COMPARISONS OF SIX DIFFERENT ESTIMATION METHODS FOR LOG-KUMARASWAMY DISTRIBUTION

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In this paper, it is considered the problem of estimation of unknown parameters of log-Kumaraswamy distribution via Monte Carlo simulations. Firstly, it is described six different estimation methods such as maximum likelihood, approximate bayesian, least-squares, weighted least-squares, percentile and Crámer–von-Mises. Then, it is performed a Monte Carlo simulation study to evaluate the performances of these methods according to the biases and mean-squared errors (MSEs) of the estimators. Furthermore, two real data applications based on carbon fibers and the gauge lengths are presented to compare the fits of log-Kumaraswamy and other fitted statistical distributions.

Key words: log-Kumaraswamy distribution, Crámer–von-Mises estimation method, Least-squares estimation, Maximum likelihood estimation, Percentile estimation, Monte Carlo Simulation.

1. Introduction

log-Kumaraswamy (LKw) distribution is a special case of log-exponentiated Kumaraswamy distribution proposed by Lemonte et al. [1]. They have generated LKw distribution by using a log-transform in cdf of Kumaraswamy (Kw) distribution suggested by Kumaraswamy [2]. Let $Y$ be a random variable having Kw distribution with parameters $a$ and $b$. LKw distribution is obtained by $X = -\log(1-Y)$ transformation. This distribution can be used in many areas such as biology, physics, chemistry, medicine, engineering, thermal science, fluid mechanics etc. The cumulative distribution function (cdf), the probability density function (pdf) and hazard rate function (hf) of this distribution are as follows:

$$f(x,a,b) = abe^{-x}(1-e^{-x})^{a-1}\left[1-(1-e^{-x})^a\right]^{b-1},$$  \hspace{1cm} (1)

$$F(x,a,b) = 1 - \left[1-(1-e^{-x})^a\right]^b,$$  \hspace{1cm} (2)

$$h(x,a,b) = \frac{ab e^{-x}(1-e^{-x})^{a-1}}{1-(1-e^{-x})^a},$$  \hspace{1cm} (3)

where $a > 0$, $b > 0$ and $x > 0$. The LKw$(a,b)$ distribution can be useful in order to model real data in areas such as hydrology, engineering, science, medicine, agriculture etc.

Some studies on LKw distribution can be listed as follows. Mohammed [3] studied inference on the log-exponentiated Kumaraswamy distribution. Further Chacko and Mohan [4] investigated the

The aim of this article is to compare the performances of methods of estimation for LW(a,b) distribution via Monte Carlo simulations and real data applications. For this reason, six different estimation methods such as the maximum likelihood, approximate bayesian, least-squares, weighted least-squares, percentile and Crámer–von-Mises are considered. This paper is organized as follows. In Section 2, these estimation methods are described. In Section 3, it is performed a Monte Carlo simulation to evaluate the performances of estimators obtained by using these estimation methods. Furthermore two empirical applications are presented in Section 4. Lastly, general results are given in Section 5.

2. Estimation Methods

2.1. Maximum Likelihood Estimates

Let \( X_1, X_2, ..., X_n \) be a random sample taken from LW(a,b) distribution. The log-likelihood function is given by

\[
\ell(a,b | x) = n(\log a + \log b) - \sum_{i=1}^{n} x_i + (a-1) \sum_{i=1}^{n} \log \left(1 - e^{-x_i}ight) \\
+ (b-1) \sum_{i=1}^{n} \log \left(1 - (1-e^{-x_i})^a\right).
\]

The maximum likelihood estimators (MLEs) of unknown parameters are derived by maximizing the log-likelihood function in Eq. (4). The likelihood equations are also obtained from the partial derivatives of log-likelihood function with respect to \( a \) and \( b \) parameters. These derivatives are

\[
\frac{\partial \ell(a,b | x)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log \left(1 - e^{-x_i}\right) - (b-1) \sum_{i=1}^{n} \left(1-e^{-x_i}\right)^a \log \left(1 - e^{-x_i}\right) \\
1 - \left(1-e^{-x_i}\right)^a,
\]

(5)
\[
\frac{\partial \ell(a,b | x)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log \left[ 1 - (1 - e^{-x_i})^a \right].
\] (6)

The MLEs, \( \hat{a}_{MLE} \) and \( \hat{b}_{MLE} \), can be obtained by solving of likelihood equations Eqs. (5)-(6). These non linear equations can be solved by numerical methods.

2.2. Approximate Bayesian Estimates

Let \( X_1, X_2, \ldots, X_n \) be a random sample with size \( n \) taken from \( LKw(a,b) \) distribution. In this study, the independent gamma priors for \( a \) and \( b \) parameters are used. These prior distributions are as follows.

\[
\pi(a) \propto a^{d_1-1}e^{-a e_i}, \quad a, e_i, d_1 > 0
\] (7)

\[
\pi(b) \propto b^{d_2-1}e^{-b e_i}, \quad b, e_i, d_2 > 0
\] (8)

The joint priors and posterior distributions of \( a \) and \( b \) parameters are given, respectively, by

\[
\pi(a,b) = \pi(a) \pi(b) \propto a^{d_1-1}b^{d_2-1}e^{-(a e_i + b e_i)},
\] (9)

\[
\pi(a,b | x) = \frac{f(x | a,b) \pi(a,b)}{f_x(x)} = \frac{w(x;a,b) \pi(a,b)}{\int \int w(x;a,b) \pi(a,b) da db},
\] (10)

where

\[
w(x;a,b) = (ab)^e e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left[ 1 - (1 - e^{-x_i})^a \right] \left[ 1 - (1 - e^{-x_i})^b \right]^{-1}.
\]

Thus, Bayes estimator (BE) under squared loss function for any function of \( a \) and \( b \), say \( u(a,b) \), is as follows:

\[
\hat{u}_B(a,b) = E[u(a,b) | x] = \frac{\int \int u(a,b | x) e^{[1(a,b)] + p(a,b)} da db}{\int \int e^{[1(a,b)] + p(a,b)} da db}.
\] (11)
where \( l(a,b|x) \) is log-likelihood function, \( \rho(a,b) \) is logarithm of joint prior distribution. It is difficult to get the integral presented in Eq. (11) in closed form. Some approximate methods to get the integrals are used. One of these methods is Tierney Kadane’s approximation method.

### 2.2.1. Bayes Estimates with Tierney and Kadane’s Method

Tierney and Kadane’s approximation introduced by Tierney and Kadane [15] to compute integral ratios in bayes analysis. This approximation has been studied by many authors such as Danish and Aslam [16], Gencer and Saraçoğlu [17], Kumar [18], Kınacı et al. [19], Tanış and Saraçoğlu [20], Jung and Chung [21]. Tierney and Kadane approximation can be summarized as follows.

\[
I(a,b) = \frac{1}{n} \left\{ \rho(a,b) + l(a,b|x) \right\},
\]

(12)

\[
I'(a,b) = \frac{1}{n} \log u(a,b) + I(a,b).
\]

(13)

where, \( \rho(a,b) \) is defined as follows.

\[
\rho(a,b) = (d_i - 1) \log (a) + (d_j - 1) \log (b) - (ae_i + be_j).
\]

(14)

With Tierney and Kadane approximation of \( u(a,b) \) under squared error loss function for \( \text{Lkw}(a,b) \) distribution can be obtained as follows

\[
\hat{u}_b(a,b) = E[u(a,b)|x] = \frac{\int_{x_0}^{x_n} e^{u(a,b)} \, da \, db}{\int_{x_0}^{x_n} e^{u(a,b)} \, da \, db}
\]

(15)

\[
= \left[ \frac{\text{det } \Sigma^*}{\text{det } \Sigma} \right]^{\frac{1}{2}} \exp \left[ n \left( I' \left( \hat{a}_i, \hat{b}_j \right) - I \left( \hat{a}_i, \hat{b}_j \right) \right) \right],
\]

where \( \left( \hat{a}_i, \hat{b}_j \right) \) and \( \left( \hat{a}_i, \hat{b}_j \right) \) maximize \( I' \left( \hat{a}_i, \hat{b}_j \right) \) and \( I \left( \hat{a}_i, \hat{b}_j \right) \), respectively. \( \Sigma^* \) and \( \Sigma \) are minus the inverse Hessians of \( I' \left( \hat{a}_i, \hat{b}_j \right) \) and \( I \left( \hat{a}_i, \hat{b}_j \right) \) at \( \left( \hat{a}_i, \hat{b}_j \right) \) and \( \left( \hat{a}_i, \hat{b}_j \right) \), respectively.

### 2.3. Least Square and Weighted Least Square Estimates

Let \( X_{1a} \leq X_{2a} \leq \ldots \leq X_{na} \) be order statistics of a random sample with \( n \) sizes having \( \text{Lkw}(a,b) \) distribution. Then, the expected value of the empirical cumulative distribution function (ecdf) is defined by

\[
E[F(X_{ia})] = \frac{i}{n+1}, \quad \text{Var}[F(X_{ia})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}, \quad i = 1, 2, \ldots, n
\]

(16)

The least square estimates of \( a \) and \( b \), \( \hat{a}_{LSE} \) and \( \hat{b}_{LSE} \), can be obtained by minimizing following Eq. (17).

\[
Z(a,b) = \sum_{i=1}^{n} \left[ 1 - \left( 1 - e^{-x_i} \right)^a \right]^{b} - \frac{i}{n+1} \right]^2.
\]

(17)
In this case, \( \hat{a}_{LSE} \) and \( \hat{b}_{LSE} \) can be obtained via the simultaneously solution of the following system of equations.

\[
\frac{\partial Z(a,b)}{\partial a} = \sum_{i=1}^{n} k(x_{ia}, a, b) \left( 1 - \left( 1 - e^{-x_{ia}} \right)^a \right) - \frac{i}{n+1} = 0, \tag{18}
\]

\[
\frac{\partial Z(a,b)}{\partial b} = \sum_{i=1}^{n} g(x_{ia}, a, b) \left( 1 - \left( 1 - e^{-x_{ia}} \right)^b \right) - \frac{i}{n+1} = 0. \tag{19}
\]

where,

\[
k(x_{ia}, a, b) = b \left[ 1 - \left( 1 - e^{-x_{ia}} \right)^a \right]^{-b-1} \left( 1 - e^{-x_{ia}} \right)^a \log(1 - e^{-x_{ia}}),
\]

\[
g(x_{ia}, a, b) = -\log \left( 1 - \left( 1 - e^{-x_{ia}} \right)^a \right) \left[ 1 - \left( 1 - e^{-x_{ia}} \right)^b \right].
\]

Eqs. (18)-(19) can be simultaneously solved using some iterative methods. The weighted least squared estimators (WLSEs) shown with \( \hat{a}_{WLSE} \) and \( \hat{b}_{WLSE} \) can be obtained by minimizing following equation with respect to \( a \) and \( b \) parameters

\[
\sigma(a, b) = \sum_{i=1}^{n} \tau_i \left[ 1 - \left( 1 - e^{-x_{ia}} \right)^a \right]^{-b} \left( 1 - e^{-x_{ia}} \right)^a - \frac{i}{n+1} \right]^2. \tag{20}
\]

where

\[
\tau_i = \frac{1}{\text{Var}[F(X_{ia})]} = \frac{(n+1)^2 (n+2)}{i(n-i+1)}.
\]

### 2.4. Percentile Estimates

In this subsection, the percentile estimators (PEs) of \( a \) and \( b \) for LKw\((a, b)\) distribution, \( \hat{a}_{PE} \) and \( \hat{b}_{PE} \) are obtained. This estimation method was firstly suggested by Kao [22, 23]. There are many studies based on percentile estimation of unknown parameters for various statistical distributions. Some of these studies are Gupta and Kundu [24], Alkasabeh and Raqab [25], Erişoğlu and Erişoğlu [26]. The quantile function of LKw\((a, b)\) distribution is

\[
Q(a, b) = -\log \left[ 1 - \left( 1 - (1 - p)^{\frac{1}{b}} \right)^{\frac{1}{a}} \right]. \tag{21}
\]

Let \( x_{ia} \) be value of \( i^{th} \) order statistics. \( \hat{a}_{PE} \) and \( \hat{b}_{PE} \) can be obtained by minimizing the following equation with respect to \( a \) and \( b \) parameters.

\[
\kappa(a, b) = \sum_{i=1}^{n} \left[ x_{ia} + \log \left( 1 - \left( 1 - \left( \frac{i}{n+1} \right)^{\frac{1}{b}} \right)^{\frac{1}{a}} \right) \right]^2. \tag{22}
\]

### 2.5. Cramér-von Mises Estimates

The Cramer-von Mises estimator is one of the goodness of-fit estimators. This method is based on the difference between the estimate of the cdf and ecdf. The bias of these estimators is smaller than
the bias of other minimum distance estimators studied by Luceno [27], Ramos and Louzada [9] and Macdonald [28]. The Cramér–von Mises estimators (CVMEs), \( \hat{a}_{CVME} \) and \( \hat{b}_{CVME} \), can be derived by minimizing following equation

\[
C(a,b) = \frac{1}{12n} + \sum_{i=1}^{n} \left[ 1 - \left(1 - e^{-x_i} \right)^a \right] ^b - \frac{2i-1}{n+1}. \tag{23}
\]

3. Simulation Study

In this section, it is performed a extensive Monte Carlo simulation study in order to compare MLEs, BEs, LSEs, WLSEs, PEs and CVMEs for LKw \((a,b)\) distribution. The biases and mean square errors (MSEs) of these estimators are simulated based on 10000 repetitions by considering different samples of sizes such as 25, 50, 100, 250 and 500 and different initial values as \((a = 0.5, b = 0.9)\), \((a = 3.3, b =1.5)\), \((a = 2.3, b = 1.2)\) and \((a = 4, b = 2)\) for LKw \((a,b)\) distribution. In the bayesian analysis, we consider \((d_1 = 0.01, e_1 = 0.01)\) and \((d_2 = 0.01, e_2 = 0.01)\) as the values of prior parameters. The results of simulation study are given in Table 1 and Table 2.

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<th>LSEs</th>
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Table 2: The biases and MSEs of $\hat{a}$ and $\hat{b}$ by using different estimation methods for $a = 2.3, b = 1.2$ and $a = 4, b = 2$

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4. Empirical Applications

In this section, it is performed two real data analysis in order to illustrate usefulness of LKw \((a,b)\) in real life. A comparison the performances of MLEs, BEs, LSEs, WLSEs, Pes and CVMEs for parameters of LKw \((a,b)\) distribution is given in this section. For these purposes it is used Anderson-Darling (A*), Cramer-Von Mises (W*), Kolmogorov-Smirnov test statistics (KS) and its (p-value).

4.1. Gauge lengths data

The first data set based on gauge lengths of 20 mm consists of 69 observations obtained by Bader and Priest [29]. These data previously used by Kundu and Raqab, [30], Ghitany et al. [31] and Nofal et al. [32]. These data are given by

\[
\]

The results of real data analysis for gauge lengths data are given in Table 3. Also, cdf and pdf curves of these estimator for LKw \((a,b)\) distribution are given in Fig. 1.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>(\hat{a})</th>
<th>(\hat{b})</th>
<th>A*</th>
<th>W*</th>
<th>K-S</th>
<th>p value</th>
</tr>
</thead>
<tbody>
<tr>
<td>MLE</td>
<td>22.3085</td>
<td>5.4072</td>
<td>0.2687</td>
<td>1.2633</td>
<td>0.0534</td>
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<tr>
<td>BE</td>
<td>22.5930</td>
<td>5.3970</td>
<td>0.3119</td>
<td>1.6777</td>
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<td>0.9647</td>
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<td>LSE</td>
<td>23.5756</td>
<td>5.6703</td>
<td>0.2444</td>
<td>0.8684</td>
<td>0.0436</td>
<td>0.9994</td>
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<tr>
<td>WLSE</td>
<td>23.7161</td>
<td>5.8354</td>
<td>0.2672</td>
<td>0.8705</td>
<td>0.0488</td>
<td>0.9966</td>
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<tr>
<td>PE</td>
<td>22.0063</td>
<td>4.9715</td>
<td>0.2735</td>
<td>1.2863</td>
<td>0.0517</td>
<td>0.9927</td>
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<tr>
<td>CVME</td>
<td>23.7881</td>
<td>5.6424</td>
<td>0.2476</td>
<td>0.7391</td>
<td>0.0432</td>
<td>0.9995</td>
</tr>
</tbody>
</table>

Figure 1. cdf (left) and density (right) curves for gauge lengths data

4.2. Carbon Fibers (in Gba) data

The second data set consists of 50 observations on breaking stress of carbon fibers (in Gba) obtained by Nichols and Padgett [33]. These data are as follows;

\[
3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92.
\]

The parameter estimates, and some selection statistics for carbon fibres data set are given in Table 4.
The cdf and pdf curves according to six different estimators are presented for carbon fibres data in Fig. 2.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>( \hat{a} )</th>
<th>( \hat{b} )</th>
<th>( A^* )</th>
<th>( W^* )</th>
<th>K-S</th>
<th>p value</th>
</tr>
</thead>
<tbody>
<tr>
<td>MLE</td>
<td>28.4793</td>
<td>3.1609</td>
<td>0.5142</td>
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<tr>
<td>BE</td>
<td>28.0556</td>
<td>3.1329</td>
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<tr>
<td>LSE</td>
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<td>0.9684</td>
<td>0.0757</td>
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<tr>
<td>WLSE</td>
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<tr>
<td>PE</td>
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<td>0.5960</td>
<td>2.4165</td>
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<td>CVME</td>
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<td>0.7860</td>
<td>0.0643</td>
<td>0.9860</td>
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</tbody>
</table>

Figure 2. cdf (left) and density (right) curves for carbon fibres data

5. Concluding Remarks

It has been considered ML, B, LS, WLS, P, and CVM estimation methods to estimate unknown parameters of LKw \((a, b)\) distribution. Then, it is performed a Monte Carlo simulation study to compare the performances of these estimators in terms of biases and MSEs at different size of samples. According to results of simulation study, it is clearly seen that approximate bayes estimator is best the estimator among all estimators. Besides, as size of samples increases, biases and MSEs of all estimators decreases. Also, it is seen that the biases and MSEs of maximum likelihood estimators and approximate bayes estimators approaches each other in big size of samples. On the other hand, we illustrate usefulness of LKw \((a, b)\) distribution for gauge lengths and carbon fibers data sets. Further, it is compared the fits of these estimators for LKw \((a, b)\) distribution via Anderson-Darling, Cramer-Von Mises, Kolmogorov-Smirnov statistics and its \(p\) values. It is seen that least square estimator is the best for gauge lengths data according to Table 3. Approximate bayes estimator is the better than other estimators in modelling carbon fiber data according to Table 4.

References


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