upsilon(x,t)}}=\frac{x^2\ t^{1-\omega(x,t)}}{\Gamma[2-\omega(x,t)]}+\frac{2\ t\ x^{2-\upsilon(x,t)}}{\Gamma[3-\upsilon(x,t)]},\qquad 0\le x,t\le 1$$

$$\chi(x,0)=0,\qquad \chi(0,t)=0,\quad \chi(1,t)=t$$

where $\omega(x,t)$ and $\upsilon(x,t)$ are $1+\sin(xt)$ and $1+\cos(xt)$, respectively. For solving this example, we applied the presented method. The exact solution [$\chi\ (xt)=x^2t$] and the absolute errors are shown in fig. 2.

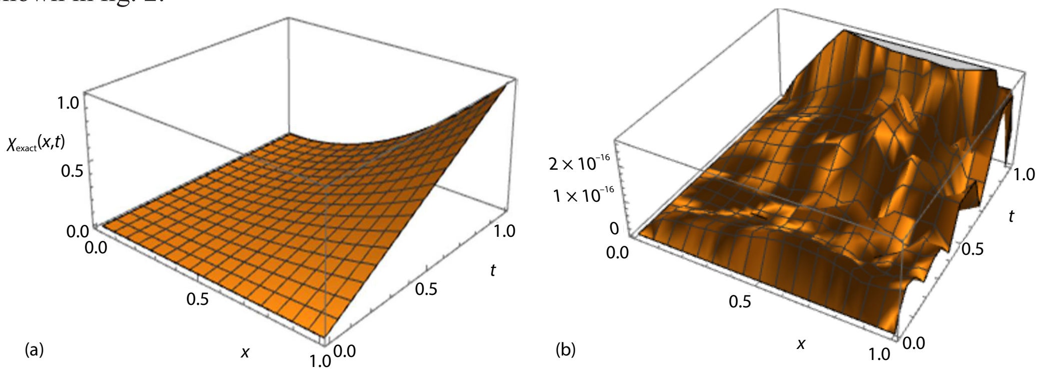


**Figure 2. (a) The exact solution, (b) the absolute errors, *n* = 5**

*Example 3*. Consider the following VODWE:

$$\frac{\partial^{\omega(x,t)}\chi(x,t)}{\partial t^{\omega(x,t)}}=\frac{1}{2}x^2\frac{\partial^2\chi(x,t)}{\partial x^2}+x^2e^t\left(\frac{\Gamma(\sin(xt))-\Gamma(\sin(xt),t)}{\Gamma(\sin(xt))}-1\right),\qquad 0<x,t<1$$

$$\chi(x,0)=x^2,\qquad \chi(0,t)=0,\quad \chi(1,t)=e^t$$

where $\omega(x,t)=1-\sin\ (xt)$. The exact solution is $\chi\ (xt)=x^2\mathrm{e}^t$. We applied the presented method and obtained approximate solution. The obtained results are plotted in fig. 3. Table 2 shows the absolute errors.

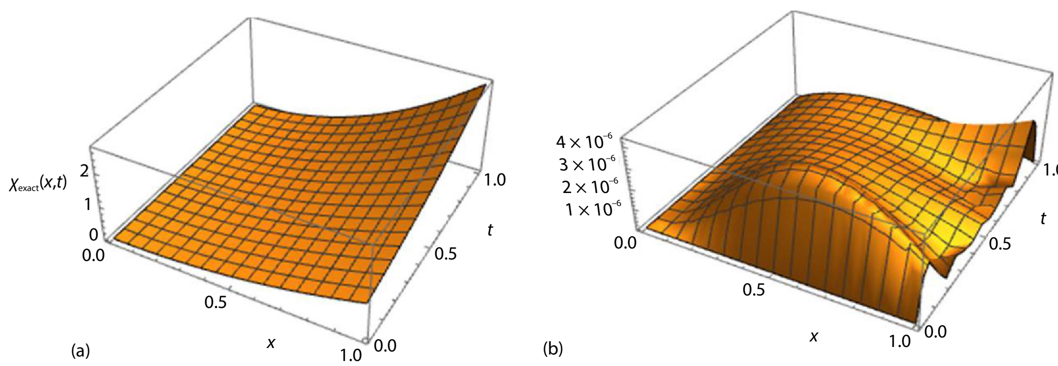


**Figure 3. (a) The exact solution, (b) the absolute errors, *n* = 5**

**Table 2. Absolute errors for various $n$ [$\omega(x, t) = 1 - \sin(x, t)$]**

| $(x, t)$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
|---|---|---|---|---|
| (0.1, 0.1) | 0.00088 | $1.04991e^{-4}$ | $5.51477e^{-6}$ | $2.51701e^{-6}$ |
| (0.2, 0.2) | 0.00299 | $3.27665e^{-4}$ | $2.20289e^{-5}$ | $1.34078e^{-6}$ |
| (0.3, 0.3) | 0.00530 | $4.95985e^{-4}$ | $3.28924e^{-5}$ | $2.04508e^{-6}$ |
| (0.4, 0.4) | 0.00666 | $4.93978e^{-4}$ | $3.05810e^{-5}$ | $2.01647e^{-6}$ |
| (0.5, 0.5) | 0.00611 | $3.33595e^{-4}$ | $2.31369e^{-5}$ | $1.72872e^{-6}$ |
| (0.6, 0.6) | 0.00317 | $1.56402e^{-4}$ | $2.02252e^{-5}$ | $1.30305e^{-6}$ |
| (0.7, 0.7) | 0.00170 | $1.58402e^{-4}$ | $1.81164e^{-5}$ | $6.00023e^{-6}$ |
| (0.8, 0.8) | 0.00670 | $4.30336e^{-4}$ | $2.91337e^{-8}$ | $3.81590e^{-6}$ |
| (0.9, 0.9) | 0.00818 | $7.04763e^{-4}$ | $3.44797e^{-5}$ | $1.84695e^{-4}$ |

## Conclusion

In this work, the numerical solution of VODWE using operational matrices based on Bernstein polynomials are investigated. We approximated the unknown function and obtained operational matrices of variable orders derivatives based on Bernstein polynomials. Then, using operational matrices and collocation method, we transferred VODWE to a system of algebraic equations and obtained the numerical solution of this system. Finally numerical examples are presented to demonstrate the high performance of the presented method. We saw that the numerical solution obtained converges to the analytical solution.

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