## From the Guest Editor

# TWO-SCALE MATHEMATICS AND FRACTIONAL CALCULUS FOR THERMODYNAMICS

by

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*A three dimensional problem can be approximated by either a two-dimensional or one-dimensional case, but some information will be lost. To reveal the lost information due to the lower dimensional approach, two-scale mathematics is needed. Generally one scale is established by usage where traditional calculus works, and the other scale is for revealing the lost information where the continuum assumption might be forbidden, and fractional calculus or fractal calculus has to be used. The two-scale transform can approximately convert the fractional calculus into its traditional partner, making the two-scale thermodynamics much promising.*

Keywords: *continuum mechanics, fractal space, fractal derivative, fractional derivative, scale-dependent law, two-scale thermodynamics*

Thermodynamics becomes extremely important in modern science and technology, it deals with thermo-properties of a matter, and everyone knows well the following heat equation:

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{1}$$

where $k$ is a diffusivity coefficient.

For simplicity, we might use 1-D or 2-D heat equation:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{2}$$

or

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{3}$$

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The lower dimensional approach, where a usage scale is used, makes the solution process much simpler, but some information might be lost. If we want not only to take the advantages of the simple solution process, but also to unveil the lost information, another scale is needed.

We always use the continuum assumption and the traditional calculus to model various problems, where the scale for continuum assumption is used. Actually all physical laws are scale-dependent, each law is valid only on the chosen scale, beyond which probabilistic properties happen. To show this we consider a piece of cloth, which can be considered as a three-dimensional object, however, it can be reduced to the 2-D case when the effect its thickness is ignored. The 2-D model can never study the effect of the cloth's thickness on its thermo-property. If we want to study the effect of porous structure on the cloth's thermo-property, fractal approach has to be adopted. If the cloth is considered as a smooth plane, the effect of its porous structure on its thermos-property becomes random and unpredictable. To fix the problem, another scale on porous size is needed, and it is called the two scale problem [1].

This issue solicits many articles on such problems. For example, an experiment on a drop of red ink's diffusion into water is published. If water were a continuum medium, the motion of red ink in water should become stochastic. Snow's porous structure is also analysed in another article in this issue to study its insulation property by using a fractional heat equation, and its astonished thermal property is revealed.

Fractional calculus and fractal calculus are good candidates for discontinuous problems [2-9]. In continuum mechanics, we have the following Green formulae to build up governing equations:

$$\oint\!\!\!\oint_{\partial V} \mathbf{n}\varphi \mathrm{d}S = \iiint_V \nabla\varphi \mathrm{d}V \tag{4}$$

$$\oint\!\!\!\oint_{\partial V} \mathbf{n}\cdot \mathrm{A}\mathrm{d}S = \iiint_V \nabla\cdot \mathrm{A}\mathrm{d}V \tag{5}$$

$$\oint\!\!\!\oint_{\partial V} \mathbf{n}\times \mathrm{A}\mathrm{d}S = \iiint_V \nabla\times \mathrm{A}\mathrm{d}V \tag{6}$$

These formulae are valid when the volume, $V$, of the studied object is continuous, and its boundary, $\partial V$, is smooth. But for a porous medium, both assumptions are invalid, so we have to use their fractal modifications to build up governing equations by the fractal derivative [2].

The fractal derivative can be generally defined [2]:

$$\frac{\partial T}{\partial x^\alpha} = \Gamma(1+\alpha) \lim_{\substack{x\to x_0 \\ \Delta x = x - x_0 \neq 0}} \frac{T - T_0}{(x-x_0)^\alpha} \tag{7}$$

where $x_0$ is the smallest scale beyond which there is no physical understanding. For the snow ball problem, $x_0$ is the porous size. This issue solicits some articles on fractional differential equations, which can be solved approximately by the homotopy perturbation method [10, 11], the variational iteration method [12], and others.

The fractional differential equation admits a non-differentiable solution, this makes solution process difficult, but if we watch the discontinuous solution as shown in fig. 1(a) for the coastline near Tianjin port on a larger scale, an approximate continuous solution can be obtained, see fig. 1(b), where the coastline becomes relatively smooth.


(a) (b)

**Figure. 1 Two scales for the coastline near Tianjin port; on a small scale, the coastline is discontinuous, but on a larger scale, it becomes relatively smooth; the scale bars for (a) and (b) are respectively 5 km and 50 km**

This observation leads to a two-scale transform to convert approximately a fractal space to a continuous partner. The two scale transform, for example, in $x$-direction, is:

$$s = x^{\alpha} \tag{8}$$

where $x$ is for the small scale and $s$ for large scale, $\alpha$ – the two-scale dimensions [1].

Using the two-scale transform, the fractional differential equations can be converted into traditional differential ones, which are easy to be solved.

This issue solicits 50 papers, shedding a promising light on two-scale thermodynamics. We hope this explanation makes the audience accessible to this issue easily.

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