The key objective of the present paper is to propose a numerical scheme based on the homotopy analysis transform technique to analyze a time-fractional nonlinear predator-prey population model. The population model are coupled fractional order nonlinear partial differential equations often employed to narrate the dynamics of biological systems in which two species interact, first is a predator and the second is a prey. The proposed scheme provides the series solution with a great freedom and flexibility by choosing appropriate parameters. The convergence of the results is free from small or large parameters. Three examples are discussed to demonstrate the correctness and efficiency of the used computational approach.

Key words: Fractional nonlinear predator-prey population model, Biological systems, Homotopy analysis transform technique

1. Introduction

In recent years, ample interest in differential equations of fractional order has been triggered due to their large number of utilities in modeling of many scientific processes such as fluid flow, heat and mass transfers, etc. [1-11]. In this work, we put up the combined form of homotopy analysis method (HAM) and classical Laplace transform to produce a modified algorithm termed as the homotopy analysis transform technique (HATT) [12-18], to analyze the time-fractional nonlinear predator-prey population model.

The dynamical connection between predator and prey is the main issue in ecology and mathematical sciences. The considerable development in population dynamic was presented independently by many research workers [19-27].

We appraise the two-species competitive model by taking into consideration population of prey \( u \) and population of predator \( v \). For population of prey \( u \rightarrow 2u \) (population of prey tends to twice of present population of prey), having the rate \( \lambda, \lambda > 0 \) indicates the rate of natural birth. For population of predator \( v \rightarrow 0 \), having the rate \( \mu, \mu > 0 \) stands for the rate of natural death. The connected term between predator and prey populations is \( u + v \rightarrow 2v \), at rate \( \sigma \), parameter \( \sigma \) designate the competitive rate. In view of an extensively accepted theory of fractional biological population models, the mathematical model of a predator-prey system of fractional order can be illustrated as

\[
\frac{\partial^\alpha \xi}{\partial t^\alpha} = \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \lambda \xi - \sigma \xi \eta, \quad \xi(x,y,0) = \varphi(x,y),
\]

\[
\frac{\partial^\beta \eta}{\partial t^\beta} = \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \sigma \xi \eta - \mu \eta, \quad \eta(x,y,0) = \phi(x,y).
\]
In the above Eq. (1) \( t > 0, x, y \in \mathbb{R}, \lambda, \sigma, \mu > 0 \), the function \( \xi \) represents the density of prey population and the function \( \eta \) indicates the density of predator population, \( \varphi(x, y), \phi(x, y) \) indicates initial conditions for the growth of population.

2. Basic concept of homotopy analysis transform technique

We assume a nonlinear PDE involving fractional derivative of the form [12-15]:

\[
D_t^\rho \xi + R \xi + N \xi = a(t), \quad n - 1 < \rho \leq n.
\]

In Eq. (2) \( D_t^\rho \xi \) is representing the fractional derivative of \( \xi \) understood in Caputo sense [1], \( R \) is standing for the linear differential operator, \( N \) is indicating the general nonlinear differential operator and \( a(t) \) is representing the term due to source.

By putting the LT operator on Eq. (2) and simplifying [2], we get

\[
L[\xi] - \frac{1}{s^\rho} \sum_{k=0}^{\rho-1} \frac{\mathcal{D}^k \xi(0)}{k!} + \frac{1}{s^\rho} [L[R \xi]] + L[N \xi] - L[a(t)] = 0.
\]

We interpret the nonlinear operator

\[
\Omega[\mathcal{H}(t; z)] = L[\mathcal{H}(t; z)] - \frac{1}{s^\rho} \sum_{k=0}^{\rho-1} \frac{\mathcal{D}^k \mathcal{H}(t; z)(0^+)}{k!} + \frac{1}{s^\rho} [L[R \mathcal{H}(t; z)] + L[N \mathcal{H}(t; z)] - L[a(t)]]
\]

where \( z \in [0, 1] \) and \( \mathcal{H}(t; z) \) is a real function of \( t \) and \( z \). We develop a homotopy as

\[
(1 - z) L[\mathcal{H}(t; z) - \xi_0(t)] = z \Omega[\xi(t)]
\]

In the expression (5), \( L \) is indicating the LT operator, \( z \in [0, 1] \) is denoting the embedding parameter, \( \xi_0 \) is standing for an auxiliary parameter, \( \xi_0 \) is denoting an initial guess of \( \xi \) and \( \mathcal{H}(t; z) \) is indicating an unknown function. Clearly, when the embedding parameter \( z = 0 \) and \( z = 1 \), it yields

\[
\mathcal{H}(t; 0) = \xi_0(t), \quad \mathcal{H}(t; 1) = \xi(t),
\]

respectively. It is to be noticed that as \( z \) increases from 0 to 1, the solution \( \mathcal{H}(t; z) \) converts from the initial guess \( \xi_0 \) to the solution \( \xi \). Now using the Taylor theorem we write the \( \mathcal{H}(t; z) \) in series form given below

\[
\mathcal{H}(t; z) = \xi_0(t) + \sum_{m=1}^{\infty} \xi_m(t) z^m,
\]

where

\[
\xi_m(t) = \frac{\partial^m \mathcal{H}(t; q)}{\partial z^m} \bigg|_{z=0}.
\]

If the initial approximation of the solution and the parameter \( h \) are taken in well manner, the series (7) converges at \( z = 1 \), then we have the subsequent series solution of the given nonlinear equation of fractional order

\[
\xi = \xi_0 + \sum_{m=1}^{\infty} \xi_m.
\]

Making use of (9), the governing equation can be derived from the Eq. (5). Now we define the following vectors

\[
\bar{\xi}_m = \{\xi_0, \xi_1, ..., \xi_m\}.
\]

On differentiation of the Eq. (5) \( m \)-times w.r.t. \( z \) and then division them by \( m! \) and then putting \( z = 0 \), we arrive at the following result:

\[
L[\xi_m - \chi_m \bar{\xi}_{m-1}] = h \mathcal{R}_m(\bar{\xi}_{m-1}).
\]

Applying the inverse LT, we get the following result

\[
\xi_m = \chi_m \bar{\xi}_{m-1} + h L^{-1}[\mathcal{R}_m(\bar{\xi}_{m-1})],
\]

where...
\( \mathcal{R}_m(\xi_m) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial z^{m-1}} \Omega[\vartheta(t; z)] \bigg|_{z=0} \tag{13} \)

and

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{14} \]

3. Implementation of the method

Here we apply the HATT to examine the fractional predator-prey system given in Eq. (1) with initial guess \( \xi_0(x, y, t) = \varphi(x, y) \) and \( \eta_0(x, y, t) = \phi(x, y) \). We express the nonlinear operator as

\[ \Omega_1[\vartheta_1(x, y; t, z), \vartheta_2(x, y; t, z)] \]

and

\[ \Omega_2[\vartheta_1(x, y; t, z), \vartheta_2(x, y; t, z)] \]

Further the LT operators are written as

\[ L[\xi_m(x, y, t) - \chi_m \xi_{m-1}(x, y, t)] = h \mathcal{R}_{1,m}[\xi_{m-1}, \eta_{m-1}], \tag{17} \]

and

\[ L[\eta_m(x, y, t) - \chi_m \eta_{m-1}(x, y, t)] = h \mathcal{R}_{2,m}[\xi_{m-1}, \eta_{m-1}]. \tag{18} \]

In the above Eqs. (17) and (18), the \( \mathcal{R}_{1,m}[\xi_{m-1}, \eta_{m-1}] \) and \( \mathcal{R}_{2,m}[\xi_{m-1}, \eta_{m-1}] \) are expressed as

\[ R_{1,m}[\xi_{m-1}, \eta_{m-1}] = L[\xi_{m-1}(x, y, t)] - (1 - \chi_m) \frac{1}{s} \xi_0 \]

and

\[ R_{2,m}[\xi_{m-1}, \eta_{m-1}] = L[\eta_{m-1}(x, y, t)] - (1 - \chi_m) \frac{1}{s} \eta_0 \]

It is obvious from Eqs. (17) and (18), we get

\[ \xi_m = \chi_m \xi_{m-1} + hL^{-1}\{R_{1,m}[\xi_{m-1}, \eta_{m-1}]\}, \tag{21} \]

and

\[ \eta_m = \chi_m \eta_{m-1} + hL^{-1}\{R_{2,m}[\xi_{m-1}, \eta_{m-1}]\}. \tag{22} \]

The approximate solutions of the original Eq. (1) can be expressed as

\[ \xi = \sum_{r=0}^{\infty} \xi_r, \quad \eta = \sum_{r=0}^{\infty} \eta_r. \tag{23} \]

4. Numerical simulation for fractional model of Predator-Prey system

In order to show and effectiveness and correctness of the HATT for analyzing the fractional model of nonlinear partial differential equation, we apply it to the subsequent several initial conditions problems.
**Case 1.** Firstly, we take the predator-prey system of fractional order along with the constant initial conditions

\[
\xi(x, y, 0) = \xi_0, \quad \eta(x, y, 0) = \eta_0.
\]  

By appealing to the HATT to solve the Eqs. (21) and (22) with initial condition (24) we get the following results

\[
\xi(x, y, 0) = \xi_0, \quad \eta(x, y, 0) = \eta_0,
\]

\[
\xi_1 = -\frac{h(\lambda \xi_0 - \sigma \xi_0 \eta_0)}{\Gamma(\rho + 1)}, \quad \eta_1 = -\frac{h(\sigma \xi_0 \eta_0 - \mu \eta_0)}{\Gamma(\beta + 1)}, \quad \xi_2 = \xi_0 + \xi_1 + \xi_2 + \cdots.
\]

In the same way the remaining terms can be obtained. The HATT solutions of the original Eq. (1) with the initial condition (30) obtained in series form as

\[
\xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \quad \eta = \eta_0 + \eta_1 + \eta_2 + \cdots
\]

In 1-4, the effect of time and order of fractional derivative is shown on prey and predator populations. The numerical results found by the HATT are very near to the anomalous biological diffusion nature seen in the real world, with high accuracy at the third-term approximations.

**Figure 1.** Time evaluation of prey $\xi$ and predator $\eta$ population density for case 1 when $\xi_0 = 100, \eta_0 = 5, \lambda = 0.08, \sigma = 0.04, \mu = 0.02$ and $h = -1$, with $\rho = \beta = 1$.

**Figure 2.** Time evaluation of prey $\xi$ and predator $\eta$ population density for case 1 when $\xi_0 = 100, \eta_0 = 5, \lambda = 0.08, \sigma = 0.04, \mu = 0.02$ and $h = -1$, with $\rho = 1$ and $\beta = 0.85$. 
Case 2. Secondly, we assume the predator-prey equation of fractional order having the initial conditions

\[ \xi(x, y, 0) = e^{x+y}, \eta(x, y, 0) = e^{x+y}. \]  

By using the HATT to solve the Eqs. (21) & (22) with initial conditions (27), we get

\[ \xi_0 = e^{x+y}, \eta_0 = e^{x+y}, \]

\[ \xi_1 = -\frac{h e^{x+y} (2 + \lambda - \sigma e^{x+y}) t^\rho}{\Gamma(\rho + 1)}, \eta_1 = -\frac{h e^{x+y} (2 + \mu + \sigma e^{x+y}) t^\beta}{\Gamma(\beta + 1)}, \]  

In this manner the remaining terms can be achieved.

The HATT solutions of the Eq. (1) having the initial condition (27) obtained in series form as

\[ \xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \quad \eta = \eta_0 + \eta_1 + \eta_2 + \cdots \]

In 5-8, the effect of order of fractional derivative is shown on prey and predator populations with respect to time and space variables. It’s clear to see the Figs. 5-8 that the fractional predator-prey model exist and continuous with space and time variables.
**Figure 5.** The shape presents the prey $\xi(x,y,t)$ population density with appropriate parameter for case 2 when $\lambda = 0.8, \sigma = 0.04, \mu = 0.3, h = -1$ and $t = 0.65$, with $\rho = \beta = 1$.

**Figure 6.** The shape presents predator $\eta(x,y,t)$ population density with appropriate parameter for case 2 when $\lambda = 0.8, \sigma = 0.04, \mu = 0.3, h = -1$ and $t = 0.65$, with $\rho = \beta = 1$.

**Figure 7.** The shape presents the prey $\xi(x,y,t)$ population density with appropriate parameter for case 2 when $\lambda = 0.8, \sigma = 0.04, \mu = 0.3, h = -1$ and $t = 0.65$, with $\rho = 0.95$ and $\beta = 0.60$.

**Figure 8.** The shape presents predator $\eta(x,y,t)$ population density with appropriate parameter for case 2 when $\lambda = 0.8, \sigma = 0.04, \mu = 0.3, h = -1$ and $t = 0.65$, with $\rho = 0.95$ and $\beta = 0.60$. 
**Case 3.** Lastly, we take the predator-prey equation of fractional order having the initial conditions
\[ \xi(x,y,0) = \sqrt{xy}, \quad \eta(x,y,0) = e^{x+y}. \]  
By using the HATT to solve the Eqs. (21) & (22) with initial condition (30), we get the following results
\[ \xi_0 = \sqrt{xy}, \quad \eta_0 = e^{x+y}, \]
\[ \xi_1 = \frac{h(x^2 + y^2 - 4\lambda x^2 y^2 + 4x^2y^2e^{x+y} \mu^\rho)}{4(xy)^{3/2} \Gamma(\rho+1)}, \quad \eta_1 = -\frac{he^{x+y}(2 - \mu + \sigma \sqrt{xy})^\beta}{\Gamma(\beta+1)}, \]  
and so on, in this manner the remaining iterations can be achieved.

The HATT solutions of the Eq. (1) having the initial condition (30) obtained in series form is presented as
\[ \xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \quad \eta = \eta_0 + \eta_1 + \eta_2 + \cdots. \]

In 9-12, the effect of order of fractional derivative is shown on prey and predator populations with respect to time and space variables.

**Figure 9.** The shape presents the prey \( \xi(x,y,t) \) population density with appropriate parameter for case 3 when \( \lambda = 0.02, \sigma = 0.06, \rho = 0.3, h = -1 \) and \( t = 0.5 \) with \( \rho = \beta = 1 \).

**Figure 10.** The shape presents predator \( \eta(x,y,t) \) population density with appropriate parameter for case 3 when \( \lambda = 0.02, \sigma = 0.06, \mu = 0.3, h = -1 \) and \( t = 0.5 \), with \( \rho = \beta = 1 \).

**Figure 11.** The shape presents the prey \( \xi(x,y,t) \) population density with appropriate parameter for case 3 when \( \lambda = 0.02, \sigma = 0.06, \mu = 0.3, h = -1 \) and \( t = 0.5 \), with \( \rho = 0.95 \) and \( \beta = 0.75 \).
Figure 12. The shape presents predator $\eta(x, y, t)$ population density with appropriate parameter for case 3 when $\lambda = 0.02, \sigma = 0.06, \mu = 0.3, h = -1$ and $t = 0.5$, with $\rho = 0.95$ and $\beta = 0.75$.

5. Conclusions

In this study, the HATT has been efficiently put up to find the approximate solution which converges to the exact solution of time-fractional nonlinear predator-prey population model subject to initial conditions. It’s a significant outcome of these fractional ordered differential systems is that they gives higher degree of freedom and flexibility in the mathematical modelling and graphs show that the solution of these fractional systems is not only depends on time, but also depends on the order of fractional derivatives. The outcomes indicate that HATT a very strong and easy computational approach in obtaining analytical solution for various kinds of nonlinear fractional differential equations.

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