ON DISCRETE FRACTIONAL SOLUTIONS OF THE HYDROGEN ATOM TYPE EQUATIONS

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Discrete fractional calculus deals with sums and differences of arbitrary orders. In this study, we acquire new discrete fractional solutions (dfs) of hydrogen atom type equations (HAEs) by using discrete fractional nabla operator \(\nabla^\alpha\) \((0 < \alpha < 1)\). This operator is applied homogeneous and nonhomogeneous HAEs. So, we obtain many particular solutions of these equations.

Keywords: Discrete fractional analysis; hydrogen atom equation; Schrodinger equation.

Introduction

Fractional analysis has many applications in diverse fields of science and engineering such as Schrodinger equation, diffusion, control theory, and statistics [1-3]. The similar theory for discrete fractional analysis was initiated and properties of the theory of fractional differences and sums were established. Recently, many books and articles related to discrete fractional analysis have been published [4-12].

In 1956 [4], differences of fractional order were introduced by Kuttner. Diaz and Osler [5], define the concept of fractional difference as follows

\[ \Delta^\rho f(\tau) = \sum_{k=0}^{\rho} (-1)^k \binom{\rho}{k} f(\tau + \rho - k) \]

where \(\rho\) is any real number. Granger, Joyeux, and Hosking [13,14] defined the concept of fractional difference as follows

\[ \nabla^\rho f(\tau) = \sum_{k=0}^{\rho} (-1)^k \binom{\rho}{k} f(\tau - k), \quad \binom{\rho}{k} = \frac{\Gamma(\rho + 1)}{\Gamma(k + 1) \Gamma(\rho - k + 1)} \]

where \(\rho\) is any real number and \(q^k f(\tau) = f(\tau - k)\) is shift operator. Gray and Zhang [15] studied on a new definition of the fractional difference through summation.

In this article, we will consider the equation:

\[ \frac{d^2\Psi}{dr^2} + \frac{n}{r} \frac{d\Psi}{dr} - \frac{\ell(\ell + 1)}{r^2} \Psi + \left( \kappa + \frac{n}{r} \right) \Psi = 0, \quad 0 < r < \infty \quad (1) \]
where $\Psi$ is the distance of the mass center to the origin, $n$ is a real number, $\ell$ is a positive integer, $\mathcal{K}$ is an energy constant and $r$ is the distance among the nucleus and electron [16]. If we take $\Psi = u / r$ in Eq. (1), then we have the HAE

$$\frac{d^2 u}{dr^2} + \left[ \mathcal{K} + \frac{n}{r} - \frac{\ell(\ell+1)}{r^2} \right] u = 0.$$  \hspace{1cm} (2)

In many works,

$$q(r) = \frac{n}{r} - \frac{\ell(\ell+1)}{r^2}$$  \hspace{1cm} (3)

this part takes a centripetal and Colomb part, the usual singularities of the nuclear problem [17]. Yilmazer [18] acquired fractional solutions of Eq. (2) by using the Nishimoto operator. Panakhov and Yilmazer [19] investigated the Hochstadt-Lieberman theorem for Eq. (2). Bas et al. [20] researched the uniqueness for the HAE (2).

The aim of this article is to obtain new dfs of the HAEs by means of fractional calculus operator.

**Preliminary and Properties**

Let $\rho \in \mathbb{R}^+$ such that $k - 1 \leq \rho < k$ where $k$ is an integer. $\rho$-th order fractional sum of $g$ is defined by

$$\nabla_{\rho}^b g(\tau) = \frac{1}{\Gamma(\rho)} \sum_{s=b}^{\tau} (\tau - \delta(s))^{\rho-1} g(s)$$  \hspace{1cm} (4)

where $\tau \in \mathbb{R}^+ = \{b\} + \mathbb{N}_0 = \{b, b+1, b+2, \ldots\}$, $b \in \mathbb{R}$, $\delta(\tau) = \tau - 1$ is jump operator.

The ascending factorial is defined by

$$\tau^\tau = \prod_{n=0}^{k-1} (\tau + n) = \tau(\tau+1)(\tau+2)\ldots(\tau+k-1), \quad k \in \mathbb{N}, \quad 0^\tau = 1.$$  \hspace{1cm} (5)

Let $\rho \in \mathbb{R}$. Then "rising" is given by

$$\tau^\tau = \frac{\Gamma(\tau + \rho)}{\Gamma(\tau)} \tau \in \mathbb{N} - \{-2, -1, 0\}, \quad 0^\tau = 0.$$  \hspace{1cm} (5)

Let us note that

$$\nabla(\tau^\tau) = \rho \tau^{\tau-1}$$  \hspace{1cm} (6)

where $\nabla u(\tau) = u(\tau) - u(\delta(\tau)) = u(\tau) - u(\tau - 1)$. 
\( \rho \)-th order fractional difference of \( g \) is given by

\[
\nabla_{b}^{\rho} g (\tau) = \nabla^{k} \left[ \nabla^{-(k+s)\rho} g (\tau) \right] 
= \nabla^{k} \left[ \frac{1}{\Gamma(k-\rho)} \sum_{s=b}^{k} (\tau-\delta(s))^{|k-\rho|} g (s) \right] 
\]  

(7)

where \( g : \bar{\mathbb{R}} \to \bar{\mathbb{R}} \) [8].

**Theorem 1** [11]. Let \( f (\tau) \) and \( g (\tau) : \bar{\mathbb{R}} \to \bar{\mathbb{R}}, \rho, \eta > 0 \) \( h, v \) are scalars. The following equality holds:

\[
\nabla^{\rho-\eta} f (\tau) = \nabla^{(\rho+\eta)} f (\tau) = \nabla^{\eta} \nabla^{\rho} f (\tau), 
\]  

(8)

\[
\nabla^{\rho} \left[ hf (\tau) + vg (\tau) \right] = h\nabla^{\rho} f (\tau) + v\nabla^{\rho} g (\tau) , 
\]  

(9)

**Lemma 2 (Power Rule)** [6]. Let \( \rho > 0 \). Then the following holds

\[
\nabla^{\rho} (\tau-b+1)^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\rho+1)} (\tau-b+1)^{\eta+\rho} 
\]  

for every \( \tau \in \bar{\mathbb{R}} \).

**Lemma 3** [7]. For any \( \rho > 0 \), the following equality holds:

\[
\nabla_{b}^{\rho} \nabla f (\tau) = \nabla^{\rho} f (\tau) - \frac{(\tau-b+1)^{\rho-1}}{\Gamma(\rho)} f (b). 
\]  

(10)

**Lemma 4 (Leibniz Rule)** [8]. For any \( \rho > 0 \), \( \rho \)-th order, the fractional difference of the product \( fg \) is given by

\[
\nabla^{\rho} (fg) (\tau) = \sum_{k=b}^{\infty} \left[ \nabla^{\rho} f (\tau-k) \right] [\nabla^{k} g (\tau)] 
\]  

(11)

where \( \left( \frac{\rho}{k} \right) = \frac{\Gamma(\rho+1)}{\Gamma(k+1) \Gamma(\rho-k+1)} \) and \( g \) is shift operator.

**Lemma 5 (Index law)** [18]. Let \( f \) is analytic and single-valued. The following equality holds:

\[
(f_{\rho} (\tau))_{\eta} = f_{\rho+\eta} (\tau) = (f_{\eta} (\tau))_{\rho} \quad (f_{\rho} (\tau) \neq 0; f_{\eta} (\tau) \neq 0) 
\]  

(12)

for every \( \rho, \eta \in \bar{\mathbb{R}} \).
Main Results

Dfs of nonhomogeneous HAE

**Theorem 6.** Let \( \Phi \in \{ \Phi : 0 \neq |\Phi| < \infty; \alpha \in \square \} \) and \( \phi \in \{ \phi : 0 \neq |\phi| < \infty; \alpha \in \square \} \). Then the nonhomogeneous HAE ( \( \tau = \ell + (1/2) \) in (2));

\[
\Phi_2 + \left[ \lambda + \frac{\gamma}{z} + \frac{1 - \tau^2}{z^2} \right] \Phi = \phi, \quad (z \neq 0)
\]

has particular solutions of the forms:

\[
\Phi^I = z^{\tau+1} e^{-\sqrt{\lambda} z} \left[ \left( \phi z^{\frac{1}{2} + \tau} e^{-\sqrt{\lambda} z} \right)_\alpha e^{-2\sqrt{\lambda} z} z^{-\tau - \frac{1}{2} - \frac{\gamma}{z}} \right]_{-(\alpha+1)},
\]

\[
\Phi^II = z^{\tau+1} e^{-\sqrt{\lambda} z} \left[ \left( \phi z^{\frac{1}{2} + \tau} e^{-\sqrt{\lambda} z} \right)_\beta e^{-2\sqrt{\lambda} z} z^{-\tau - \frac{1}{2} - \frac{\gamma}{z}} \right]_{-(\beta+1)},
\]

\[
\Phi^III = z^{-\tau+1} e^{-\sqrt{\lambda} z} \left[ \left( \phi z^{\frac{1}{2} + \tau} e^{-\sqrt{\lambda} z} \right)_\alpha e^{2\sqrt{\lambda} z} z^{-\tau - \frac{1}{2} + \frac{\gamma}{z}} \right]_{-(\alpha+1)},
\]

\[
\Phi^IV = z^{-\tau+1} e^{-\sqrt{\lambda} z} \left[ \left( \phi z^{\frac{1}{2} + \tau} e^{-\sqrt{\lambda} z} \right)_\beta e^{2\sqrt{\lambda} z} z^{-\tau - \frac{1}{2} + \frac{\gamma}{z}} \right]_{-(\beta+1)},
\]

where \( \Phi_2 = d^2 \Phi / dz^2 \), \( \Phi_n = \Phi(\Phi(z))(z \neq 0, z \in \square) \), \( \phi = \phi(z) \) (an arbitrary given function); \( \tau, \gamma \) are given constants, \( Q \) is a shift operator and \( \alpha, \beta \) are given by

\[
\alpha = -Q^{-1} \left( \frac{i \gamma}{2 \sqrt{\lambda}} + \tau + \frac{1}{2} \right), \quad \alpha = -Q^{-1} \left( -\frac{i \gamma}{2 \sqrt{\lambda}} + \tau + \frac{1}{2} \right) = \beta.
\]

**Proof.** Set

\[
\Phi = z^\sigma \psi, \quad \psi = \psi(z)
\]

we have then

\[
z \psi_2 + 2z \psi_1 + \left[ \left( \sigma^2 - \sigma + \frac{1}{4} - \tau^2 \right)z^{-1} + \sqrt{\lambda} z + \gamma \right] \psi = \phi z^{1-\sigma}
\]

(19)

from (13). Choose \( \sigma \) such that

\[
\sigma = \frac{1}{2} \pm \tau.
\]

Case of \( \sigma = \tau + \frac{1}{2} \). From (18) and (19), we have
\[
\Phi = z^{\tau \over 4} \psi \tag{21}
\]

and

\[
z\psi_2 + (2\tau + 1)\psi_1 + (\lambda z + \gamma)\psi = \phi z^{1/2 - \tau} \tag{22}
\]

respectively.

Next, set

\[
\psi = e^{\nu^2} \phi \left[ \varphi = \varphi(z) \right] \tag{23}
\]

we have then

\[
z\phi_2 + (2\nu z + 2\tau + 1)\phi_1 + \left[ (\nu^2 + \lambda)z + (2\tau + 1)\nu + \gamma \right]\phi = \phi z^{1/2 - \nu} e^{-\nu z} \tag{24}
\]

from (22), applying (23). Choose \( \nu \) such that

\[
\nu = \pm \sqrt{\lambda i}, \quad \lambda > 0. \tag{25}
\]

\( i \) When \( \nu = -\sqrt{\lambda i} \), we have

\[
\psi = e^{-\sqrt{\lambda} \delta} \phi \tag{26}
\]

and

\[
z\phi_2 + \left( -2\sqrt{\lambda} iz + 2\tau + 1 \right)\phi_1 + \left[ -i\sqrt{\lambda} \left( 2\tau + 1 \right) + \gamma \right]\phi = \phi z^{1/2 + \nu} e^{\sqrt{\lambda} \delta} \tag{27}
\]

from (23) and (24).

Applying the discrete operator \( \nabla^\mu \) to both sides of (27), we obtain

\[
z\phi_{2,\alpha} + \left( -2\sqrt{\lambda} iz + 2\tau + 1 + \alpha Q \right)\phi_{1,\alpha} + \left[ \gamma - i\sqrt{\lambda} \left( 2\tau + 1 + 2\alpha Q \right) \right]\phi_{\alpha} = \left( \phi z^{1/2 + \nu} e^{\sqrt{\lambda} \delta} \right)_{\alpha} \tag{28}
\]

from (9), (11) and (12).

Choose \( \alpha \) such that
\[ \alpha = -Q^{-1}\left(\frac{iy}{2\sqrt{\lambda}} + \tau + \frac{1}{2}\right) \]  \hspace{1cm} (29)

then we have

\[
z\varphi_{z\varphi^{-1}(\frac{i}{2\sqrt{\lambda}+\tau+i})} + \left(\frac{-2\sqrt{\lambda}iz + \tau + \frac{1}{2} - \frac{iy}{2\sqrt{\lambda}}}{z}\right)\varphi_{z\varphi^{-1}(\frac{i}{2\sqrt{\lambda}+\tau+i})} = \left(\phi z^{\frac{i}{z}} e^{\sqrt{\lambda}z}\right) \varphi_{z\varphi^{-1}(\frac{i}{2\sqrt{\lambda}+\tau+i})} \hspace{1cm} (30)
\]

from (28).

Next, by writing

\[
\varphi_{z\varphi^{-1}(\frac{i}{2\sqrt{\lambda}+\tau+i})} = w = w(z) \hspace{1cm} (31)
\]

we obtain

\[
w_i + \left[\frac{-2\sqrt{\lambda}i + \frac{\tau + \frac{1}{2} - \frac{iy}{2\sqrt{\lambda}}}{z}}{w} \right]w = \frac{1}{z} \left(\phi z^{\frac{i}{z}} e^{\sqrt{\lambda}z}\right) \varphi_{z\varphi^{-1}(\frac{i}{2\sqrt{\lambda}+\tau+i})} \hspace{1cm} (32)
\]

from (30). A solution to this differential equation is given by

\[
w = \left[\left(\phi z^{\frac{i}{z}} e^{\sqrt{\lambda}z}\right) \varphi_{z\varphi^{-1}(\frac{i}{2\sqrt{\lambda}+\tau+i})} e^{2i\sqrt{\lambda}z} \right] z^{-\frac{i}{z}} \varphi_{z\varphi^{-1}(\frac{i}{2\sqrt{\lambda}+\tau+i})} \hspace{1cm} (33)
\]

Making use of the reverse process to obtain \( \Phi' \), we finally obtain the solution (14) from (21), (26), (31) and (33).

\textit{ii}-) When \( \nu = \sqrt{\lambda i} \), we have

\[
\psi = e^{\sqrt{\lambda}z} \varphi, \quad \varphi = \varphi(z) \hspace{1cm} (34)
\]

and

\[
z\varphi_2 + \left(2\sqrt{\lambda}iz + 2\tau + 1\right)\varphi_1 + \left[i\sqrt{\lambda}(2\tau + 1) + \gamma\right] \varphi = \phi z^{\frac{i}{z}} e^{-\sqrt{\lambda}z} \hspace{1cm} (35)
\]

from (23) and (24), respectively.

Applying the discrete operator \( \nabla^\alpha \) to both members of (35), we have

\[
z\varphi_{2\varphi^\alpha} + \left(2\sqrt{\lambda}iz + 2\tau + 1 + \alpha Q\right)\varphi_{1\varphi^\alpha} + \left[\gamma + i\sqrt{\lambda}(2\tau + 1 + 2\alpha Q)\right] \varphi_{\alpha} = \left(\phi z^{\frac{i}{z}} e^{-\sqrt{\lambda}z}\right)_{\varphi^\alpha} \hspace{1cm} (36)
\]
Choosing $\alpha$ such that
\[ \alpha = -Q^{-1} \left( -\frac{i\gamma}{2\sqrt{\lambda}} + \tau + \frac{1}{2} \right) \equiv \beta \]  
(37)
and replacing
\[ \varphi_{1-\varphi^{-1}(\frac{-\sqrt{\lambda}-\tau+\frac{1}{2}}{z})} = \vartheta = \vartheta(z) \]  
(38)
we have
\[ \vartheta + \left[ 2\sqrt{\lambda} + \tau + \frac{1}{2} + \frac{i\gamma}{z} \right] \vartheta = \frac{1}{z} \left( \phi z^{-\tau} e^{-i\sqrt{\lambda}z} \right)_{-\varphi^{-1}(\frac{-\sqrt{\lambda}-\tau+\frac{1}{2}}{z})} \]  
(39)
from (36). A solution to this differential equation is given by
\[ \vartheta = \left[ \phi z^{-\tau} e^{-i\sqrt{\lambda}z} e^{i\sqrt{\lambda}z^{-\frac{\tau}{2}+\frac{1}{2}} \frac{z}{z^2}} \right]_{-1} e^{-i\sqrt{\lambda}z^{-\frac{\tau}{2}+\frac{1}{2}}}. \]  
(40)
Therefore, we have (15) from (21), (34), (38) and (40).

**Remark 1.** In the same way as the procedure in subsections, we use $\sigma = -\tau + (1/2)$ and replacing $\tau$ by $-\tau$, then we have other solutions (16) and (17) different from (14) and (15), respectively, if $\tau \neq 0$.

**Dfs of the homogeneous HAE**

**Theorem 7.** Let $\Phi \in \left\{ \Phi \, : \, 0 \neq \Phi_{\alpha} \left| q ; \alpha \in \square \right. \right\}$. Then the homogeneous HAE;
\[ \Phi_z + \left[ \lambda + \frac{\gamma}{z} + \frac{1}{2} \frac{\tau}{z^2} \right] \Phi = 0, \]  
(41)
has particular solutions of the forms:
\[ \Phi^{(1)} = k z^{-\frac{\tau}{2}+\frac{1}{2}} e^{-i\sqrt{\lambda}z} \left( e^{i\sqrt{\lambda}z^{-\frac{\tau}{2}+\frac{1}{2}} \frac{z}{z^2}} \right)_{-1} \varphi^{-1}(\frac{-\sqrt{\lambda}-\tau+\frac{1}{2}}{z}), \]  
(42)
\[ \Phi^{(2)} = k z^{-\frac{\tau}{2}+\frac{1}{2}} e^{-i\sqrt{\lambda}z} \left( e^{i\sqrt{\lambda}z^{-\frac{\tau}{2}+\frac{1}{2}} \frac{z}{z^2}} \right)_{-1} \varphi^{-1}(\frac{\sqrt{\lambda}+\tau-\frac{1}{2}}{z}), \]  
(43)
\[ \Phi^{(3)} = k z^{-\frac{\tau}{2}+\frac{1}{2}} e^{-i\sqrt{\lambda}z} \left( e^{i\sqrt{\lambda}z^{-\frac{\tau}{2}+\frac{1}{2}} \frac{z}{z^2}} \right)_{-1} \varphi^{-1}(\frac{-\sqrt{\lambda}+\tau-\frac{1}{2}}{z}), \]  
(44)
\[ \Phi^{(4)} = k z^{-\frac{\tau}{2}+\frac{1}{2}} e^{-i\sqrt{\lambda}z} \left( e^{i\sqrt{\lambda}z^{-\frac{\tau}{2}+\frac{1}{2}} \frac{z}{z^2}} \right)_{-1} \varphi^{-1}(\frac{-\sqrt{\lambda}+\tau+\frac{1}{2}}{z}). \]  
(45)

**Proof.** Taking $\phi = 0$ in Theorem 6, we have
\[ w_i + \left[ -2\sqrt{\lambda}i + \frac{\tau + \frac{1}{2} - \frac{i\gamma}{2\sqrt{\lambda}}}{z} \right] w = 0 \]  \hspace{1cm} (46)

and

\[ \varrho_i + \left[ 2\sqrt{\lambda}i + \frac{\tau + \frac{1}{2} + \frac{i\gamma}{2\sqrt{\lambda}}}{z} \right] \varrho = 0 \]  \hspace{1cm} (47)

for \( \nu = -i\sqrt{\lambda} \) and \( \nu = i\sqrt{\lambda} \), instead of (32) and (39), respectively. Therefore, we obtain (42) for (46) and (43) for (47).

Remark 2. In the same way, we use \( \sigma = -\tau + \frac{1}{2} \) and replacing \( \tau \) by \( -\tau \), then we have other solutions (44) and (45) different from (42) and (43), if \( \tau \neq 0 \).

Example. In the case \( \gamma = 0, \ \tau = -1/2 \) and \( \phi(z) = z \), we have

\[ \Phi_2 + \lambda \Phi = z \]  \hspace{1cm} (48)

from (13) the solution of equation (48) is obtained as

\[ \Phi(z) = e^{-\sqrt{\lambda}z} \left\{ \left( zze^{\sqrt{\lambda}z} \right) e^{-2\sqrt{\lambda}z} z^{-1} \right\} \]  \hspace{1cm} (49)

by using (14). The function obtained in (49) provides the equation (48).

We plotted two-dimensional graphs of Eq. (49), as shown in Fig. 1.
Fig. 1. 2D-dimension solutions of Eq. (49), using $h_1 = 0.2$ and $h_2 = 0.3$.

Conclusions

In this article, we first considered the HAE obtained from the radial part of the reduced equation by applying the method of separating variables in the spherical coordinates of the Schrödinger equation. We use the nabla discrete fractional operator for hydrogen atom type equations. We consider homogeneous and nonhomogeneous HAE. We have obtained many different dfs for these equations. In previous time, no one achieved these solutions of HAEs. We will obtain particular solutions of the same type singular ordinary and partial differential equations by using the discrete fractional nabla operator in future works.

References


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