GLOBAL WELL-POSEDNESS OF A CLASS OF DISSIPATIVE THERMOELASTIC FLUIDS BASED ON FRACTAL THEORY AND THERMAL SCIENCE ANALYSIS

by

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Thermodynamics and fluid mechanics are used to study the mechanical properties of a class of thermoelastic fluid materials. Using the law of thermodynamics and the law of conservation of energy, thermal science analysis of dissipative thermoelastic fluid materials is performed in a planar 2-D flow field, and a corresponding mathematical model is established. Fractal theory, operator semigroup theory and fractional calculus are used to study the overall well-posedness of a dissipative thermoelastic flow.

Key words: *fractal, fractional order, thermoelastic, fluid, well-posedness*

Introduction

Thermoelastic fluid materials are widely used in various advanced applications, for examples, stab-proof vests, and microelectromechanical system (MEMS) resonators. Due to their dissipative property, the diffusion is different from that of molecules. The molecular diffusion is a representative of the Fick-type diffusion. As early as 1855, Fick first proposed the first law of molecular diffusion, followed by Fick's second law. Sir G. I. Taylor, a British fluid mechanics expert, began studying the turbulent diffusion law of fluids in 1921, which laid the foundation for the study of fluid diffusion and transport movement [1]. On the other hand, some fluids are dissipative in nature, and their density is different, the fluids will also form different repetitive or stratified flows.

The mathematical model for the dissipative thermoelastic fluid develops from 0-D, 1-D steady-state models, to 3-D and 3-D dynamic models. The simulated variables have evolved from non-living substances to aquatic organisms such as bacteria, algae, zooplankton and benthic animals. The scope of its applications has evolved from a single fluid to an integrated fluid with different fluid mechanical properties, and the calculated number of spatiotemporal grids has grown geometrically.

Instructions

The fluid model with dissipative properties is influenced by many factors physically, chemically, and dynamically, it is usually more comprehensive and complex [2, 3] than classic fluid models, because the model focuses on the essential characteristics of real fluids [4].

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Let $\Omega \subset R^N$, $N = 2$, 3 be a bounded Lipschitz domain filled with a fluid. The time evolution of the density $\rho = \sigma(t, x)$, and the velocity $u = u(x, t)$ are governed by the thermoelastic and hydrodynamic system with a dissipative property:

$$\partial_t \rho + \operatorname{div}(\rho u + \nabla u) = f \tag{1}$$

$$\partial_t(\rho u) + \nabla(\rho u \times u) + \operatorname{div}(\rho u) = \operatorname{div}(T + \rho f) \tag{2}$$

where T is the viscous stress tensor, which can be expressed:

$$T = T(u) = \varsigma(\nabla u + \partial_t u) + \delta \operatorname{div} \varepsilon u \tag{3}$$

where $f = f(t, x)$ represents a given external force density and σ stands for the isentropic pressure determined by the constitutive relation:

$$\sigma(\rho) = \lambda p^{\gamma}, \quad \gamma > \frac{2}{3} N \tag{4}$$

The viscosity coefficients λ, ς are assumed to be constant throughout the whole text, and they should satisfy the physically relevant hypothesis, such as energy conservation law.

According to the law of conservation of energy [5], due to the thermoelastic action of fluids and the result of energy decay, coupled with disturbances and turbulence, the fluid itself undergoes anisotropic flow and satisfies the following relationship:

$$\Delta u = \frac{n_0 U_h^{n_1} H^{n_2}}{\mu} \tag{5}$$

$$U_h = u_p V_0 (T_s - T_g) \tag{6}$$

The initial conditions are:

$$U(x,0) = u_0, \quad U_t(x,0) = u_1 \tag{7}$$

where n_0 is the thermal status and surface coefficient, n_1 – the heat release rate index, n_2 – the height index, U_h – the heat release rate, H – the height of fluid from the ground, u_p – the average constant pressure specific heat under standard conditions, V_0 – the emissions under standard conditions, V_s – the outflow rate, T_s – the temperature at which the fluid is discharged, T_g – the ambient temperature of the contaminated area, $f(x)$ – an arbitrary infinitely differentiable function, and μ – the average speed of the ambient wind at the exit of the fluid.

For some positive a, $p \in [1/2,2]$, $C > 0$, and $q > 0$, we have a weak solution, which satisfies the physically relevant hypothesis:

$$\rho > 0, \quad \sigma + \frac{u}{2} \geq 0 \tag{8}$$

Note that constant coefficient equations also admit more sophisticated solutions, it presents the case of incomplete separation of variables where the solution is separated with respect to the space variables but is not separated with respect to time t. The energy equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\iint_{\Omega}[\eta|\nabla u|^{2}+(\delta+\nu)|\mathrm{div}u|\mathrm{d}x\mathrm{d}t \le \iint_{\Omega}\vartheta gu\mathrm{d}x\mathrm{d}t \tag{9}$$

Starting from the fluid system as a whole, the mechanical properties of fluid diffusion were used to study the horizontal and vertical diffusion rules of the pipeline, and the effects of external forces and thermal elasticity on the 2-D flow field and the 2-D flow field in the horizontal direction were analyzed:

$$\frac{\partial}{\partial t}\omega(t)+\eta|\nabla u|^{2}=M(\omega)|\mathrm{div}u| \tag{10}$$

$$\omega(x,t)=f(x)+\sum_{k=1}^{\infty}\frac{t^{k}}{k!}M^{k}f(x) \tag{11}$$

where M is an arbitrary linear differential operator of the second order that only depends on the space variables, and has the formal series solution:

$$E(t)=E_{(\sigma,u)}(t)=\iint_{\Omega}\left(\frac{2}{3}\eta|u|^{3/2}+\frac{\tau}{\gamma+1}\rho^{\gamma}\right)\mathrm{d}x\mathrm{d}t \tag{12}$$

It can be seen that the total energy is limited in an attenuated state.

In what follows, we shall deal with the finite energy global well-posedness for some constants ρ_0, $M>0$, $\tau\in H^{4}[0,+\infty)$ satisfying that:

$$x'(t)\ +\tau_{0}(1+t)^{\left(\ell+\frac{d}{2}\right)}x(t)^{\left(2+\frac{\gamma}{p}\right)}\le\sigma y(t)^{\frac{\upsilon}{2}}+\tau_{1}(1+t)^{\delta}y(t)^{\gamma} \tag{13}$$

$$\tau(v^{2})\ +\ 2\sigma_{0}(v^{2})v+\tau''(v^{2})\le M\ <\infty \tag{14}$$

Main results

Basic knowledge and definitions

From famous Cauchy formula, extending the following formula from a positive integer to a normal positive real number, the most common method is to use a special function Gamma function [6]:

$$G_{0^{+}}^{n}g(t)=\frac{1}{n!}\int_{0}^{t}(t-s)^{n}g(s)\mathrm{d}s,\quad t>0,\quad n\in N \tag{15}$$

$$G_{0^{+}}^{n}f(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}f(s)\mathrm{d}s,\quad t>0,\quad n\in N \tag{16}$$

$$\Gamma(t)=\int_{0}^{\infty}e^{-s}s^{t-1}\mathrm{d}s \tag{17}$$

$$\Gamma(t+1)=t\Gamma(t) \tag{18}$$

This is the Riemann-Liouville fractional integral x:(0, +∞) → R. The Riemann-Liouville fractional derivative is defined:

$$G_{0^+}^n x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{x(s)}{(t-s)^{n-\alpha-1}} \,\mathrm{d}s \tag{19}$$

In which $\alpha \in R$, $\alpha > 0$, and $n - 1 < \alpha \le n$, $n \in N$.

The α^{th} order Caputo fractional derivative of continuous function x is defined:

$$G_{0^+}^n x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^n(s) \,\mathrm{d}s \tag{20}$$

In which x: (0, +∞) → R, $\alpha \in R$, $\alpha > 0$, and $n - 1 < \alpha \le n$, $n \in N$.

Suppose E is a real Banach Space [7], mapping γ is called a non-negative continuous convex function, if and only if:

$$\gamma[tx + (1-t)y] \le t\gamma(x) + (1-t)\gamma(y) \tag{21}$$

where γ: $P \to [0, +\infty)$, $P \subset E$, $x, y \in P$, $t \in [0, 1]$.

Basic result

Theorem: suppose that $y(t) \in C(0, 1) \cup L^2 (0, 1)$, and $y(t) \ge 0$, then the system of eqs. (1)-(3) satisfies the global well-posedness, and we have:

$$\varphi_p\{[{}^cD_{0^+}^\alpha x(t)]\} + \lambda y(t) = (1-\tau)^{\alpha-1}\Gamma(\alpha)(1-\eta), \quad 0 < t < 1 \tag{22}$$

$$x(\alpha) = \eta \iint_\Omega \frac{f(t,x)}{\varphi_p(x)} 10(s-\tau)^{\alpha-1} \varphi q(\lambda\tau) g(s) x(s) \,\mathrm{d}s\mathrm{d}t$$

$$u''(0) = u^{2/3}(0) = 0, \quad f(t,x) > 0, \quad 1 - \eta \iint_\Omega g(s)\mathrm{d}s > 0$$

$$f_\varphi^0, f_\varphi^\infty \in [0,\infty], \quad t \in [0,1] \quad \text{and} \quad x > 0$$

$$0 < \iint_\Omega \omega(s,t) \,\mathrm{d}s\mathrm{d}t < +\infty$$

$$u(t) = \frac{2}{3\Gamma(\alpha)} \iint_\Omega (1-s)^{\alpha-1} \varphi_q(t) y(\tau) \,\mathrm{d}\tau\mathrm{d}t + \frac{\eta^{2/3}}{\Gamma(\alpha)\left[1 - \eta \iint_\Omega g(s) y(\tau) \mathrm{d}\tau\mathrm{d}s\right]} -$$

$$- \iint_\Omega (s-\tau)^{\frac{\alpha-1}{2}} \lambda_\tau(\tau) y(v) \,\mathrm{d}v\mathrm{d}\tau \tag{27}$$

where

$$x(t)\ge 0,\quad \min_{0\le t\le 1} u(t)\ge \sigma\|x\|,\quad g(s)(1-\tau)^{\alpha-1}\le \varphi_q[\lambda y(v)\mathrm{d}v]^{\tau}(t-s)^{\alpha-1}$$

$$2<\alpha\le 3\quad 0<\eta<1\quad 1-\eta\iint_{\Omega} g(s)\mathrm{d}s>0$$

Proof of main result

We focus on the global well-posedness of the class of dissipative thermoelastic fluid materials based on fractal theory and thermal science analysis. Without loss of generality, we consider the initial-boundary value problem if the initial data satisfies (5)-(7).

Assuming that $T(P)\subset P$, $\forall x_n(t)\in E$, $x_n(t)\to x(t)$ $(n\to\infty)$, according to the continuity of the function $f[t, x(t)]$, we can get $f[t, x_n(t)]\to f[t, x(t)]$, that is $\phi_i(t)\to\phi(t)$ if $n\to\infty$, therefore:

$$\sup_{t\in[0,1]}\left|\phi_n(t)-\phi(t)\right|\to 0,\quad n\to\infty \tag{28}$$

Let $I=\left|(Tx_n)(t)\ -(Tx)(t)\right|$, then the following formula exists:

$$I=\Lambda\Upsilon\iint_{\Omega} g(s)(1-\tau)^{\alpha-1}\left|\phi_n(\tau)-\phi(\tau)\right|\mathrm{d}\tau\mathrm{d}s+\iint_{\Omega} g(s-\tau)^{\alpha-1}\left|\phi_n(\tau)-\phi(\tau)\right|\mathrm{d}\tau\mathrm{d}s\le$$

$$\le\frac{2}{3\Gamma(\alpha)}\iint_{\Omega}(1-s)^{\alpha-1}\left|\phi_n(s)-\phi(s)\right|^q x_n(t)\mathrm{d}s\mathrm{d}t+\iint_{\Omega}(t-s)^{\frac{\alpha-1}{2}}\left|\phi_n(s)-\phi(s)\right|^p y_n(t)\mathrm{d}s\mathrm{d}t\le$$

$$\le\sup_{s\in[0,1]}\left|\phi_n(s)-\phi(s)\right|\frac{2\alpha}{3\Gamma(\alpha+1)}\left[1+\Upsilon\Gamma(\alpha)^{\frac{p-1}{p}}\iint_{\Omega} g(s)\mathrm{d}s\right] \tag{29}$$

This means $\left|(Tx_n)(t)\ -(Tx)(t)\right|\to 0,(n\to\infty)$. That implies that the operator T is continuous.

The two ends of eq. (1) are simultaneously integrated, one can get:

$$\varphi_p[D_{0^+}^{\alpha}x(t)]-\varphi_p\left[D_{0^+}^{\frac{\alpha-1}{p}}x(0)\right]=-\gamma\iint_{\Omega}\lambda(x)ty(s)\mathrm{d}s$$

From the boundary conditions $x'(0)=x''(0)=0$, one obtains $D_{0^+}^{\alpha}x(0)=0$, therefore:

$$\varphi_p[D_{0^+}^{\alpha}x(t)]=-\rho\iint_{\Omega}\tau(\lambda)ty(s)\mathrm{d}s\mathrm{d}\lambda$$

$$\varphi_p[D_{0^+}^{\alpha}x(t)]=-\varphi_p\left[\sigma\iint_{\Omega}\tau(\lambda)y(s)\mathrm{d}s\mathrm{d}\lambda\right]^{\frac{q-1}{q}}$$

One can immediately get:

$$x(t) = -I^{\alpha}\varphi_p\left[\lambda\iint_{\Omega} g(t)y(s)\mathrm{d}s\mathrm{d}t\right] + \rho_1 + \rho_2 t + \rho_2 t^2 \tag{30}$$

Let $M = \max_{0\le t\le 1, x\le r} f[t, x(t)],\quad N(t) = \varphi_q\left[M\iint_{\Omega} u(s)\omega(t)\mathrm{d}s\mathrm{d}t\right].$

Using Young inequality, Grounwell inequality [8], since $x(t)$ has the summation representation, for all fixed x_0 and y_0, the function $u_1, u_2, \ldots, u_m$ are non-decreasing with respect to all t and s, we have:

$$\rho_1 \le \Upsilon\left[\iint_{\Omega} g(s)(1-\tau)^{\frac{2}{3}\alpha-1}\left|\phi(\tau)\right|\mathrm{d}\tau\mathrm{d}s + \iint_{\Omega}(s-\tau)^{\frac{\alpha-1}{p}}\left|\phi(\tau)\right|\mathrm{d}\tau\mathrm{d}s\right] +$$

$$+\frac{2}{3\Gamma(\alpha)}\iint_{\Omega} x(t)(1-s)^{\frac{\alpha-1}{q}}\left|\phi(s)\right|\mathrm{d}s\mathrm{d}t + \iint_{\Omega} y(t)(t-s)^{\frac{\lambda}{2}(\alpha-1)}\left|\phi(s)\right|\mathrm{d}s\mathrm{d}t \tag{31}$$

$$\rho_2 \le \lambda\iint_{\Omega} g(s)(1-\tau)^{\frac{\alpha-1}{2}}\left|N(\tau)\right|\mathrm{d}\tau\mathrm{d}s + \iint_{\Omega}(s-\tau)^{\frac{\alpha-1}{p}}\left|N(\tau)\right|^{\frac{2q}{3p}}\mathrm{d}\tau\mathrm{d}s +$$

$$+\frac{2}{3\Gamma(\alpha)}\iint_{\Omega}(1-s)^{\frac{p-1}{p}}\varphi(t)\left|M(s)\right|\mathrm{d}s\mathrm{d}t + \iint_{\Omega}(t-s)^{\frac{\alpha-1}{3}}\left|N(s)\right|\mathrm{d}s\mathrm{d}t \tag{32}$$

$$\rho_3 \le KM(\Omega)\iint_{\Omega} g(s)(1-\tau)^{\frac{\alpha-1}{3}}\mathrm{d}\tau\mathrm{d}s + \iint_{\Omega}(s-\tau)^{\frac{1}{q}(\alpha-1)}\left|N(\partial_{\Omega})\right|\mathrm{d}\tau\mathrm{d}s +$$

$$+\frac{M}{3\Gamma(\alpha)}\iint_{\Omega}(1-s)^{\frac{2}{q}(\alpha-1)}(1-t)\mathrm{d}s\mathrm{d}t + N\iint_{\Omega} t(t-s)^{\frac{1}{2(p-1)}\alpha-1}\mathrm{d}s\mathrm{d}t \tag{33}$$

From eqs. (31)-(33), and in view of eq. (30), one can get:

$$\left|(Tx)(t)\right| \le \frac{2\min(M,N)}{3\Gamma(\alpha+1)\left[1-\eta\iint_{\Omega} g(\tau)\varphi(s)\mathrm{d}\tau\mathrm{d}s\right]} \tag{34}$$

This means the operator T maps the bounded set in P to a bounded set, and the operator T is equicontinuous.

Differentiating eq. (2) results in:

$$u'(t) = -\frac{\sigma y(t)}{\Gamma(\alpha-1)}\iint_{\Omega}(t-s)^{\frac{1}{2}(\alpha-2)}\varphi(\tau)\mathrm{d}s\mathrm{d}\tau \le 0 \tag{35}$$

Using the Stieltjes transform inequality and Natanson inequality, we can get that:

$$\|u(t)\| = u(0), \quad \min_{0\le t\le 1} u(t) = u(1) \tag{36}$$

From the boundary conditions:

$$u'(0) = u''(0) = 0, \quad y(1) = \eta \iint_\Omega g(s)x(t)\mathrm{d}s\mathrm{d}t$$

one can get that $\rho_1 = 0$, $\rho_2 = 1/3\sigma$, using the nature of the concave function and Araela-Ascoli theorem [9]:

$$\rho_3 = \frac{\eta}{2\Gamma(1-\alpha)(1-\eta)} \iint_\Omega g(t)(t-s)^{\frac{\alpha-1}{p}} \varphi(\lambda) y(\tau)\mathrm{d}\tau\mathrm{d}s \le$$

$$\le \frac{p-1}{3\Gamma(\alpha)(1-\eta)^{p-1}} \iint_\Omega (1-s)^{\frac{1}{2}(\alpha-1)} y(\tau)\mathrm{d}\tau\mathrm{d}s \tag{37}$$

On the other hand, for $\forall x(t) \in \Omega_E$, making $0 < t_1 < t_2 < 1$, we have:

$$\|(Tx)(t_2) - (Tx)(t_1)\| \le \frac{p}{3\Gamma(\alpha)(1-\eta)^p} \iint_\Omega (t-s)^{\frac{1}{2}(\alpha-1)} \varphi(t)\mathrm{d}t\mathrm{d}s \le$$

$$\le \frac{\alpha-1}{3\Gamma(\alpha)N(1)} \iint_\Omega \left[(t_2-s)^{\frac{(\alpha-1)m}{p}} - (t_1-s)^{\frac{(\alpha-1)n}{q}} \right] \varphi(t)\mathrm{d}t\mathrm{d}s \tag{38}$$

By the mean value theorem and Cauchy fixed point theorem, we obtain:

$$\iint_\Omega \left[(t_2-s)^{\frac{(\alpha-1)m}{p}} - (t_1-s)^{\frac{(\alpha-1)n}{q}} \right] \mathrm{d}t\mathrm{d}s \le$$

$$\le \iint_\Omega \left[\frac{\alpha-1}{p}(t_2-s)^m - \frac{(\alpha-1)}{q}(t_1-s)^n \right] \mathrm{d}t\mathrm{d}s \le$$

$$\le \iint_\Omega \delta \left[\frac{\alpha-1}{p}(t_2-s)^m - \frac{(\alpha-1)}{q}(t_1-s)^n \right] (t_2-t_1)\mathrm{d}t\mathrm{d}s \le$$

$$\le \partial\Omega |t_2-t_1| \le \kappa |t_2-t_1| \tag{39}$$

According to the characteristic of the function on the convex set:

$$u(t) \le tu(1) + (1-t)u(0) \tag{40}$$

$$u(1) \le \frac{2\eta}{3\Gamma(\alpha)(1-\eta)} \iint_\Omega g(s)(1-\tau)^{\frac{\lambda}{2\tau}(\alpha-1)} y(\upsilon)\varphi_q(0)\mathrm{d}\upsilon\mathrm{d}\tau \le$$

$$\leq \iint_{\Omega} (s-\tau)^{\alpha-1} \varphi_q(\lambda) y(\upsilon) \mathrm{d}\upsilon \mathrm{d}s +$$

$$+ \frac{\eta(p-1)}{\Gamma(1-\alpha)(1-\eta)^p} \iint_{\Omega} g(s) \left[(\lambda-\tau)^{\frac{3}{2}\alpha-1} - (s-\tau)^{\frac{3}{2}\alpha-1} \right] \varphi_q(\lambda\tau) y(\upsilon) \mathrm{d}\upsilon \mathrm{d}\tau \leq$$

$$\leq \iint_{\Omega} (1-\tau)^{\alpha-1} y(t) \varphi_q(\lambda\tau) \mathrm{d}\tau \mathrm{d}t \leq \lambda C \tag{41}$$

There holds the following inequality:

$$\beta \iint_{\Omega} x(t) g(s) \mathrm{d}s \mathrm{d}t + \beta u(1) \iint_{\Omega} t g(t) u(x) \mathrm{d}x \mathrm{d}t \leq$$

$$\leq \beta u(0) \iint_{\Omega} (1-t)^{\frac{\alpha-1}{3}} g(t) u(x) \mathrm{d}x \mathrm{d}t \tag{42}$$

Because of:

$$u(1) = \beta \iint_{\Omega} x(t) g(s) \mathrm{d}s \mathrm{d}t, \quad 1 - \beta \iint_{\Omega} g(t) \eta(s) \mathrm{d}s \mathrm{d}t < \lambda \tau_0$$

$$u(1) \leq \frac{\beta \iint_{\Omega} (1-t)^{\frac{\alpha-1}{2}} g(t) \eta^{\tau}(s) \mathrm{d}s \mathrm{d}t}{1 - \beta \iint_{\Omega} t^{\alpha} g(t) y(s) \mathrm{d}s \mathrm{d}t} \tag{43}$$

That is:

$$\min_{\substack{0<t<1 \\ x(0)<s<x(1)}} u(t) \leq x(1) + \frac{\beta \iint_{\Omega} (1-t)^{\alpha-1} \eta(1-\tau) g(t) \mathrm{d}\tau \mathrm{d}t}{1 - \beta \iint_{\Omega} t^{\alpha} g(t) \eta(\tau) \mathrm{d}\tau \mathrm{d}t} x(0) \leq \rho \|x\| \tag{44}$$

This means that the global well-posedness of eqs. (1)-(4) hold.

Conclusion

In this paper, we exploited some theorems such as the fixed point theorem, and Liouville theorem, operator semigroup method to investigate some properties of non-linear infinite dimensional PDE with impulse perturbations concentrated on the surfaces for a class of thermoelastic fluids with dissipation, we proved its overall well-posedness, laying the foundation for real fluid testing and simulation. In our study, Riemann-Liouville fractional derivative is adopted, and our results can easily be extended to other definitions of the fractional derivative or fractal derivatives [10-17].

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