GROWTH OF N-DIMENSIONAL SPHERICAL BUBBLE WITHIN VISCOUS, SUPERHEATED LIQUID: ANALYTICAL SOLUTION

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In this paper, we present the study of the bevaiour of spherical bubble in N-dimensions fluid. The fluid is a mixture of vapour and superheated liquid. The mathematical model is formulated in N-dimensions fluid on the basis of continuity and momentum equations, and solved its analytically. The variable viscosity is taken in an account problem. The obtained results show that the radius of bubble increases with the decreasing of the value of N-dimensions.

Key words*: spherical bubbles; N-dimensions fluid; analytical solution, growth problem; vapour and superheated liquid.*

1. Introduction

The dynamics of a growth or a collapse of vapour/gas bubbles are an area of great interest with several applications of science as dynamics of fluids, biomedical physics, and engineering and bioengineering applications [1-5]. The dynamics of vapour/gas bubbles are strongly dependent on many physical parameters such as the parameter of pressure of the vapour/gas that contained on it. In principle, these quantities must be determined. These determinations represent the solution of the conservation equations like continuity, momentum, and energy equations inside and outside the bubble joined together by suitable boundary conditions at the bubble interface.

Rayleigh [6] considers the first one who derived the Rayleigh equation for the dynamics of a gas-filled cavity, where this work [6] presented without consideration thermal effect, viscosity of liquid and surface tension. In addition to that, there are many efforts that they developed the bubble problem, as Plesset [7] who studied the Rayleigh equation in a large wide by including viscosity of liquid and surface tension. The Rayleigh- Plesset equation describes the models dynamics of an empty or gas/vapour –filled spherical cavity in an infinite volume of a liquid [7]. It is considered that the Rayleigh-Plesset equation is a special case of the Navier-Stokes equation, that describes the bubble dynamics in an infinite incompressible fluid. In general, there is no known general solution to the Rayleigh-Plesset equation. The equations are often solved numerically, but there are some efforts and attempts to solve it analytically in a special approximation like efforts in refs. [6-13], i.e., authors in ref. [13] have investigated the solution of Rayleigh-Plesset equation analytically by using the transformation of Cole-Hopf in the bubble growth. And also, in another work [14,15], authors studied the analytical approximation of Rayleigh equation in the collapse of an empty spherical bubble. For our knowledge, There are no generalized and analytically solutions of Rayleigh-Plesset equation in N-dimensions for vapour bubble growth where N-dimensional bubble is the generalized solutions of

Rayleigh-Plesset equation in N-dimensions fluid and estimate the bubble radius for any value of N-dimensions. It is considered that the Rayleigh-Plesset equation in N-dimensions is satisfied at N ≥ 3.

The main aim of the present work deals to study the analytical solution of the mathematical model of bubble dynamics in N-dimensions. The variable viscosity is taken into account. We interest in knowing the the mechanism of bubble behaviour in N-dimensions and compare the presented results with the previous available studies.

In this work we concern to formulate the mathematical model of the growth of bubble in the N-dimensions fluid that depends on the basis of Navier-stokes equation. When we study the Navier-Stokes equation in N-dimension [16], we should take under our considerations the regularity criteria established [17] that have been used to setup the bubble dynamics problem. The present work is to derive the Rayleigh- Plesset equation in N-dimensions fluid. This equation describes the growth of vapour bubble in N-dimensions. The analytical solution of the mathematical model is solved analytically by using the modifications of Plesset and Zwick [2], and Mohammadein [8]. We also take into account a surface tension and viscosity in the current work.

The introduction is presented in section one. In the section two, we introduce the mathematical formulation that describes the velocity of the mixture (vapour and superheated liquid) fluid in N-dimensions and derivation also the Rayleigh-Plesset equation for a gas–filled cavity in N-dimensions. The analytical solution is presented in section three. In sections four and five, we introduce the results, discussions and conclusions.

2. Mathematical Formulation

In the current section we introduce the derivation of the main equations as continuity equation and momentum equation in N-dimensions spherical fluid, that describe the problem of bubble dynamics in N-dimensions.

2.1. Continuity equation in N-dimensions fluid

The continuity equation of liquids has the form:

$$\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{u}) = 0, \tag{1}$$

where, $\boldsymbol{u}$ is the vector of velocity, ρ is the density of liquid. In the case of incompressible liquid that means $\rho = constant$, and for the case of spherically symmetric liquid in N-dimensions, the equation of continuity (1) takes the form

$$div(\boldsymbol{u}) = \frac{1}{r^{N-1}} \frac{d}{dr} (r^{N-1} u) \ = 0, \tag{2}$$

where, N represents the N-dimensions of fluid. r is the radial coordinate. Note that, the point of radial coordnates r equals zero ($r = 0$), that corresponds to the center of bubble.
The equation of vapour continuity in terms of the velocities reads as

$$\rho_v \dot{R} = \rho \dot{R} - \rho (u)_{r=R}. \tag{3}$$

Here, ρ_v is a density of vapour in flow. $\dot{R}$ is the velocity of bubble wall. $(u)_{r=R}$ is the velocity of liquid at the bubble boundary which can be expressed in the following form

$$(u)_{r=R} = \varepsilon \dot{R}, \tag{4}$$

where, $\varepsilon = 1 - \frac{\rho_v}{\rho}$.
Integrating (2) w.r.t. r, the solution takes

$$u\, r^{N-1} = c_1(t). \tag{5}$$

where; $c_1(t)$ is a function of time and it is independent with r. At $r = R$ and using Eq. (4) then $c_1(t) = \varepsilon \dot{R} R^{N-1}$, the velocity in the mixture of the vapour and superheated liquid and in N-dimensions becomes

$$u = \frac{\varepsilon \dot{R} R^{N-1}}{r^{N-1}}. \tag{6}$$

Here u is the mixture velocity in N-dimensions.

2.2. Derivation of Rayleigh-Plesset equation in N-dimensions fluid

The equation of Navier-Stokes in spherical coordinates for the case of spherically symmetric and incompressible fluid has the form

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) = -\frac{\partial P_l'}{\partial r} + \eta(\nabla^2 u - \frac{(N-1)u}{r^2}). \tag{7}$$

Here, u is the velocity of mixture of vapour and superheated liquid in N-dimensions fluid. η is the viscosity. t is the time. P_l' can be defined as

$$P_l' = P_l + 2\eta \frac{\partial u}{\partial r}. \tag{8}$$

To evaluate the term $(\nabla^2 u - \frac{(N-1)u}{r^2})$ in Eq. (9), we use this relation $\frac{1}{r^{N-1}}\frac{d}{dr}(r^{N-1}u)$, then we get

$$\nabla^2 u - \frac{(N-1)u}{r^2} = 0\ . \tag{9}$$

Differentiating Eq. (8) w. r. to r, and using Eq. (6), the term $\frac{\partial P_l'}{\partial r}$ can be rewritten in the form:

$$\frac{\partial P_l'}{\partial r} = \frac{\partial P_l}{\partial r} - 2\eta N(N-1)\varepsilon \frac{\dot{R}R^{N-1}}{r^{N+1}}, \tag{10a}$$

also,

$$\frac{\partial u}{\partial t} = \frac{\varepsilon}{r^{N-1}}\left(\ddot{R}R^{N-1} + (N-1)R^{N-2}\dot{R}^2\right), \tag{10b}$$

and

$$u\frac{\partial u}{\partial r} = -\varepsilon(N-1)\frac{R^{2(N-2)}\dot{R}^2}{r^{2N-1}}. \tag{10c}$$

Substituting from Eqs. (9,10) into Eq. (7), we get

$$\rho\left(\frac{\varepsilon}{r^{N-1}}\left(\ddot{R}R^{N-1} + (N-1)R^{N-2}\dot{R}^2\right) - \varepsilon(N-)\frac{R^{2(N-2)}\dot{R}^2}{r^{2N-1}}\right) = -\frac{\partial P_l}{\partial r} + 2\eta N(N-1)\varepsilon\frac{\dot{R}R^{N-1}}{r^{N+1}} \tag{11}$$

Assuming that the vapour bubble is in an infinite sea of liquid and integrate Eq. (11) w. r. to r from $r = R$ to $r = \infty$, then we get

$$\left(\frac{1}{-(N-2)}\left(\ddot{R}R^{N-1}+(N-1)R^{N-2}\dot{R}^2\right)\frac{1}{r^{N-2}}\Big|_{r=R}^{\infty}-\frac{(N-1)}{2(N-1)}\frac{R^{2(N-2)}\dot{R}^2}{r^{2N-1}}\Big|_{r=R}^{\infty}\right)=$$
$$-\frac{1}{\varepsilon\rho}P_l|_{r=R}^{\infty}+\frac{2\eta N(N-1)}{N\rho}\dot{R}R^{N-1}\frac{1}{r^N}\Big|_{r=R}^{\infty}. \quad (12)$$

Note that, $P_l|_{r=R}^{\infty}=P_\infty-(\Delta P_l)_{r=R}^{\infty}$.
In the Pressure balance at the bubble surface, A simple force balance at the surface of spherical bubble (at $r=R$) is defined as $\pi R^2(P_v-(\Delta P_l)_{r=R}^{\infty})=2\pi R\sigma$ and the pressure at the bubble boundary (an ambient pressure) takes the form

$$(\Delta P_l)_{r=R}^{\infty}=P_v-\frac{2\sigma}{R} \quad (13)$$

After some calculation, Eq. (12) can be put in the form

$$\frac{1}{(N-2)}R\ddot{R}+\frac{N}{2(N-2)}\dot{R}^2=\frac{1}{\varepsilon\rho}\left(P_v-P_\infty-\frac{2\sigma}{R}\right)-\frac{2(N-1)}{\rho}\frac{\dot{R}}{R}\eta. \quad (14)$$

The Eq. (14) is called the modified of Rayleigh-Plesset equation in N-dimensions for the growth (or collapse) of bubble in a viscous liquid.

In Eq. (14), we remark that N is the number of space dimensions, and $N \geq 3$ satisfies in the Rayleigh-Plesset equation in N-dimensions. Also, we note that we do not use the case of $N=2$ in our study, since Eq. (14) contains logarithmic terms and, therefore, are singular. Now, the main aim is to find the generalized solutions of Eq. (14). It seems impossible to construct solutions of (14) by their direct method. However, it is possible to solve it by using the method that will introduce at next section.

3. Method of the solution

To complete the study of bubble dynamics, we suppose that the vapour bubble is a spherical shape. The liquid is a mixture of vapour and superheated liquid where the mixture is in an incompressible. The pressure inside the bubble is to be uniform. The distribution of vapour density inside the bubble also is to be uniform except for a thin boundary layer near the bubble wall. Any gravitational effects are omitted in the equation of motion. We take the viscosity in an account. So, let us to start, suppose the viscosity of the liquid reads

$$\eta=\frac{C\,R(t)}{\dot{R}\,(t)}. \quad (15)$$

Here $C=\frac{\eta_0\dot{R}_0}{R_0}$, where R_0 is the ambient radius of bubble. $\dot{R}_0$ is the initial velocity of bubble. η_0 is the initial viscosity.

In our study, we note that the pressure inside the bubble is P_v, and the pressure P_∞ in the liquid is constant. Assume that the difference of pressure ΔP is constant in time, and should be $P_v>P_\infty$,

In order to complete the soluton Eq. (14) we assume the difference between the pressure at the bubble boundary and the pressure in the liquid $\Delta P=P_v-P_\infty$ takes

$$\Delta P=A(T_R-T_\infty). \quad (16)$$

Here, T_R is the temperature inside the bubble. T_∞ is the temperature in the liquid, A is the constant, will be estimated in below.
Combining Eqs. (14-16), eq. (14) becomes

$$\frac{1}{(N-2)}R\ddot{R}+\frac{N}{2(N-2)}\dot{R}^2=\frac{1}{\varepsilon\rho}\left(A(T_R-T_\infty)-\frac{2\sigma}{R}\right)-\frac{2(N-1)}{\rho}C. \quad (17)$$

In the following the initial conditions, when the bubble starts to appear that the initial conditions becomes as

$$\begin{aligned} &t = 0, \\ &R(0) = R_0\,, \\ &\dot{R}(0) = \dot{R}_0, \\ &\ddot{R}(0) = 0\,, \\ &T_R = T_0. \end{aligned} \tag{18}$$

Applying the initial conditions in Eq. (18) into Eq. (17), then we can calculate A. After some simplification of calculations. Eq. (17) reads

$$R\ddot{R} + \frac{N}{2}\dot{R}^2 = \frac{2(N-2)\sigma}{\varepsilon\rho R_0}\left(\frac{\Delta T_R^*}{\Delta\theta_0} + 1\right)\left[1 + \frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right] - \frac{2(N-2)}{\varepsilon\rho}\frac{\sigma}{R} - \frac{2(N-1)(N-2)}{\rho}C, \tag{19}$$

where $\Delta T_R^* = T_R - T_0$ is defined in down , and $\Delta\theta_0 = T_0 - T_\infty$ is the superheating liquid.
From above equations, the Eq. (19) can rewrite in the form

$$\frac{1}{d_1 R^{d_2-1}\dot{R}^{d_3-1}}\frac{d}{dt}\left(R^{d_2}\dot{R}^{d_3}\right) = \frac{2(N-2)\sigma}{\varepsilon\rho R_0}\left(\frac{\Delta T_R^*}{\Delta\theta_0} + 1\right)\left[1 + \frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right] - \frac{2(N-2)}{\varepsilon\rho}\frac{\sigma}{R} - \frac{2(N-1)(N-2)}{\rho}C, \tag{20}$$

where, $\frac{N}{2} = \frac{d_2}{d_1}$, $d_1 = d_3$.
The parameter ΔT_R^* is defined by Plesset et al [7] in the form

$$\Delta T_R^* = -\left(\frac{a_l}{\pi}\right)^{\frac{1}{2}}\int_0^t \left(\frac{R^2(y_1)\left(\frac{\partial T}{\partial r}\right)_{r=R(y_1)}}{\left(\int_0^t R^4(y_2)dy_2\right)^{\frac{1}{2}}}\right)dy_1. \tag{21}$$

where, a_l is the thermal diffusivity.

Eq. (21) represents a solution of the heat equation. Authors in ref. [7] achive under the assumption where the bubble motion is sufficiently rapid because of any translational motion may be omitted which leads to vanish the convection term in heat equation and using the thin layer approximation around the growing bubble with error less than 10% [18].
To solve the system, we convert the system to dimensionless

$$\begin{aligned} &\chi = \left(\frac{R}{R_0}\right)^3, \\ &\nu = \frac{\gamma}{R_0^4}\int_0^t R^4(y_2)\,dy_2, \\ &\gamma = \left(\frac{2\sigma}{\rho R_0^3}\right)^{\frac{1}{2}}. \end{aligned} \tag{22}$$

The gradient of temperature in the mixture of vapour and superheated liquid at the bubble boundary reads as

$$(4\pi R^2)k_l\left(\frac{\partial \mathrm{T}}{\partial r}\right)_{r=R} = \frac{d}{dt}\left(\rho_v\left(\frac{4}{3}\pi R^3\right)\mathrm{L}\right). \tag{23}$$

Here, L is latent heat, k_l is thermal conductivity. ρ_v is the density of vapour.
By using Eq. (22) into Eq. (23), then

$$\left(\frac{\partial \mathrm{T}}{\partial \mathrm{r}}\right)_{\mathrm{r=R}} = \frac{\gamma\rho_v \mathrm{L R_0}}{3\mathrm{k_l}}\chi^{\frac{2}{3}}\chi'. \tag{24}$$

Substituting from Eq. (22) into the R.H.S of Eq. (20), then

$$\frac{1}{d_1 R^{d_2-1}\dot{R}^{d_3-1}}\frac{d}{dt}\left(R^{d_2}\dot{R}^{d_3}\right)=\frac{\gamma^2 R_0^2}{3d_1}\frac{1}{\chi^{\frac{1}{3}(d_2+2d_3-7)}\chi'^{(d_3-1)}}\frac{d}{d\nu}\left(\chi^{\frac{1}{3}(d_2+d_3)}\chi'^{d_3}\right). \tag{25}$$

After some simplification calculation ΔT_R^* will be taken

$$\Delta T_R^* = -\frac{\rho_v L R_0}{k_l}\left(\frac{\gamma a_l}{9\pi}\right)^{\frac{1}{2}}\int_0^{\nu}\frac{\chi'(V)}{(\nu-\xi)^{\frac{1}{2}}}d\xi. \tag{26}$$

From Eqs. (22, 25, 26) into Eq. (20), then we get

$$\frac{\gamma^2 R_0^2}{3d_1}\frac{1}{\chi^{\frac{1}{3}(d_2+2d_3-7)}\chi'^{(d_3-1)}}\frac{d}{d\nu}\left(\chi^{\frac{1}{3}(d_2+d_3)}\chi'^{d_3}\right) = \frac{2(N-2)\sigma}{\varepsilon\rho R_0}\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2+\frac{2(N-1)}{\rho}C\right)\right] - \frac{2(N-2)\sigma K}{\varepsilon\rho R_0}\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2+\frac{2(N-1)}{\rho}C\right)\right]\frac{\rho_v L R_0}{k_l\Delta\theta_0}\left(\frac{\gamma a_l}{9\pi}\right)^{\frac{1}{2}}\int_0^{\nu}\frac{\chi'(V)}{(\nu-\xi)^{\frac{1}{2}}}d\xi - \frac{2(N-2)}{\varepsilon\rho}\frac{\sigma}{\mathrm{R}_0\chi^{\frac{1}{3}}} - \frac{2(N-1)(N-2)}{\rho}C, \tag{27}$$

Dividing above Eq. by $\gamma^2 R_0^2\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2+\frac{2(N-1)}{\rho}C\right)\right]$ and make some calculations, then Eq. (27)

$$\frac{1}{3d_1\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2+\frac{2(N-1)}{\rho}C\right)\right]}\frac{1}{\chi^{\frac{1}{3}(d_2+2d_3-7)}\chi'^{(d_3-1)}}\frac{d}{d\nu}\left(\chi^{\frac{1}{3}(d_2+d_3)}\chi'^{d_3}\right) = \frac{(N-2)}{\varepsilon} - \lambda\int_0^{\nu}\frac{\chi'(V)}{(\nu-\xi)^{\frac{1}{2}}}d\xi - \frac{(N-2)}{\varepsilon\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2+\frac{2(N-1)}{\rho}C\right)\right]\chi^{\frac{1}{3}}} - \frac{(N-1)(N-2)R_0}{\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2+\frac{2(N-1)}{\rho}C\right)\right]\sigma}C, \tag{28}$$

where

$$\lambda = \frac{2(N-2)\sigma}{\varepsilon\rho R_0^2}\frac{\rho_v L}{k_l\Delta\theta_0}\left(\frac{a_l}{9\pi\gamma^3}\right)^{\frac{1}{2}}.$$

The L.H.S of the Eq. (28) represents essentially the acceleration effects of the bubble growth in the liquid. As the bubble grows, this acceleration tends towards zero (see ref. [8]) at complete growth under these conditions

$$t = t_{\mathrm{f}}, \qquad R = R_{\mathrm{m}},\ \dot{R}\neq 0,\ \ \ddot{R}=0,\ \text{then}\ \chi=\chi_m. \tag{29}$$

Thus, Eq. (28) takes the form:

$$\int_0^{\nu}\frac{\chi'(\xi)}{(\nu-\xi)^{\frac{1}{2}}}d\xi \to \frac{1}{\lambda}\left(\frac{(\mathrm{N}-2)}{\varepsilon}\left(1-\frac{1}{\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{\mathrm{N}}{2(\mathrm{N}-2)}\dot{\mathrm{R}}_0^2+\frac{2(\mathrm{N}-1)}{\rho}\mathrm{C}\right)\right]\chi_{\mathrm{f}}^{\frac{1}{3}}}\right) - \frac{(\mathrm{N}-1)(\mathrm{N}-2)\mathrm{R}_0}{\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{\mathrm{N}}{2(\mathrm{N}-2)}\dot{\mathrm{R}}_0^2+\frac{2(\mathrm{N}-1)}{\rho}\mathrm{C}\right)\right]\sigma}\mathrm{C}\right). \tag{30}$$

To integrate this term $\int_0^{\nu}\frac{\chi'(\xi)}{(\nu-\xi)^{\frac{1}{2}}}d\xi$, we suppose that $\xi = z\nu$, then we get

$$\int_0^{\nu}\frac{\chi'(\xi)}{(\nu-\xi)^{\frac{1}{2}}}d\xi = \nu^{-\frac{-1}{2}}\int_0^1(1-z)^{\frac{-1}{2}}\frac{d\chi(z\nu)}{dz}\frac{dz}{d\xi}\nu dz = \nu^{-\frac{-1}{2}}\int_0^1(1-z)^{\frac{-1}{2}}\frac{d\chi(z\nu)}{dz}\,dz. \tag{31}$$

Since the Eq. (31) is independent of the time, we assume $\chi = h_0\nu^{\frac{1}{2}}$, where h_0 is a constant. Hence,

$$\chi(z\nu) = h_0 (z\nu)^{\frac{1}{2}}, \text{ and } \frac{d\chi(z\nu)}{dz} = \frac{1}{2} h_0 \nu^{\frac{1}{2}} z^{-\frac{1}{2}}. \tag{32}$$

The integration (31) can be reduced to

$$\int_0^{\nu} \frac{\chi'(\xi)}{(\nu-\xi)^{\frac{1}{2}}} d\xi = \frac{h_0}{2} \int_0^1 (1-z)^{\frac{-1}{2}} z^{-\frac{1}{2}}\, dz = \pi \frac{h_0}{2}. \tag{33}$$

Using (33) into (30), we get

$$\mathrm{h}_0 \to \frac{2}{\pi\lambda}\left(\frac{(\mathrm{N}-2)}{\varepsilon}\left(1 - \frac{1}{\left[1+\frac{\varepsilon\rho \mathrm{R}_0}{2\sigma}\left(\frac{\mathrm{N}}{2(\mathrm{N}-2)}\dot{\mathrm{R}}_0^2 + \frac{2(\mathrm{N}-1)}{\rho}\mathrm{C}\right)\right]\chi_{\mathrm{f}}^{\frac{1}{3}}}\right) - \frac{(\mathrm{N}-1)(\mathrm{N}-2)\mathrm{R}_0}{\left[1+\frac{\varepsilon\rho \mathrm{R}_0}{2\sigma}\left(\frac{\mathrm{N}}{2(\mathrm{N}-2)}\dot{\mathrm{R}}_0^2 + \frac{2(\mathrm{N}-1)}{\rho}\mathrm{C}\right)\right]\sigma}\mathrm{C}\right). \tag{34}$$

Substituting from Eq. (30) into this assumption $\chi = h_0 \nu^{\frac{1}{2}}$, then we get

$$\chi = \nu^{\frac{1}{2}} \frac{2}{\pi\lambda}\left(\frac{(N-2)}{\varepsilon}\left(1 - \frac{1}{\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right]\chi_f^{\frac{1}{3}}}\right) - \frac{(N-1)(N-2)R_0}{\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right]\sigma}C\right). \tag{35}$$

Combining Eqs. (22,35) and using the value λ from Eq. (28), then we obtain

$$R = \left(\frac{12}{\pi} a_l\right)^{\frac{1}{2}} J_a \left(\left(1 - \frac{1}{\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right]\chi_f^{\frac{1}{3}}}\right) - \frac{(N-1)R_0}{\left[1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right]\sigma}C\right) t^{\frac{1}{2}}. \tag{36}$$

Since the Jacob number, thermal diffusivity and initial void fraction are defined in the form $\varphi_0 = \left(\frac{R_0}{R_m}\right)^3$, $0 < \varphi_0 < 1$, and ,$J_a = \frac{\rho_l C_P \Delta\theta_0}{\rho_v L}$, is a Jacob number.
By using the above relations, the Eq. (36) reads

$$R = \left(\frac{12}{\pi} a_l\right)^{\frac{1}{2}} J_a \left(1 - \frac{\varphi_0^{\frac{1}{3}}}{\left(1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right)} - \frac{(N-1)R_0}{\left(1+\frac{\varepsilon\rho R_0}{2\sigma}\left(\frac{N}{2(N-2)}\dot{R}_0^2 + \frac{2(N-1)}{\rho}C\right)\right)\sigma}C\right) t^{\frac{1}{2}}. \tag{37}$$

The above equation represents the behaviour of growth vapour bubble in N-dimensions fluid for the variable viscosity between two finite boundaries. The growth of bubble depends on initial void fraction, N-dimensions fluid and parameter C.

4. Results and discussions

The Rayleigh-Plesset equation (14) is derived in N-dimensions, that is included the effect of the viscosity, density ratio, pressure and surface tension. This equation is solved analytically using the modified method in Ref. [7]. The Eq. (14) is reduced to the model of Rayleigh-Plesset [7] when *N=3*, and we can obtain the Rayleigh-Plesset [7] for considering viscosity in the form:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\Delta P - \frac{4\eta}{\rho}\frac{\dot{R}}{R}.$$

Note that, Eq. (14) is satisfied for every value bigger than and equal three, $N \geq 3$. The velocity of the mixture of vapour and superheated liquid in N-dimensions is obtained in (6). The general analytical solution of N-dimensional Rayleigh-Plesset equation for bubble growth is given in Eq. (37). The obtained solution in (37) represents the behaviour the growth of bubble in N-dimensions fluid. The numerical values of physical parameters, which are used in the simulating and calculations for the bubble growth as: $\rho = 937.5\ [kg/m^3]$, $\rho_v = 0.579\ [kg/m^3]$, $\sigma = 0.0535\ [m^3/sec]$, $L = 533 \times 10^3 [J/kg]$, $\Delta\theta_0 = 1.0\ [^0K]$, $\eta_0 = 0.89 \times 10^{-3} [Pa.sec]$, $C_{pl} = 4240 [J/kg^0K]$, $C_{pv} = 2160 [J/kg^0K]$ *and* $k_l = 0.6857\ [W/m^0K]$. These values of physical parameters are in ref. [19], and $R_0 = 10^{-5} [m], R_m = 10^{-4}\ [m], \dot{R}_0 = 0.2 [m/sec]$. The void fraction φ_0 is estimated as $\varphi_0 = 1.0 \times 10^{-3}$, and Jacob number $J_a$is also estimateded as $J_a = 12.88$.

Figure 1 shows the results of the analytical solution of Rayleigh-Plesset equation in N-dimensions liquid for the growth of vapour bubble. We note that Fig.1 illustrates the radius of bubble for some different values of N-dimensions. In other words, we can say that we obtain the behaviour of vapour bubble at $N \geq 3$; N is denoted the N-dimensional spherical bubbles, This results are satisfied when we can find for certain value of N that means $N = 3$, in this case for $N = 3$, we can get the Rayleigh equation [6] and its solution for neglecting viscosity of liquid and surface tension, For the following the effect of N on bubble dynaimic in Fig. 1, we can illustrate that it can see, the radius and the period of the bubble decrease with the increasing of the space N-dimension. These results are agreement with the results in ref. [14], although results in ref. [14] have been obtained in empty–filled spherical cavity in an infinite volume.

Figure 2 refers the results of the behaviour of vapour bubble in viscous and non-viscous liquid at N-dimensions (i.e., we choose $N = 3$), and we get the behviour of vapour bubble in viscous fluid is smaller than one. This result should be taken when we study cavitations and bubble dynamics that we remark, the liquid viscosity plays an important role on growth of bubbles. Figures 3a and 3b reflect the comparison between the present model for different values of N and several previous models in [7-9, 20, 21] for the growth of vapour bubbles in water at relatively high superheats. To the best of our knowledge, the presented solution of the radius of bubble growth $R(t)$ in N-dimensions in Eq. (37) has been obtained for the first time and we can reach to a special cases of the previous solutions of $R(t)$, i.e. in the models [7-9,20,21]. It is clear that, if we put $R(t) = \Omega_1 \sqrt{t}$, we can get

- ❖ the solution of Plesset and Zwick model [7], at $N = 3$, $\varphi_0 = 0$, $C = 0$, $\Omega_1 = J_a \left(\frac{12}{\pi} a_l\right)^{\frac{1}{2}}$,
- ❖ the solution of Mohammadein model [8], at $N = 3$, $C = 0$, $\Omega_1 = J_a \left(\frac{12}{\pi} a_l\right)^{\frac{1}{2}} \left(1 - \frac{4\pi\varphi_0^{\frac{1}{3}}}{3\dot{\rho} R_0^2 R_0}\right)$,
- ❖ the solution of Olek model [9], at $N = 3$, $C = 0$, $\varphi_0 = 0$, $\Omega_1 = \frac{1+(1-\frac{2\pi}{Ja})}{\pi^{\frac{1}{2}}} J_a a_l^{\frac{1}{2}}$,
- ❖ the solution of Forster model [20], at $N = 3$, $\varphi_0 = 0$, $C = 0$, $\Omega_1 = \pi J_a a_l^{\frac{1}{2}}$.
- ❖ the solution of Bosnjakovic model [21], at $N = 3$, $\varphi_0 = 0$, $C = 0$, $\Omega_1 = 2(J_a a_l)^{\frac{1}{2}}$,

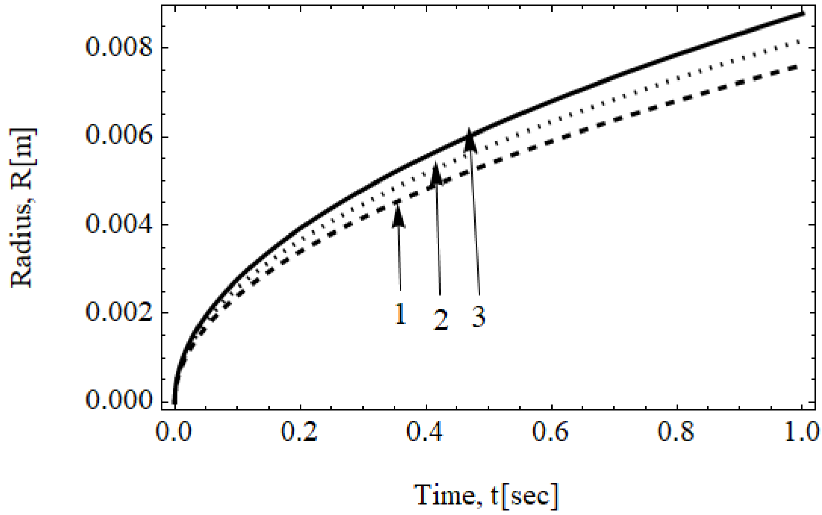

Figure 1. The behaviour of bubble growth $R(t)$ versus the time t for some different of N-dimensions. 1) at $N = 7$, 2) at $N = 5$, 3) at $N = 3$

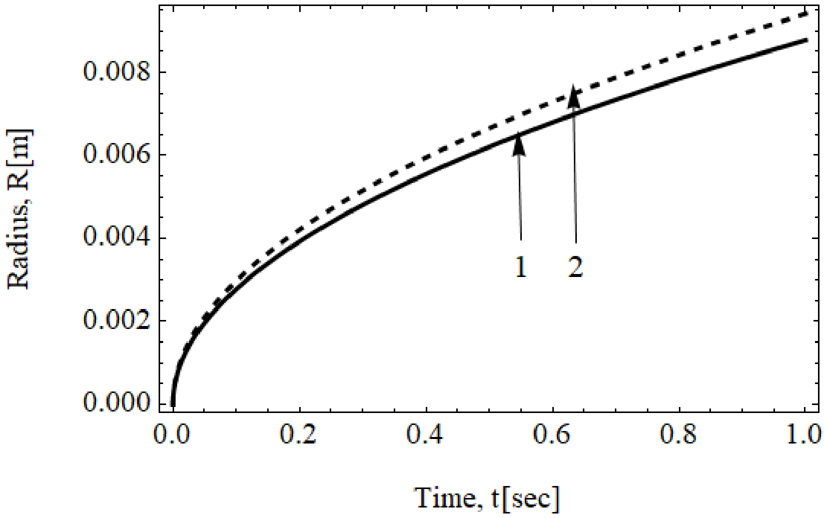

Figure 2. The behaviour of bubble growth $R(t)$ versus the time t. 1) viscous fluid, 2) non-viscous fluid; $N = 3$

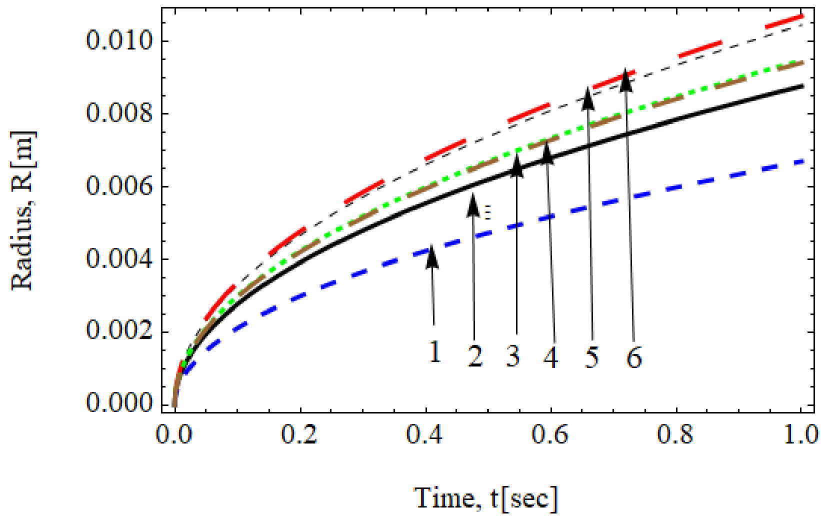

Figure 3a. Comparison the present model with the previous models at N=3. **1) Olek model [9], 2) Results for present model, 3) Bosnjakovic model [21], 4) Mohammadein model [8], 5) Plesset and Zwick model [7], 6) Forster model [20]**

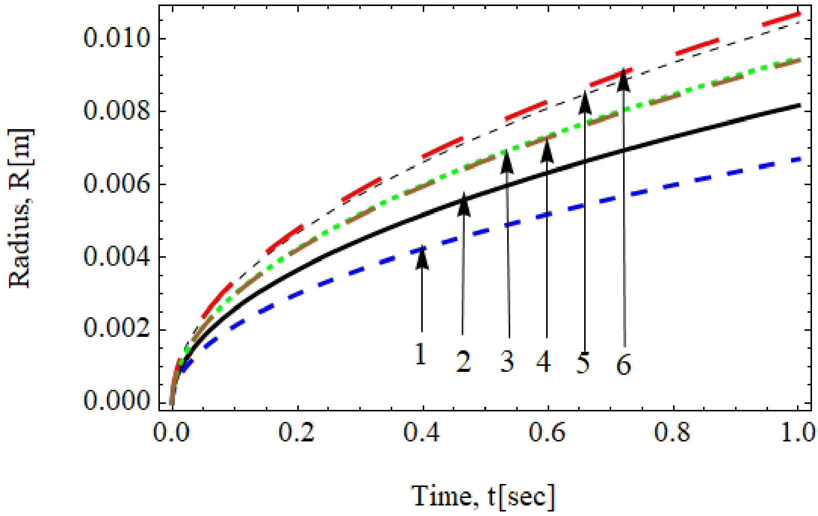

Figure 3b. Same the Fig. 3a at N=5

5. Conclusions

We have studied the continuity equation and Rayleigh-Plesset equation in N-dimension for the description of a vapour/gas spherical bubble dynamics. We have analyzed the general solutions of the Rayleigh-Plesset equation in N- space dimensions for the terms of surface tension and viscosity. We have discussed dependence of these solutions on N-dimensions. We have also considered influence of the viscosity on the bubble motion. We concluded that the bubble radius decrease with the increasing of the space N-dimensions. The growth of vapour bubble in viscous liquid is smaller than in non-viscous liquid. These conclusions should be used in the applications of bubble dynamics

Nomenclature

A	- constants in Eq.(16), [-]	***Greek symbols***	
a_l	- thermal diffusivity, $[m^2/\mathrm{sec}]$	χ	- dimensionless parameter defined by Eq. (22), [-]
$\mathrm{C_p}$	- specific heat capacity, $[\mathrm{J/kg^0K}]$	k_l	- thermal conductivity, $[Kg/m\ \mathrm{sec}]$
$\mathrm{c_1}$	- constants in Eq.(5), [-]	σ	- surface tension, $[\mathrm{m^3/sec}]$
d_1, d_2, d_3	- constants in Eq. 20, [-]	ρ, ρ_v	- density $[Kg/m^3]$
h_0	- constant in Eq. (32), [-]	η	- viscosity, $[\mathrm{Pa.\,sec}]$
L	- latent heat, $[J/kg]$	ν	- constants in Eq. (22), [-]
N	- N-dimensions fluidin Eq. (2), [-]	γ	- constants in Eq. (22), [-]
u	- velocity in mixture $[m/sec]$	λ	- constant is defined by (28)
$\Delta\mathrm{P}$	- difference between the external pressure at infinity and in mixture, $[\mathrm{Kg/m\ sec^2}]$	ξ	- variables in Eq. (31)
r	- radial Coordinate, [m]	J_a	- Jacob number
R	- radius of bubble. $[m]$	φ	- void fraction
$\dot{R}$	- velocity of bubble $[m/sec]$	$\Delta\theta_0$	- superheating liquid, $[^0K]$
$\ddot{R}$	- acceleration of bubble $[m/sec^2]$	ε	- ratio of density [-]
T	- temperature, $[^0K]$	***Subscripts***	

t	- time $[sec]$	0	- initial value
y_1, y_2	- constants in Eq. (21)	m	- maximum value
z	- variables in Eq. (31)	v	- vapour
		l	- liquid
		∞	- infinity value

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