DIFFERENTIAL REPRESENTATION OF THE LORENTZIAN SPHERICAL TIMELIKE CURVES BY USING BISHOP FRAME

by

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In this study, we will give the differential representation of the Lorentzian spherical timelike curves according to Bishop frame and we obtain a third-order linear differential equation which represents the position vector of a timelike curve lying on a Lorentzian sphere.

Key words: *timelike curve, spherical curve, Bishop frame, Lorentzian sphere*

Introduction

We all know that the fundamental structure of differential geometry is the curves. In time, the best part of classical differential geometry topics has been expanded to space curves. There are many studies that imply different characterizations of these curves. Also, spherical curves are the special space curves which lie on the sphere. There are also many studies on spherical curves. Firstly, Wong [1] gave a universal formulation of the condition for a curve to be on a sphere in 1963. Then, Breuer and Gottlieb [2] proved that the differential equation characterizing a spherical curve can be solved explicitly to express the radius of curvature of the curve in terms of its torsion in 1971. Wong [3] obtained a necessary and sufficient condition for a curve to be a spherical as Breuer and Gottlieb but he proved this without any precondition on the curvature and torsion. The proof is based on an earlier result of the author's earlier paper on spherical curves. Mehlum and Wimp [4] proved that the position vector of any 3-space curve lying on a sphere provides a third-order linear differential equation in 1985. Kose [5] gave an explicit characterization of the dual spherical curve. Abdel Bakey [6] studied with dual spherical curves and obtained a differential characterization of dual spherical curves. Then he gave the explicit solution of this differential equation without the precondition on the dual torsion and also necessary and sufficient condition for a dual curve to be spherical curve is given by him in 2002. Ilarslan *et al.* [7] presented the spherical characterization of non-null regular curves in 3-D Lorentzian space. Furthermore, the differential equation which expresses the mentioned characterization is solved in 2003. Kocayigit *et al.* [8] showed that the differential equation characterizing a spherical curve in n-D Euclidean space $n \geq 3$ can be solved explicitly to express nth curvature function of the curve in terms of its curvatures and its other curvature functions in 2003. Ayyildiz *et al.* [9] presented the differential equation that is characterizing the dual Lorentzian spherical curves and then gave an explicit solution of this differential equation in 2007. Camci *et al.* [10] studied with regular curves in 3-D Sasakian space and gave the spherical characterizations of them. Furthermore, the differential equation which expresses the

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aforesaid characterization is solved in 2007. Okullu [11] gave an explicit characterization of spherical curves by using Bishop frame. The Lorentzian spherical spacelike and timelike curves are studied by the authors, respectively in [12, 13].

In this paper, we give the differential equation characterizing the timelike spherical curves by using Bishop frame. Then, we present that the position vector of a timelike curve lying on the Lorentzian sphere satisfies a third-order linear differential equation.

Instructions

The Minkowski 3-space E_1^3, is the Euclidean 3-space provided with the Lorentzian inner product:

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$. Let remember that an arbitrary vector $\vec{x} = (x_1, x_2, x_3)$ in E_1^3 can have one of three Lorentzian characters. The Lorentzian product of the vector $\vec{x} = (x_1, x_2, x_3)$ by itself can be $\langle \vec{x}, \vec{x} \rangle > 0$, $\langle \vec{x}, \vec{x} \rangle < 0$, and $\langle \vec{x}, \vec{x} \rangle = 0$. Then the vector named, respectively, as spacelike, timelike, lightlike. The norm of $\vec{x} \in E_1^3$ is given by $\| \vec{x} \| = \langle \vec{x}, \vec{x} \rangle^{1/2}$. Let $\vec{\alpha} : I \subset IR \to E_1^3$ be an arbitrary curve in the Euclidean space E_1^3. Let the curve $\vec{\alpha}$ is said to be of unit speed if $\langle \vec{\alpha}', \vec{\alpha}' \rangle = \pm 1$, the derivatives of the Frenet frame:

$$\vec{T}' = \kappa \vec{N}'$$
$$\vec{N}' = -\kappa \vec{T}' + \tau \vec{B}$$
$$\vec{B}' = \tau \vec{N}$$

where $\{\vec{T}, \vec{N}, \vec{B}\}$ is Frenet frame of $\vec{\alpha}$ and κ, τ are the curvature and the torsion of the curve $\vec{\alpha}$, respectively [3]:

$$\langle \vec{T}, \vec{T} \rangle = \langle \vec{N}, \vec{N} \rangle = \langle \vec{B}, \vec{B} \rangle = 1, \quad \langle \vec{T}, \vec{N} \rangle = \langle \vec{T}, \vec{B} \rangle = \langle \vec{N}, \vec{B} \rangle = 0$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \| \vec{T}'(s) \|$, $\tau(s) = -\langle \vec{N}, \vec{B}' \rangle$. Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ be vectors in E_1^3. Cross product of $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ is defined:

$$\vec{x} \times \vec{y} = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_2, x_1 y_2 - x_2 y_1)$$

The Lorentzian sphere of center $m = (m_1, m_2, m_3)$ radius $r > 0$ in the space E_1^3 is defined:

$$S_1^2 = \{ \vec{x} = (x_1, x_2, x_3) \in E_1^3 : \langle \vec{x}, \vec{x} \rangle = r^2 \}$$

Denote by $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$ the moving Bishop frame along the timelike curve $\vec{x}(s)$ in Minkowski 3-space E_1^3, the following Bishop formula are given:

$$\frac{\mathrm{d}\vec{T}}{\mathrm{d}s} = k_1 N_1 + k_2 N_2, \quad \frac{\mathrm{d}\vec{N}_1}{\mathrm{d}s} = k_1 \vec{T}, \quad \frac{\mathrm{d}\vec{N}_2}{\mathrm{d}s} = k_2 \vec{T}$$

The relations between κ, τ, θ, and k_1, k_2 are given:

$$\theta(s) = \arctan\left(\frac{k_2}{k_1}\right), \quad \tau(s) = \theta'(s) \text{ and } \kappa(s) = \sqrt{k_1^2 + k_2^2} \quad [14]$$

Characterization of timelike spherical curves according to Bishop frame

Theorem 1. Let $\vec{x}(s)$ be a unit speed timelike curve with first bishop curvature $k_1(s) \neq 0$ and second bishop curvature $k_2(s) \neq 0$ for each $s \in I \subset IR$. Then $\vec{x}(s)$ lies on the Lorentzian sphere of center m and radius $r \in IR^+$ if and only if:

$$\left(\frac{-k_2'}{k_1'k_2 - k_1k_2'}\right)^2 + \left(\frac{k_1'}{k_1'k_2 - k_1k_2'}\right)^2 = r^2$$

Proof. Firstly, let assume that $\vec{x}$ lies on the Lorentzian sphere of center m and radius r:

$$\langle \vec{x} - \vec{m}, \vec{x} - \vec{m} \rangle = r^2 \tag{1}$$

If we differentiate eq. (1) with respect to s and we use corresponding Frenet formulas:

$$\vec{x} - \vec{m} = \frac{-k_2'}{k_1'k_2 - k_1k_2'}\vec{N}_1 + \frac{k'}{k'k_2 - k_1k'}\vec{N}_2$$

Afterward, $\vec{x} - \vec{m}$ is written as the linear combination of Bishop frame elements:

$$\vec{x} - \vec{m} = c_1\vec{T} + c_2\vec{N}_1 + c_3\vec{N}_2 \tag{2}$$

where $c_1 = c_1(s)$, $c_2 = c_2(s)$, and $c_3 = c_3(s)$ are arbitrary functions. Then:

$$\langle T, x - m \rangle = -c_1 = 0,\ \langle N_1, x - m \rangle = c_2 = \frac{-k_2'}{k_1'k_2 - k_1k_2'},\ \langle N_2, x - m \rangle = c_3 = \frac{k'}{k'k_2 - k_1k_2'}$$

are obtained.

Therefore, the substitution of the coefficients c_1 , c_2, and c_3 in eq. (2):

$$\vec{x} - \vec{m} = \frac{-k_2'}{k_1'k_2 - k_1k_2'}\vec{N}_1 + \frac{k'}{k'k_2 - k_1k_2'}\vec{N}_2 \tag{3}$$

Thus, from eqs. (1) and (3):

$$\left(\frac{-k_2'}{k_1'k_2 - k_1k_2'}\right)^2 + \left(\frac{k_1'}{k'k_2 - k_1k_2'}\right)^2 = r^2$$

On the other hand:

$$\left(\frac{-k_2'}{k_1'k_2 - k_1k_2'}\right)^2 + \left(\frac{k_1'}{k'k_2 - k_1k_2'}\right)^2 = r^2 \tag{4}$$

where $r \in IR^+$, we may take into consideration that the vector $m \in E_1^3$:

$$\vec{m} = \vec{x} + \frac{k_2'}{k_1'k_2 - k_1k_2'}\vec{N}_1 - \frac{k_1'}{k_1'k_2 - k_1k_2'}\vec{N}_2 \tag{5}$$

we can prove that $m = \text{constant}$. By differentiating eq. (5) with respect to s:

$$\vec{m}' = \frac{k_2\left(k_1'k_2'' - k_1''k_2'\right)}{\left(k_1'k_2 - k_1k_2'\right)^2}\vec{N}_1 - \frac{k_1\left(k_1'k_2'' - k_1''k_2'\right)}{\left(k_1'k_2 - k_1k_2'\right)^2}\vec{N}_2 \tag{6}$$

If we differentiate eq. (4):

$$\frac{2\left(k_1''k_2'-k_2''k_1'\right)\left(k_1k_1'+k_2k_2'\right)}{-\left(k_1'k_2-k_1k_2'\right)^3}=0,\quad \left(k_1''k_2'-k_2''k_1'\right)\left(k_1k_1'+k_2k_2'\right)=0$$

$$\left(\frac{k_1}{k_2'}\right)'(k_2')^2\left[\left(\frac{1}{k_1}\right)'(-k_1)^3+\left(\frac{1}{k_2}\right)'(-k_2)^3\right]=0,\quad \left(\frac{k_1}{k_2'}\right)'(k_2')^2=0 \tag{7}$$

Substituting eq. (7) in eq. (6), we find that $m'=0$ for each $s \in I \subset IR$ and thus $m=$ constant. The eq. (6) implies:

$$\left\langle \vec{x}-\vec{m},\vec{x}-\vec{m}\right\rangle=\left(\frac{-k_2'}{k_1'k_2-k_1k_2'}\right)^2+\left(\frac{k_1'}{k_1'k_2-k_1k_2'}\right)^2=r^2$$

Hence $\vec{x}(s)$ lies on the Lorentzian sphere.

Corollary 1. Let $\vec{x}(s): I \to E_1^3$ be a timelike spherical curve in E_1^3 with arc length parameter s. Then the expression of position vector $\vec{x}(s)$ in terms of Bishop frame elements:

$$\left\langle \vec{x}-\vec{m},\vec{x}-\vec{m}\right\rangle=\left(\frac{-k_2'}{k_1'k_2-k_1k_2'}\right)^2+\left(\frac{k_1'}{k_1'k_2-k_1k_2'}\right)^2=r^2$$

Theorem 2. Let $\vec{x}(s): I \to E_1^3$ be a curve in E_1^3 with arc length parameter s. If $\vec{x}=\vec{x}(s)$ is timelike spherical curve, then there is a correlation between the bishop curvatures:

$$\frac{k_1'-k_2'}{k_1'k_2-k_1k_2'}=c,\quad c=c_1+c_2$$

Proof. Let $\vec{x}$ be a timelike spherical curve with the arc length parameter s in E_1^3. If we take the derivative of eq. (3) according to s:

$$\vec{T}=\left(\frac{-k_2'}{k_1'k_2-k_1k_2'}\right)'\vec{N}_1+\left(\frac{-k_2'}{k_1'k_2-k_1k_2'}\right)\left(k_1\vec{T}\right)+\left(\frac{k_1'}{k_1'k_2-k_1k'}\right)'\vec{N}_2+\left(\frac{k_1'}{k_1'k_2-k_1k'}\right)\left(k_2\vec{T}\right) \tag{8}$$

From the previous equation:

$$\left(\frac{-k_2'}{k_1'k_2-k_1k_2'}\right)'=0,\ \left(\frac{k_1'}{k_1'k_2-k_1k_2'}\right)'=0$$

$$\left(\frac{-k_2'}{k_1'k_2-k_1k_2'}\right)'=c_1,\quad \left(\frac{k_1'}{k_1'k_2-k_1k_2'}\right)'=c_2,\quad c_1,c_2=\text{constant}$$

$$\frac{k_1'-k_2'}{k_1'k_2-k_1k_2'}=c,\quad c=c_1+c_2$$

Theorem 3. The position vector of a timelike Lorentzian spherical curve satisfies a third-order linear differential equation.

Proof. If we take the derivative of position vector three times, respectively:

$$\vec{x}'=\vec{T} \tag{9}$$

$$\vec{x}''=\vec{T}'=k_1\vec{N}_1+k_2\vec{N}_2 \tag{10}$$

$$\vec{x}''' = \left(k_1^2 + k_2^2\right)\vec{T} + k_1'\vec{N}_1 + k_2'\vec{N}_2 \tag{11}$$

From eqs. (9)-(11):

$$\vec{N}_1 = \frac{k_2\vec{x}''' - k_2'\vec{x}'' - k_2\left(k_1^2 + k_2^2\right)\vec{x}'}{k_1'k_2 - k_1k_2'} \tag{12}$$

and

$$\vec{N}_2 = \frac{k_1\vec{x}''' - k_1'\vec{x}'' - k_1\left(k_1^2 + k_2^2\right)\vec{x}'}{k_1'k_2 - k_1k_2'} \tag{13}$$

If we put eqs. (12) and (13) in eq. (3):

$$\vec{x} - \vec{m} = \frac{k_1k_1' + k_2k_2'}{\left(k_1'k_2 - k_1k_2'\right)^2}\vec{x}''' - \frac{k_1'^2 + k_2'^2}{\left(k_1'k_2 - k_1k_2'\right)^2}\vec{x}'' + \frac{\left(k_1^2 + k_2^2\right)\left(k_1k_1' + k_2k_2'\right)}{\left(k_1'k_2 - k_1k_2'\right)^2}\vec{x}' \tag{14}$$

If the eq. (14) is rearranged:

$$\left(k_1k_1' - k_2k_2'\right)\vec{x}''' + \left(-k_1'^2 + k_2'^2\right)\vec{x}'' + \left(k_1^2 + k_2^2\right)\left(-k_1k_1' + k_2k_2'\right)\vec{x}' - \left(k_1'k_2 - k_1k_2'\right)^2\vec{x} = m \tag{15}$$

Equation (15) shows that the position vector of a Lorentzian spherical timelike curve satisfies a third-order linear differential equation.

Corollary 1. Let $\vec{x}(s) : I \to E_1^3$ is a timelike spherical curve in E_1^3 with arc length parameter s. The centers of spheres with 3-contact points with $\vec{x}(s)$ in the point $\vec{x}(s)$ lie on straight line:

$$\vec{m} = \vec{x} + \left(\frac{k_2'}{k_1'k_2 - k_1k_2'}\right)\vec{N}_1 - \left(\frac{k_1'}{k_1'k_2 - k_1k_2'}\right)\vec{N}_2$$

Corollary 2. Let $\vec{x}(s) : I \to E_1^3$ is a timelike spherical curve in E_1^3 with arc length parameter s. The radius of spheres with 3-contact points with $\vec{x}(s)$:

$$r = \sqrt{\left(\frac{k_2'}{k_1'k_2 - k_1k_2'}\right)^2 + \left(\frac{k_1'}{k_1'k_2 - k_1k_2'}\right)^2}$$

Theorem 4. Let $\vec{x}(s)$ be a plane unit speed curve with a first bishop curvature $k_1 = k_1(s)$.Then $\vec{x}(s)$ lies on the Lorentzian sphere of center m and radius $r \in IR^+$ in E_1^3 if and only if k_1 = constant:

$$\vec{x} - \vec{m} = \frac{1}{k_1}\vec{N}_1 \pm \sqrt{r^2 - \frac{1}{k_1^2}}\vec{N}_2$$

Proof. Let us suppose that $\vec{x}$ lies on the Lorentzian sphere of center m radius $r \in IR^+$ then $\langle \vec{x} - \vec{m}, \vec{x} - \vec{m}\rangle$, for each $s \in I \subset IR$. By differentiation with respect to s of the previous relation two times:

$$k_1\left\langle \vec{N}_1, \vec{x} - \vec{m}\right\rangle + k_2\left\langle \vec{N}_2, \vec{x} - \vec{m}\right\rangle = 1 \tag{16}$$

By assumption $\vec{x}$ is a planar curve. Hence $\tau = 0$ and using $\theta = \int \tau \mathrm{d}s, \tan\theta = k_2 / k_1$, we get $k_2 = 0$ and $k_1 = \kappa$. By using the previous equations, eq. (16) reduces to:

$$k_1\left\langle \vec{N}_1, \vec{x}-\vec{m}\right\rangle = 1$$

From here we can write:

$$\left\langle \vec{N}_1, \vec{x}-\vec{m}\right\rangle = \frac{1}{k_1} \tag{17}$$

If we differentiate eq. (17) with respect to s, we get:

$$\left\langle \vec{N}_1', \vec{x}-\vec{m}\right\rangle + \left\langle \vec{N}_1, \vec{T}\right\rangle = \left(\frac{1}{k_1}\right)'$$

$$k_1\left\langle \vec{T}, \vec{x}-\vec{m}\right\rangle = \left(\frac{1}{k_1}\right)', \quad 0 = \left(\frac{1}{k_1}\right), \quad \frac{1}{k_1} = \text{constant}, \; k_1 = \text{constant}$$

Next, decompose the vector $\vec{x}-\vec{m}$:

$$\vec{x}-\vec{m} = aT + bN_1 + cN_2 \tag{18}$$

where $a = a(s)$, $b = b(s)$, $c = c(s)$ are arbitrary functions. By using eqs. (17) and (18) we find the coefficients and substituting the coefficients a, b, and c in eq. (18):

$$\vec{x}-\vec{m} = \frac{1}{k_1}\vec{N}_1 + c\vec{N}_2$$

Now it is easy to see that $\left\langle \vec{x}-\vec{m}, \vec{x}-\vec{m}\right\rangle = \left(\frac{1}{k_1}\right)^2 + c^2 = r^2$, so it follows $c = \pm\sqrt{r^2 - \left(\frac{1}{k_1}\right)^2}$
Consequently:

$$\vec{x}-\vec{m} = \frac{1}{k_1}\vec{N}_1 \pm \sqrt{r^2 - \left(\frac{1}{k_1}\right)^2}\,\vec{N}_2$$

Conversely, let k_1 = constant $\neq 0$:

$$\vec{x}-\vec{m} = \frac{1}{k_1}\vec{N}_1 \pm \sqrt{r^2 - \left(\frac{1}{k_1}\right)^2}\,\vec{N}_2$$

then $\vec{m} \in E_1^3$ is an arbitrary vector and $r \in IR^+$. We shall prove that $\vec{m}$ constant. Since:

$$\vec{m} = \vec{x} - \frac{1}{k_1}\vec{N}_1 \pm \sqrt{r^2 - \left(\frac{1}{k_1}\right)^2}\,\vec{N}_2$$

By differentiation with respect to s of the previous equation and using the corresponding derivative formulae we get $\vec{m}' = 0$. It follows that $\vec{m}$ = constant and that $\langle \vec{x}-\vec{m}, \vec{x}-\vec{m}\rangle = r^2$. Therefore, $\vec{x}$ lies on the Lorentzian sphere of center $\vec{m}$ and radius r.

Conclusion

In the present paper we dealt with Lorentzian spherical timelike curves. We give the condition for a timelike curve to be on Lorentzian Sphere. We introduce the representation of the position vector of a Lorentzian spherical timelike curve in terms of Bishop frame elements and the relation between the first and second Bishop curvatures for the spherical curve. Then we obtain a third order linear differential equation which is characterizing the position vector of the Lorentzian spherical curve.

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