ON TWO-SCALE DIMENSION AND ITS APPLICATIONS

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Dimension or scale is everything. When a thing is observed by different scales, different results can be obtained. Two scales are enough for most of practical problems, and a new definition of a two-scale dimension instead of the fractal dimension is given to deal with discontinuous problems. Fractal theory considers a self-similarity pattern, which cannot be found in any a real problem, while the two-scale theory observes each problem with two scales, the large scale is for an approximate continuous problem, where the classic calculus can be fully applied, and on the smaller scale, the effect of the porous structure on the properties can be easily elucidated. This paper sheds a new light on applications of fractal theory to real problems.

Keywords: fractal calculus, fractal dimension, two-scale dimension, two-scale transform, fractal Fourier's law, hierarchical structure, magic, uncertainty

1. Introduction

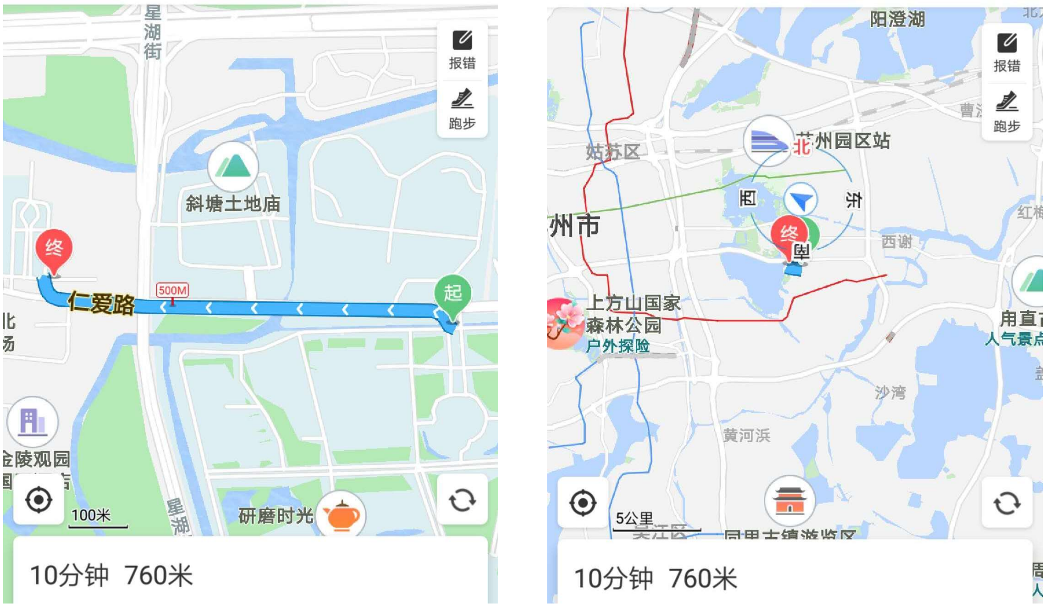
The analysis of fractal objects has grown increasingly during last decades. Fractal theory and fractal patterns are being used by diverse scientific areas, for examples, computer graphics, geological media, medical imaging and biology. It wouldn't be wrong to say that everything in the universe is fractal. Each particle in the universe, no matter how small, is so extremely complex that any an attempt to unveil something supposed to be fundamental will certainly reveal more complexities. Mathematical fractals are mostly derived from some construction process iterated up to infinity, for examples, Koch curve and Cantor set. Natural fractals are approximately fractals that occur in nature, for example, porosity of fabrics and soil, the cost of Brittany. Natural fractals resemble themselves roughly at all level of resolution, and their level of complexity becomes greater with each step.

The fractal geometry was proposed by Mandelbrot[1], this influential idea has contributed remarkably to mathematics and has found many practical applications from social science, economics to engineering and life. Fractal is mathematically beautiful and physically imperative to deal with discontinuous problems, such as heat conduction through a porous medium or a hierarchical structure[2-10]. Many hierarchical structures can be approximately considered as fractals though they are not, a porous medium is always considered as a fractal space though it is not. This fractal assumption in practical applications leads to some but not all useful results. During design of a hierarchical structure using the fractal theory, it is difficult to determine the last cascade due to restrictions from cost and technology, so a modified fractal theory is much needed to deal with real problems.

2. Two-scale dimension

Dimension or scale is everything, different scales result in different results or laws for a same phenomenon. We begin with a blackboard demonstrating that physics laws are different at different levels of observation. The blackboard can be zero dimensional if we observe it on the Sun, it can be also one-dimensional if we measure it by the scale of its length. It becomes unsmooth on an extremely small scale, saying 100 nanometers, and fractal dimensions have to be used.

We always use a map to find the origin-to-destination line. Fig.1(a) shows the origin-to-destination distance is 760 meters on a scale of 100 meters, but when we see the map on a scale of 5km, as shown in Fig.1(b), the origin-to-destination line becomes a point.



(a) (b)

Fig.1 The origin-to-destination line on different scales.
(a) the scale bar is 100 m, (b) The scale bar is 5 km.

Scale-dependent laws work only on an assumed scale. Water on any observation scales is continuous, and the continuum mechanics works, however, water becomes discontinuous on a molecule scale. The continuum mechanics cannot describe a molecule's motion in water, it is chaotic though its motion is determinate on the molecule scale. Uncertainty occurs when a wrong scale is used. We, therefore, need a new mathematical tool to deal with the two-scale problems instead of the fractal dimension.

For a fractal geometry, the fractal dimension is defined as

$$D = \frac{\ln N}{\ln M} \tag{1}$$

where $M=L/l$, L and l are two scales, on the larger scale, we have a unit, while on the smaller scale, the units are N.

However, many practical problems lack a self-similarity structure. Just consider a cloth, it is two-dimensional on a large scale, however when it is observed on a small scale, its warp-weft structure can be seen. If we want study the effect of warp-weft structure on the air permeability, we have to use the small scale. The cloth is far from mathematical fractal, so it is necessary to give a new definition for dimension.

The two-scale dimension is defined as

$$\alpha = \alpha_0 \frac{V}{V_0} \tag{2}$$

where α is the two scale dimension for the small scale, α_0 is the dimension for the large scale, for the cloth α_0=2, V/V_0 is the measured ratio using two different scales.

For the first iteration of the Koch curve, see Fig.2(a) $\alpha_0 = 1$, the two-scale dimension is

$$\alpha = \alpha_0 \frac{V}{V_0} = 1 \times \frac{4}{3} = 1.333 \tag{3}$$

while the fractal dimension for Koch curve is

$$D = \frac{\ln 4}{\ln 3} = 1.261 \tag{4}$$

For the first iteration of Sierpinski carpet, see Fig.2(b), $\alpha_0 = 2$, the two-scale dimension can be calculated as

$$\alpha = \alpha_0 \frac{V}{V_0} = 2 \times \frac{8}{9} = 1.777 \tag{5}$$

while the fractal dimension for the Sierpinski carpet reads

$$D = \frac{\ln 8}{\ln 3} = 1.892 \tag{6}$$

(a) (b) (c)

Fig. 2 First iteration of the Koch curve (a) and the Sierpinski carpet(b), Second iteration of the Sierpinski carpet(c).

The small difference in the values of dimension calculated from two different definitions is evidence of effectiveness of new definition though there is no apparent relation between two estimators, i.e., 1.33>1.23 for Koch curve and 1.77<1.89 for Sierpinski Carpet. The two scale dimension is, however, purely a physical property without any useless mathematical interference. It simply provides the ratio of complexity of given discontinuous pattern between two adjacent different scales of observation.

To further illustrate the physical understanding of the two-scale dimension, we consider a magic game. The magic uses actually two scales of time, one is larger, the other is smaller. Our eyes can determine a motion of 24 pictures per second, faster than the threshold, the eyes see nothing, this is the magic!

4. Application

We consider heat conduction through a porous medium as illustrated in Fig.3. For simple illustration, one-dimensional Fourier's law is used in this paper. The local heat flux density reads

$$q = k\frac{\partial T}{\partial x} \tag{7}$$

where q is heat flux density, k is the thermal conductivity, T is the temperature. The heat equation for a continuous medium can be written in the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) \tag{8}$$

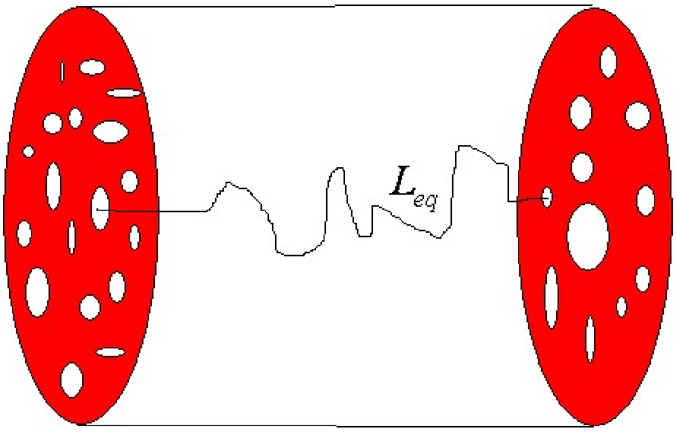


Fig. 3 Heat conduction through a porous medium

For a porous medium, a fractal modification of the Fourier's law has to be made, and Eq.(7) is modified as

$$q = k\frac{\partial T}{\partial x^{\alpha}} \tag{7}$$

where α is the two scale dimension, $\partial T / \partial x^{\alpha}$ is the fractal derivative defined as[11,12]

$$\frac{\partial T}{\partial x^{\alpha}} = \Gamma(1+\alpha)\lim_{x \to x_0}\frac{T - T_0}{(x - x_0)^{\alpha}} \tag{8}$$

where x_0 is the smallest scale beyond which there is no physical understanding. For the air permeability of a cloth, x_0 is the porous size of the warp-weft structure, for a porous medium, x_0 can be the smallest porosity size, smaller than this size, the effect on heat conduction is ignored.

For one-dimensional heat conduction through a porous medium, the two scale dimension can be calculated as

$$\alpha = \alpha_0\frac{L_{eq}}{L} = \frac{L_{eq}}{L} \tag{9}$$

where L_{eq} is illustrated in Fig.3, L is the length of the porous tube. As L_{eq} is difficult to be measured, we use the following formula to calculate the two-scale dimension:

$$\alpha = \frac{V}{V - V_{porosity}} \tag{10}$$

where V is the total volume of the tube, $V_{porosity}$ is the porosity volume. The heat equation in a porous medium can be written as

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x^{\alpha}}(k\frac{\partial T}{\partial x^{\alpha}}) \tag{11}$$

In Ref.[13,14] the following transform is used

$$s=x^{\alpha} \tag{12}$$

We can call it as the two-scale transform, where x can be considered as a small scale, while s is the large scale. Using the two-scale transform, Eq.(12) becomes

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial s}(k\frac{\partial T}{\partial s}) \tag{13}$$

It is exactly same as Eq.(8) for continuous medium. We give the following explanation: for the large scale, the problem can be approximately considered as a continuous problem, but it cannot study the effect of the porosity patterns on the heat property; for the small scale, the fractal calculus has to be used, and it can elucidate the effect of porous structure on the heat conduction property.

5. Discussion and Conclusion

For the very first time ever, we propose an alternative definition of fractal dimension, that is the two-scale dimension. As the dimension is everything in our every study, when you observe our Earth from an infinite far point, it becomes a point though it is large enough in our observation. Newton's gravity considers the Earth as a point, so it cannot deal with the earthquake. When we study a problem, the scale used is of great importance. For example, if we want to study the effect of the warp-weft structure on the cloth's properties, we have to use the scale that can measure the warp-weft structure, below the scale, any effect is ignored. If we want further study the effect of the yarn structure on the cloth's properties, we have to use a much smaller scale that can measure the fiber diameter, and we can use three-scale dimension, which we will discuss the definition in a forthcoming paper.

Now we return the second iteration of the Sierpinski carpet as shown in Fig.1(c), the two-scale dimension is

$$\alpha=\alpha_0\frac{V}{V_0}=2\times\frac{81-9-8}{81}=1.580 \tag{14}$$

while its fractal dimension keeps unchanged, that is

$$D=\frac{\ln 64}{\ln 9}=1.892 \tag{15}$$

If we consider the second iteration of the Sierpinski carpet, it will be something like a porous medium, it is of course different from that of the first iteration of the Sierpinski carpet given in Fig.1(b), so the two-scale dimension should be different.

Natural phenomena should follow simple laws, which should be of mathematical simplicity when interpreted, and unnecessary assumptions should be completely avoided. A tree is of hierarchical structure, it is not a mathematical fractal, each adjacent cascades have inherent bio-functions. If we consider a tree is a fractal pattern, some hidden pearls cannot be revealed. If we want to unveil the life secret of the Kleiber's 3/4 law, we have to reject the Rubner's law if only the metabolic rate is considered, however, if we view the life on a cell scale, and the cell is non-smooth, the Rubner law becomes totally valid again[15]. The spider silk's mechanical properties cannot be explained by the continuum mechanics, but its hidden secret can be fully revealed on a molecule scale[16], where the geometric potential[17-19] works.

In this paper, we, therefore, have critically re-conceptualized the method in which the fractal dimension is commonly calculated, revealing that the inherently dull process of calculation induces the impractical mathematics. This situation opens door to either wrong or meaningless results. We have developed a simple and physical model for the fractal dimension with simple definition but physical meaning. The original idea given in this paper sheds a new light of applications of fractal to real problems without unnecessary assumptions.

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