FICTITIOUS TIME INTEGRATION METHOD FOR SOLVING THE TIME FRACTIONAL GAS DYNAMICS EQUATION

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In this work a powerful approach is presented to solve the time-fractional gas dynamics equation. In fact, we use a fictitious time variable \( y \) to convert the dependent variable \( w(x, t) \) into a new one with one more dimension. Then by taking an initial guess and implementing the group preserving scheme we solve the problem. Finally four examples are solved to illustrate the power of the offered method.

Key words: Fictitious time integration method; Group preserving scheme; Time fractional Gas dynamics equation; Caputo derivative;

Introduction

Calculus of fractional order is increasingly being worked to model various physical systems. Since many physical phenomena growing in engineering as well as in allied sciences can be depicted by developing models with the help of the fractional calculus. The fractional partial equations response ultimately converges to the non fractional equations, fulfilling a notable care in the present times. The fractional derivatives are important due to broad scope of applications for mathematical modeling of problems such as traffic flow models, control, and relaxation processes \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. There are some some analytical and numerical methods which are implemented to solve the fractional equations such as Group preserving scheme \cite{12, 13}, differential transform methods \cite{14, 15, 16}, Homotopy Perturbation Methods \cite{17, 18, 19, 20}. This presented work is dedicated to study the following time fractional gas dynamics

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equation(TFGD)

\[
\begin{cases}
C^{\alpha}_{0+} w(x, t) + w(x, t) w_x(x, t) - w(x, t) (1 - w(x, t)) = K(x, t), \\
w(x, 0) = g_1(x), \ x \in \Omega_x, \\
w(x, T) = g_2(x), \ x \in \Omega_x, \\
w(0, t) = h_1(t), \ t \in \Omega_t, \\
w(b, t) = h_2(t), \ t \in \Omega_t, \\
\Omega := \{(x, t) : a \leq x \leq b, \ 0 \leq t \leq T\},
\end{cases}
\]

(1)

The gas dynamics equations are mathematical terminology that are adjunct on the physical laws of conservation such as the conservation of momentum, conservation of mass, and conservation of energy. Many authors solved the fractional gas dynamics equations using different numerical and analytical methods [21, 22, 23, 24, 25, 26, 27, 28, 29]. The differential transform method is implemented for solving TFGD [30, 31] and Fractional homotopy analysis transform method [32].

In this presented work, we create a powerful and reliable numerical approach to obtain the numerical solution of TFGD equation. This approach is firstly presented by Liu [35].

The fictitious time integration method (FTIM)

The Caputo fractional derivative of for fractional order \( \alpha > 0 \) is described by [33, 34]

\[
C^{\alpha}_{0+} w(x, t) = \frac{\partial^\alpha w(x, t)}{\partial t^\alpha} = \begin{cases}
\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\sigma)^{m-\alpha-1} \frac{\partial^m w(x, \sigma)}{\partial \sigma^m} \, d\sigma, \quad m - 1 < \alpha < m, \\
\frac{\partial w(x, t)}{\partial t^m}, \quad \alpha = m,
\end{cases}
\]

(2)

By using Caputo fractional derivative definition and \( 0 \leq \alpha < 1 \) for Eq. (1) we have

\[
\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w_\sigma(x, \sigma)}{(t-\sigma)^\alpha} \, d\sigma + \eta w w_x - w (1 - w) - K(x, t) = 0.
\]

(3)

Now, we multiply the Eq. (3) into the parameter \( \eta \) as a fictitious damping coefficient which can help our to raise the stability of numerical integration:

\[
\frac{\eta}{\Gamma(1-\alpha)} \int_0^t \frac{w_\sigma(x, \sigma)}{(t-\sigma)^\alpha} \, d\sigma + \eta w w_x - \eta w (1 - w) - \eta K(x, t) = 0.
\]

(4)

Now, we impose the following transformation:

\[
z(x, t, y) = (1 + y)^d w(x, t), \quad 0 < d \leq 1,
\]

(5)

By using above transformation, Eq. (4) converts to a new form:

\[
\frac{\eta}{(1 + y)^d} \left[ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z_\sigma(x, \sigma, y)}{(t-\sigma)^\alpha} \, d\sigma + \eta z z_x - \eta z (1 - z) - \eta K(x, t) \right] = 0.
\]

(6)

From Eq. (5) we can get:

\[
\frac{\partial z}{\partial y} = d(1 + y)^{d-1} w(x, t).
\]

(7)
A combination of Eq. (7) and Eq. (6), concludes:

\[
\frac{\partial z}{\partial y} = \eta \frac{1}{(1+y)^d} \left[ \Gamma(1-\alpha) \int_0^t \frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha} \, d\sigma + z z_x - z(1-z) \right] - \eta K(x,t) + d (1+y)^{d-1} w. \tag{8}
\]

Then, Eq. (8) can be transformed into a new form of PDE for \( z \), by using \( w = z/(1+y)^d \):

\[
\frac{\partial z}{\partial y} = \frac{\eta}{(1+y)^d} \left[ \Gamma(1-\alpha) \int_0^t \frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha} \, d\sigma + z z_x - z(1-z) \right] - \eta K(x,t)
+ \frac{k z}{1+y}. \tag{9}
\]

By using

\[
\frac{\partial}{\partial y} \left( \frac{z}{(1+y)^d} \right) = \frac{z_y}{(1+y)^d} - \frac{dz}{(1+y)^{1+d}}, \tag{10}
\]

Next, by multiplying \(1/(1+y)^d\) on both sides of Eq. (9), we obtain:

\[
\frac{\partial}{\partial y} \left( \frac{z}{(1+y)^d} \right) = \frac{\eta}{(1+y)^{2d}} \left[ \Gamma(1-\alpha) \int_0^t \frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha} \, d\sigma + z z_x - z(1-z) \right]
- \frac{\eta K(x,t)}{(1+y)^d}. \tag{11}
\]

Using again the transformation \( w = \frac{z}{(1+y)^d} \), we get:

\[
w_y = \frac{\eta}{(1+y)^d} \left[ \Gamma(1-\alpha) \int_0^t \frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha} \, d\sigma + w(x,t,y) w_x(x,t,y) \right.
- w(x,t,y)(1-w(x,t,y)) \left] - \eta K(x,t). \tag{12}
\]

We have to emphasis that \( y \) plays the fictitious coordinate role which able us to embed Eq.(3) into a new PDE form in a space called 3-space, denoted \( R^3 \). As well as, by a initially guess \( w(x,t,0) \), for all \( y \geq 0 \), \( w = w(x,t,y) \) is an undetermined function with regard to the conditions in Eq. (1).

Supposing \( w^i_j(x) := w(x_i,t_j,y) \) and \( K^i_j := K(x_i,t_j) \) as the discrete values of \( w \) and \( K \) at a point \( (x_i,t_j) \). Implementing a semi-discretization to the Eq.(12) concludes:

\[
\frac{d}{dy} w^i_j(y) = \frac{\eta}{(1+y)^d} \left[ \Gamma(1-\alpha) \int_0^{t_j} \frac{y_\sigma(x_i,\sigma,y)}{(t_j-\sigma)^\alpha} \, d\sigma \right.
+ w^i_j(y) \frac{w^{i+1}_j(y) - w^i_j(y)}{\Delta x}
- w^i_j(y)(1-w^i_j(y)) \left] - K^i_j. \tag{13}
\]
To calculate the above integral terms we can write the following approximation:

\[
\int_0^{t_j} \frac{w_\sigma(x_i, \sigma, y)}{(t_j - \sigma)^\alpha} \, d\sigma \approx \sum_{l=1}^{j-1} \frac{w(x_i, t_{l+1}, y) - w(x_i, t_l, y)}{\Delta t (t_j - t_l)\alpha},
\]

(14)

Which stepsize \( \Delta x = \frac{b-a}{M_1} \), \( \Delta t = \frac{T}{M_2} \), \( x_i = a + i\Delta x \) and \( t_j = j\Delta t \).

### The GPS for extracted system of ODEs

In this stage, with \( w = (w_1, w_2, \ldots, w_n)^T \) we can write the Eq. (13) in the following form:

\[
w' = E(w, y), \quad w \in \mathbb{R}^N, \quad y \in \mathbb{R},
\]

(15)

Where \( E \) indicates a vector with \( ij \)-elements being the right-hand side of Eq. (13) and \( w' \) denotes the differential of \( w \) with regard to \( y \), and \( N = M_1 \times M_2 \) is the number of total grid point.

In this step we can use of group-preserving scheme (GPS) that introduced by Liu [35].

Let

\[
X_{l+1} = B(l)X_l,
\]

(16)

Where \( X_l \) indicates the value of \( X \) at the \( y_l \) and \( B(l) \) is a component of \( SO_0(N, 1) \) which represents the group value of \( B \) at \( y_l \).

The Lie group can be created from \( C \) which is a element of \( so(N, 1) \).

\[
B_l = \exp[\Delta y C(l)] = \begin{bmatrix} I_N + \frac{(\Psi_l - 1)}{\|E_l\|}E_l^T & \frac{\Phi_l}{\|E_l\|} \\ \frac{\Phi_l}{\|E_l\|} & \Psi_l \end{bmatrix},
\]

where

\[
\Psi_l = \cosh \left( \frac{\Delta y \|E_l\|}{\|w_l\|} \right),
\]

\[
\Phi_l = \sinh \left( \frac{\Delta y \|E_l\|}{\|w_l\|} \right).
\]

(17)

\( X := (w^T, \|w\|)^T \) is a vector in Minkowskian space which converts Eq. (15) into \( \frac{\partial X}{\partial y} = CX \).

Where

\[
C := \begin{pmatrix} \Theta_{N \times N} & \frac{E(w, y)}{\|w\|} \\ E^T(w, y) & 0 \end{pmatrix} \in so(N, 1),
\]

(18)

is a Liu algebra of the proper orthochronous Lorentz group \( SO_0(N, 1) \). By replacing Eq. (17) for \( B_l \) into Eq. (16), we have:

\[
w_{l+1} = w_l + \frac{(\Psi_l - 1)E_l w_l + \Phi_l \|w_l\| \|E_l\| E_l}{\|E_l\|^2} = w_l + \Pi_l E_l.
\]

(19)
In this stage, by selecting an initial value $u^j_i(0)$ we can apply GPS to solve numerical solution of Eq. (15) from the initial fictitious $y_0$ to a chosen final fictitious time $y_f$. Moreover, we can control the convergence of $w^j_i$ at the $l$ and $l+1$ steps by the following criterion:

$$\sqrt{M_1M_2 \sum_{i,j=1}^{M_1,M_2} [w^j_i(l) - w^j_i(l+1)]^2} \leq \varepsilon,$$

where $\varepsilon$ is the convergence criterion.

**Numerical examples**

To show the power of our method 4 examples are solved.

**Example 1:**

In order to show the ability of presented method we consider the following fractional TFGD equation with fractional order $\alpha = 0.9$

$${^C_D}_0^\alpha \mathcal{D} w(x, t) + w(x, t)w_x(x, t) - w(x, t)(1 - w(x, t)) = 0.$$  

We implement the presented method to solve this problem under parameters $\eta = 35$ and $d = 0.1$. The initial guess and stepsize for $y$ are supposed as $w^j_i(0) = 1e-5$ and $\Delta y = 1e-3$. We use the number of knots $M_1 = 25$ and $M_2 = 25$ in each coordinates of space and time respectively. Also, considered domain in this example is $\Omega = [0, 1] \times [0, 1]$. Fig. 1 is dedicated to show the exact solution $w(x, t) = e^{-x+t}$ and approximate solutions obtained by presented scheme. Power of the method with maximum absolute error $1.4 \times 10^{-17}$ is shown in Fig. 2.

**Example 2:**

Suppose following problem of TFGD with $\alpha = 1.5$ and $a = 2$

$${^C_D}_0^\alpha \mathcal{D} w(x, t) + w(x, t)w_x(x, t) - w(x, t)(1 - w(x, t))\log(a) = 0, \quad a > 0,$$

In order to manage the stability and convergency of the approach we choose $\eta = 5, d = 0.001$, respectively. Initial guess is $w^j_i(0) = 0.001$ and stepsize of method is same with example1. For $M_1 = M_2 = 39$ and $\Delta y = 10^{-10}$. Exact $w(x, t) = a^{t-x}$ and numerical solutions are plotted in Fig. 3. Absolute numerical errors for this example $1 \times 10^{-17}$ which are depicted in Fig. 4.
We take the TFGD equation with
\[ C \mathcal{D}^\alpha_{0^+, t} w(x, t) + w(x, t)w_x(x, t) - (1 + t)^2 w^2(x, t) - x^2 = 0, \quad \alpha > 0, \]
Under parameters \( \alpha = 0.3, \eta = 2, d = 0.001, \Delta y = 10^{-5}, M_1 = M_2 = 19 \) and initial guess \( w_i^0(0) = 0.0001 \). The solutions and maximum absolute errors are demonstrated in Figs. 5 and 6, respectively. Moreover, the exact solution of this example is \( w(x, t) = x/(1 + t) \) and \( \Omega = [0, 1] \times [0, 1] \).

Conclusion

In this work we have converted TFGD equation into a new type of functional PDE in one more dimension by implementing a fictitious coordinate. Then by using a semi-discretization for original equation, the GPS as a geometric approach is imposed to solve the obtained system of first order ODEs. Four numerical examples are solved, which demonstrate that our presented scheme is powerful and applicable to gain the numerical solutions of TFGD equation.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( C )</td>
<td>augmented matrix</td>
</tr>
<tr>
<td>( \mathcal{K} )</td>
<td>source term</td>
</tr>
<tr>
<td>( B )</td>
<td>an element of Lorentz group</td>
</tr>
<tr>
<td>( h )</td>
<td>boundary solute concentration</td>
</tr>
<tr>
<td>( I_M )</td>
<td>( M )-dimensional unit matrix</td>
</tr>
<tr>
<td>( N )</td>
<td>number of discretized points</td>
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<tr>
<td>( SO_0(M, 1) )</td>
<td>( M )-dimensional Lorentz group</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>time stepsize</td>
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<tr>
<td>( w )</td>
<td>solute concentration</td>
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<tr>
<td>( g_1 )</td>
<td>initial solute concentration</td>
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<tr>
<td>( x )</td>
<td>space dimension</td>
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<tr>
<td>( E )</td>
<td>( M )-dimensional vector field in eq. (15)</td>
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<tr>
<td>Greek symbols</td>
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<tr>
<td>( \Delta x )</td>
<td>space stepsize</td>
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<tr>
<td>( \eta )</td>
<td>fictitious damping coefficient</td>
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<td>( \alpha )</td>
<td>fractional derivative order</td>
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References


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Figure 1: Plots of the exact, approximate solutions and error for example 1.
Figure 2: Plot of the exact, approximate solutions and error for example 2.
Figure 3: Plot of the exact, approximate solutions and error for example 3.