A NUMERICAL SCHEME TO SOLVE VARIABLE ORDER DIFFUSION-WAVE EQUATIONS

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In this work, we consider variable order diffusion-wave equations. We choose variable order derivative in the Caputo sense. First, we approximate the unknown functions and its derivatives using Bernstein basis. Then, we obtain operational matrices based on Bernstein polynomials. Finally, with the help of these operational matrices and collocation method, we can convert variable order diffusion-wave equations to an algebraic system. Few examples are given to demonstrate the accuracy and the competence of the presented technique.

Keywords: Variable order diffusion-wave equations; Bernstein polynomials; Operational matrix; Collocation method.

1. Introduction

Fractional calculus (FC) can be used to simulate various real phenomena involving long memory accurately [1]. Many problems in various fields such as physics, chemistry, biology and engineering such as viscoelasticity and damping, diffusion equations, electromagnetic waves can be modeling via systems of fractional ordinary/partial/integro-differential equations [2, 3, 4, 5, 6, 7]. In most cases, it is difficult to obtain exact solution of most ordinary/partial/integro-differential equations. Therefore, we must use of the approximate and numerical methods. There are several numerical methods for solving such equations (see [8, 9, 10, 11, 12, 13, 14, 15, 16]).

Variable order derivative is a new definition (Samko and Ross [17] in 1993) in FC which means the order is a function of time, space or other variables. Since derivative operator has a kernel of the variable order, it is not simply task to find the solution of such equations. Recently, several numerical and approximate methods have been presented to solve variable order of differential equations (VODEs). Yu and Ertürk applied a finite difference method to solve VO integro-differential equations [18]. Jafari et al. obtained the approximate solution of differential equations with variable order using operational matices [19]. In [20] and [21], the functional boundary value problems with variable order are solved by reproducing kernel method. Ganji and Jafari applied Jacobi polynomials to obtain solution of multi VODEs [22]. Hassani and Naraghirad solved variable-order time fractional Burgers equation via generalized polynomials [23]. Heydari et al. obtained a approximate solution for VO diffusion-wave equation by Chebyshev wavelets [24]. Jiang and Guo applied the reproducing kernel method to solve two-dimensional VO anomalous sub-diffusion equation [25] and see [26, 27].
The aim of this work is obtaining a numerical scheme to solve variable order diffusion-wave equations (VODWEs) by Bernstein polynomials (BPs).

We investigate the following type of VODWEs

\[
\frac{\partial^{\alpha(x,t)}}{\partial t^{\alpha(x,t)}} \chi(x,t) + \gamma(x,t) \frac{\partial^{\nu(x,t)}}{\partial x^{\nu(x,t)}} \chi(x,t) = H(x,t,\chi),
\]

(1)

\[
\chi(x,0) = f_0(x), \quad \chi(0,t) = f_1(t), \quad \chi(1,t) = f_2(t), \quad 0 < x, t < 1,
\]

(2)

where \( q - 1 < \omega(x,t) \leq q \) and \( q' - 1 < \nu(x,t) \leq q' \). \( q \) and \( q' \) are positive integer numbers. \( \gamma(x,t) \in L^2([0,1]^2) \) is a known function. \( \chi(x,t) \) is an unknown function. \( \partial^{\alpha(x,t)} \chi(x,t) / \partial t^{\nu(x,t)} \) and \( \partial^{\nu(x,t)} \chi(x,t) / \partial x^{\nu(x,t)} \) indicate the VO derivatives respect to space and time. They are defined

\[
\frac{\partial^{\alpha(x,t)}}{\partial t^{\alpha(x,t)}} \chi(x,t) = \frac{1}{\Gamma(q - \omega(x,t))} \int_0^t (t - s)^{q - \omega(x,t) - 1} \frac{\partial^r \chi(x,s)}{\partial s^r} \, ds,
\]

\[
\frac{\partial^{\nu(x,t)}}{\partial x^{\nu(x,t)}} \chi(x,t) = \frac{1}{\Gamma(q' - \nu(x,t))} \int_0^x (x - s)^{q' - \nu(x,t) - 1} \frac{\partial^{r'} \chi(s,t)}{\partial s'^{r'}} \, ds.
\]

We can show easily that

\[
\frac{\partial^{\alpha(x,t)}}{\partial t^{\alpha(x,t)}} I^r = \begin{cases} 
\frac{\Gamma(r + 1)}{\Gamma(r - \omega(x,t) + 1)}, & r \in \mathbb{N}, r \geq \lceil \omega(x,t) \rceil \text{ or } r \not\in \mathbb{N}, r > \lceil \omega(x,t) \rceil, \\
0, & r \in \mathbb{N} \cup \{0\}, r < \lceil \omega(x,t) \rceil.
\end{cases}
\]

(3)

Similarly

\[
\frac{\partial^{\nu(x,t)}}{\partial x^{\nu(x,t)}} I^r = \begin{cases} 
\frac{\Gamma(r + 1)}{\Gamma(r - \omega(x,t) + 1)}, & r \in \mathbb{N}, r \geq \lceil \omega(x,t) \rceil \text{ or } r \not\in \mathbb{N}, r > \lceil \omega(x,t) \rceil, \\
0, & r \in \mathbb{N} \cup \{0\}, r < \lceil \omega(x,t) \rceil.
\end{cases}
\]

(4)

2. Bernstein polynomials

The BPs are important in numerous area of mathematics. These polynomials are positive and their sum is unit.

The n-th degree BPs is defined as

\[
B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad 0 \leq t \leq 1, \quad k = 0, 1, 2, \ldots, n.
\]

(5)

Applying the binomial expansion for \((1-t)^{n-k}\), we have

\[
B_{k,n}(t) = \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} t^{k+p}, \quad k = 0, 1, \ldots, n.
\]

(6)

We can write Bernstein basis polynomials in the matrix form

\[
\varphi(t) = [B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t)]^T = A T_n(t),
\]

(7)

where
\[ A = [a_{ij}]_{(n+1) \times (n+1)} = \begin{cases} \binom{n}{i-j} \binom{n-i}{j-i}, & i \leq j, \\
 0, & i > j, \end{cases} \quad \text{and} \quad T_n(t) = \begin{bmatrix} 1 \\
 \vdots \\
 t^n \end{bmatrix}. \]

3. Function approximation

We can approximate \( \chi(x,t) \in L^2([0,1]^2) \) by the first \( n+1 \) terms of BPs as

\[ \chi(x,t) \approx \chi_n(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} B_{i,n}(x) B_{j,n}(t) = \varphi(x)^T C \varphi(t), \]

where

\[ C = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0n} \\
 c_{10} & c_{11} & \cdots & c_{1n} \\
 \vdots & \vdots & \ddots & \vdots \\
 c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}, \]

and \( c_{ij} \) elements are computed by

\[ c_{ij} = \frac{\left( B_{i,n}(x), \left( \chi(x,t), B_{j,n}(t) \right) \right)}{\left( B_{i,n}(x), B_{i,n}(x) \right) \left( B_{j,n}(t), B_{j,n}(t) \right)}, \quad i, j = 0,1,...,n. \]

We can rewrite (9) as

\[ C = Q^{-1} \left( \varphi(x)(\chi_n(x,t), \varphi(t)) \right) Q^{-1}, \]

where

\[ Q = (\varphi(t), \varphi(t)) = \int_0^1 \varphi(t) \varphi(t)^T dt = \int_0^1 B_{i,n}(t) B_{j,n}(t) dt = \frac{\binom{n}{i} \binom{n}{j}}{(2n+1) \binom{2n}{i+j}}, \quad i, j = 0,1,...,n. \]

3.1. Convergence analysis

Let \( \Pi_n = \text{span}\{ B_{i,n}(x) B_{j,n}(t), i, j = 0,1,...,n \} \). Suppose that \( \chi(x,t) \in I = [0,1]^2 \) be a smooth function and \( \chi_n(x,t) \in \Pi_n \) is the best approximation of \( \chi(x,t) \). We obtain an analytic expression for error.

In view of definition of the best approximation, we have

\[ \forall \chi_n(x,t) \in \Pi_n, \quad \| \chi(x,t) - \chi_n(x,t) \|_\infty \leq \| \chi(x,t) - \chi_n(x,t) \|_\infty. \]

If \( \chi_n(x,t) \) denotes the interpolating polynomial for \( \chi(x,t) \) at points \( \left( x_i, t_j \right) \), where \( x_i, i = 0,1,...,n \) are the roots of \( B_{i,n}(x) \), while \( t_i, j = 0,1,...,n \) are the roots of \( B_{j,n}(t) \), then the previous inequality is true. Then by similar procedures as in [25,28,29,30]
\[ \chi(x,t) - \chi_n(x,t) = \frac{\partial^{n+1} \chi(\xi,t)}{\partial x^{n+1}} (n+1)! \sum_{i=0}^{n} (x-x_i) + \frac{\partial^{n+1} \chi(x,\eta)}{\partial t^{n+1}} (n+1)! \sum_{j=0}^{n} (t-t_j), \]

where \( \xi, \xi', \eta, \eta' \in [0,1] \). Then we obtain

\[ \| \chi(x,t) - \chi_n(x,t) \|_\infty \leq \max_{(x,t) \in I} \left| \frac{\partial^{n+1} \chi(\xi,t)}{\partial x^{n+1}} \right| \| \sum_{i=0}^{n} (x-x_i) \|_\infty \]

\[ + \max_{(x,t) \in I} \left| \frac{\partial^{n+1} \chi(x,\eta)}{\partial t^{n+1}} \right| \| \sum_{j=0}^{n} (t-t_j) \|_\infty \]

\[ + \max_{(x,t) \in I} \left| \frac{\partial^{2n+2} \chi(\xi',\eta')}{\partial x^{n+1} \partial t^{n+1}} \right| \| \sum_{j=0}^{n} (t-t_j) \|_\infty \| \sum_{i=0}^{n} (x-x_i) \|_\infty \]

Since \( \chi(x,t) \) is a smooth function on \( I \), then there exist constants \( s_1, s_2 \) and \( s_3 \), such that

\[ \max_{(x,t) \in I} \left| \frac{\partial^{n+1} \chi(x,t)}{\partial x^{n+1}} \right| \leq s_1, \quad \max_{(x,t) \in I} \left| \frac{\partial^{n+1} \chi(x,t)}{\partial t^{n+1}} \right| \leq s_2, \quad \max_{(x,t) \in I} \left| \frac{\partial^{2n+2} \chi(x,t)}{\partial x^{n+1} \partial t^{n+1}} \right| \leq s_3. \]

The factor \( \| \sum_{i=0}^{n} (x-x_i) \|_\infty \) by the mapping \( x = (\tau + 1)/2 \) between \([-1,1]\) and \([0,1]\), one has

\[ \min_{x_i \in [0,1]} \max_{x \in [0,1]} \| \sum_{i=0}^{n} (x-x_i) \|_\infty \leq \min_{\tau_i \in [-1,1]} \max_{x \in [0,1]} \left| \frac{1}{2^{n+1}} \sum_{i=0}^{n} (x-x_i) \right| = \frac{1}{2^{n+1}} \min_{\tau_i \in [-1,1]} \max_{x \in [0,1]} \left| \frac{1}{2^{n+1}} \sum_{i=0}^{n} (x-x_i) \right| = \frac{1}{2^{n+1}}, \]

where \( \tau_i \) are the roots of Chebyshev polynomials. Then we obtain

\[ \| \chi(x,t) - \chi_n(x,t) \|_\infty \leq \max_{x \in [0,1]} \left| \frac{\partial^{n+1} \chi(x,t)}{\partial x^{n+1}} \right| \| \sum_{i=0}^{n} (x-x_i) \|_\infty \leq \frac{s_1 + s_2}{2^{n+1} (n+1)!} + \frac{s_3}{2^{4n+2} ((n+1)!)^2}. \] (10)

By using of the approximate \( \chi(x,t) \) as (8), we obtain

\[ \| \chi(x,t) - \varphi(x)^\tau \mathrm{Cf}(t) \|_2 \leq \int_0^1 \int_0^1 \| \chi(x,t) - \varphi(x)^\tau \mathrm{Cf}(t) \|^2 \, dx \, dt \]

\[ \leq \int_0^1 \int_0^1 \| \chi(x,t) - \chi^*(x,t) \|^2 \, dx \, dt, \]

where \( \chi^* \) indicates the interpolating polynomial of degree \( n \) on \( I \). Using (10) have

\[ \| \chi(x,t) - \chi^*(x,t) \|_\infty \leq \frac{s_1 + s_2}{2^{n+1} (n+1)!} + \frac{s_3}{2^{4n+2} ((n+1)!)^2}. \]

We know that \( \| \| \leq \sqrt{n} \| \|_\infty \), then we obtain

\[ \| \chi(x,t) - \varphi(x)^\tau \mathrm{Cf}(t) \|_2 \leq \sqrt{n} \left( \frac{s_1 + s_2}{2^{n+1} (n+1)!} + \frac{s_3}{2^{4n+2} ((n+1)!)^2} \right). \]

Then, we have

\[ \| \chi(x,t) - \varphi(x)^\tau \mathrm{Cf}(t) \|_2 \to 0, \quad \text{as} \quad n \to \infty. \]
4. Operational matrix

4.1. Operational matrix of the integer order derivatives

The differentiation of vectors \( \varphi(t) \) and \( \varphi(x) \) can be given as

\[
\frac{d}{dt} \varphi(t) = D \varphi(t) \quad \text{and} \quad \frac{d}{dx} \varphi(x) = D \varphi(x),
\]

where \( D \) is the \( (n+1) \times (n+1) \) operational matrix for derivative based on BPs. For \( l \geq 2 \), where \( l \) is the order of derivative, we get

\[
\frac{d^l}{dt^l} \varphi(t) = D^l \varphi(t) \quad \text{and} \quad \frac{d^l}{dx^l} \varphi(x) = D^l \varphi(x).
\]

The details of obtaining this matrix are given in [19].

4.2. Operational matrix of the VO derivative

Here, we obtain the operational matrix of VO derivative for vector \( \varphi(t) \):

\[
\frac{\partial \varphi(t)}{\partial t} = \frac{\partial \varphi(t)}{\partial t^o(t)} [A T_n(t)] = A \frac{\partial \varphi(t)}{\partial t^o(t)} [1 \ t \ t^{q-1} \ t^q \ \ldots \ t^n]^T.
\]

According to (3), we take \( q = \left[ \omega(x,t) \right] \) and \( q < n \), then

\[
\frac{\partial \varphi(t)}{\partial t^o(t)} = A [0 \ 0 \ \ldots \ 0 \ \frac{\Gamma(q+1)}{\Gamma(q+1-\omega(x,t))} \ t^{q-\omega(x,t)} \ \ldots \ \frac{\Gamma(n+1)}{\Gamma(n+1-\omega(x,t))} t^n]^T = A' T_n(t),
\]

where

\[
Y = \begin{bmatrix} \rho_{ij} \end{bmatrix}_{(n+1) \times (n+1)} = \begin{cases} \frac{\Gamma(s+1)t^{-\omega(x,t)}}{\Gamma(s+1-\omega(x,t))}, & s = j \geq q, \\ 0, & \text{otherwise.} \end{cases}
\]

From (7), \( T_n(t) = A^{-1} \varphi(t) \), then

\[
\frac{\partial \varphi(t)}{\partial t^o(t)} = A' \varphi(t).
\]

We rewrite

\[
A' \varphi(t) = \Psi \varphi(t).
\]

Similarly, we can write

\[
\frac{\partial \varphi(x)}{\partial x} = Y \varphi(x),
\]

where \( Y = A \Phi A^{-1} \) and

\[
\Phi = [\sigma_{ij}]_{(n+1) \times (n+1)} = \begin{cases} \frac{\Gamma(s+1)x^{-\omega(x,t)}}{\Gamma(s+1-\omega(x,t))}, & s = j \geq q' = \left[ \nu(x,t) \right], \\ 0, & \text{otherwise.} \end{cases}
\]

\( \Psi \) and \( Y \) are the operational matrices for variable orders derivatives based on Bernstein polynomials.
5. The method for solving Eq. (1)

Substituting (8), (11) and (12) in Eq. (1), we have

\[ \varphi(x)^T C \Psi \varphi(t) + \gamma(x,t) Y \varphi(x)^T C \varphi(t) = H\left(x,t, \varphi(x)^T C \varphi(t)\right), \]

\[ \varphi(x)^T C \varphi(0) = f_0(x), \quad 0 < x < 1, \quad (13) \]

\[ \varphi(0)^T C \varphi(t) = f_1(t), \quad \varphi(1)^T C \varphi(t) = f_2(t), \quad 0 < t < 1. \quad (14) \]

We define the residual function

\[ R(x,t) = \varphi(x)^T C \Psi \varphi(t) + \gamma(x,t) Y \varphi(x)^T C \varphi(t) - H\left(x,t, \varphi(x)^T C \varphi(t)\right). \quad (15) \]

Substituting points \( x_i \) and \( t_j \) in (13), (14) and (15), we obtain

\[ R(x_i,t_j) = \varphi(x_i)^T C \Psi \varphi(t_j) + \gamma(x_i,t_j) Y \varphi(x_i)^T C \varphi(t_j) \]

\[ - H\left(x_i,t_j, \varphi(x_i)^T C \varphi(t_j)\right) = 0, \quad i = 1,2,\ldots,n - 1, \quad j = 1,2,\ldots,n, \quad (16) \]

\[ \varphi(x_i)^T C \varphi(0) = f_0(x_i), \quad i = 0,1,\ldots,n, \]

\[ \varphi(0)^T C \varphi(t_j) = f_1(t_j), \quad \varphi(1)^T C \varphi(t_j) = f_2(t_j), \quad j = 1,2,\ldots,n. \]

By solving system Eq. (16), coefficients \( c_{ij} \) can be calculated. Finally, we obtain the approximate solution for Eq. (1).

6. Test Examples

We present three examples to show the efficiency of this method. We compare the exact and approximate solutions. Here the absolute errors are defined as

\[ Error = | \chi(x,t) - \varphi(x)^T C \varphi(t) |, \quad x,t \in [0,1]. \quad (17) \]

**Example 1.** Consider the following VODWEs

\[ \frac{\partial^{\omega(x,t)} \chi(x,t)}{\partial t^{\omega(x,t)}} = \frac{\partial^2 \chi(x,t)}{\partial x^2} + \frac{2x t^{2-\omega(x,t)}}{\Gamma(3-\omega(x,t))}, \]

\[ \chi(x,0) = 0, \quad \chi(0,t) = 0, \quad \chi(1,t) = t^2, \quad x,t \in (0,1), \]

where \( \omega(x,t) = \sin(x,t) \). The exact solution is \( \chi(x,t) = x t^2 \). The numerical results using the presented method are shown in Figure 1 and Table 1 show the absolute error for various \( \omega(x,t) \).

![Figure 1](image1.png)  
(a) The exact solution  
(b) The absolute errors \((n = 3)\).
Table 1: Comparison absolute errors for various $\omega(x,t)$ ($n = 3$).

<table>
<thead>
<tr>
<th>$(x,t)$</th>
<th>$\omega = (1-(x,t)^4)/5$</th>
<th>$\omega = x^2 t^2$</th>
<th>$\omega = \cos(x,t)$</th>
</tr>
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<tbody>
<tr>
<td>(0.1,0.1)</td>
<td>$1.51788e-18$</td>
<td>$1.30104e-18$</td>
<td>$8.67362e-19$</td>
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<tr>
<td>(0.2,0.2)</td>
<td>$6.93889e-18$</td>
<td>$6.93889e-18$</td>
<td>$5.20417e-18$</td>
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<td>$1.38779e-17$</td>
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<td>(0.8,0.8)</td>
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<td>$0.00000$</td>
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</tr>
</tbody>
</table>

**Example 2.** Consider the following VODWEs

$$\frac{\partial^{(n)}\chi(x,t)}{\partial t^{(n)}(x,t)} + \frac{\partial^{(n)}\chi(x,t)}{\partial x^{(n)}(x,t)} = \frac{2t}{\Gamma(3-u(x,t))},$$

$$\chi(x,0) = 0, \quad \chi(0,t) = 0, \quad \chi(1,t) = t, \quad 0 < x, t < 1,$$

where $\omega(x,t)$ and $u(x,t)$ are $1 + \sin(x,t)$ and $\cos(x,t)$ respectively. For solving this example, we applied the presented method. The exact solution ($\chi(x,t) = x^2 t$) and the absolute errors are shown in Figure 2.

![Figure 2](image1.png)

(a) The exact solution (b) The absolute errors ($n = 5$).

**Example 3.** Consider the following VODWEs

$$\frac{\partial^{(n)}\chi(x,t)}{\partial t^{(n)}(x,t)} = \frac{1}{2} x^2 \frac{\partial^2 \chi(x,t)}{\partial x^2} + x^2 e^t \left( \frac{\Gamma(\sin(x,t)) - \Gamma(\sin(x,t),t)}{\Gamma(\sin(x,t))} - 1 \right),$$

$$\chi(x,0) = x^2, \quad \chi(0,t) = 0, \quad \chi(1,t) = e^t, \quad 0 < x, t < 1,$$

where $0 \leq t \leq 1$, $\omega(x,t) = 1 - \sin(x,t)$. The exact solution is $\chi(x,t) = x^2 e^t$. We applied the presented method and obtained approximate solution. The obtained results are plotted in Figure 3. Table 2 shows the absolute errors.
Figure 3: (a) The exact solution (b) The absolute errors ($n = 5$).

Table 2: Absolute errors for various $n$ ($\omega(x,t) = 1 - \sin(x,t)$).

<table>
<thead>
<tr>
<th>$(x,t)$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
</tr>
</thead>
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<tr>
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<td>1.84695e-6</td>
</tr>
</tbody>
</table>

7. Conclusion

In this work, the numerical solution of VODWEs using operational matrices based on Bernstein polynomials are investigated. We approximated the unknown function and obtained operational matrices of variable orders derivatives based on Bernstein polynomials. Then, using operational matrices and collocation method, we transfer VODWEs to a system of algebraic equations and obtained the numerical solution of this system. Finally numerical examples are presented to demonstrate the high performance of the presented method. We saw that the numerical solution obtained converges to the analytical solution.

References


