NEW MATHEMATICAL MODELS IN ANOMALOUS VISCOELASTICITY FROM THE DERIVATIVE WITH RESPECT TO ANOTHER FUNCTION VIEW POINT

by

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In this article, we address the mathematical models in anomalous viscoelasticity containing the derivatives with respect to another function for the first time. The Newton-like, Maxwell-like, Kelvin-Voigt-like, Burgers-like, and Zener-like models via the new derivatives with respect to another functions are discussed in detail. The results for the calculus with respect to another function are as a new perspective proposed to present the better accuracy and efficiency in the descriptions of the complex behaviors of the materials.

Key words: *viscoelasticity, derivative with respect to another function, integral with respect to another function, calculus with respect to another function*

Introduction

The Newton-Leibniz calculus, see [1], have the important applications in viscoelasticity [2]. The Newtonian dashpot element, proposed in 1701 by Newton, is given [3]:

$$\sigma(\tau) = \gamma \mathrm{D}^{(1)} \varepsilon(\tau)$$

where γ is the viscosity of the material, $\mathrm{D}^{(1)}$ – the Newton-Leibniz derivative, see the *Calculus with respect to another function*, $\varepsilon(t)$ – the strain, $\sigma(t)$ – the stress, and τ – the time. The constitutive equation for the Maxwell model can be written [4]:

$$\mathrm{D}^{(1)} \varepsilon(\tau) = \frac{\sigma(\tau)}{\gamma} + \frac{\mathrm{D}^{(1)} \sigma(\tau)}{\zeta}$$

where γ is the viscosity of the material and ζ is the elastic modulus of the material. The constitutive equation for the Kelvin-Voigt model can be given [5, 6]:

$$\sigma(\tau) = \gamma \varepsilon(\tau) + \zeta \mathrm{D}^{(1)} \varepsilon(\tau)$$

where γ is the viscosity of the material and ζ is the elastic modulus of the material. The constitutive equation for the Burgers model can be reported as [7]:

$$\left(1 + a\mathrm{D}^{(1)} + b\mathrm{D}^{(2)}\right)\sigma(\tau) = \left(c\mathrm{D}^{(1)} + d\mathrm{D}^{(2)}\right)\varepsilon(\tau)$$

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where a, b, c, and d are the material constants. The constitutive equation for the Zener model can be given [8]:

$$\left(1+a\mathrm{D}^{(1)}\right)\sigma(\tau)=\left(b+c\mathrm{D}^{(1)}\right)\varepsilon(\tau)$$

where a, b, and c are the material constants.

Recently, as an extended version of the Newton-Leibniz calculus, the calculus with respect to another function was reported [9]. The main goal of the paper is to structure the mathematical models in anomalous viscoelasticity containing the derivatives with respect to another function.

Calculus with respect to another function

The Newton-Leibniz calculus

The Newton-Leibniz derivative is defined [1]:

$$\mathrm{D}^{(1)}\Lambda(t)=\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t}=\Lambda^{(1)}(t)=\lim_{\Delta t\to 0}\frac{\Theta(t+\Delta t)-\Theta(t)}{\Delta t} \tag{1}$$

The Newton-Leibniz integral is defined [1]:

$${}_aI_t^{(1)}\mathrm{M}(t)=\int_a^t \mathrm{M}(t)\mathrm{d}t \tag{2}$$

The relations between them are given [1]:

$$\Lambda(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \Lambda(t)\mathrm{d}t \tag{3}$$

and

$$\Lambda(t)=\int_0^\tau\left[\mathrm{D}^{(1)}\Lambda(t)\right]\mathrm{d}t+\Lambda(0) \tag{4}$$

Calculus with respect to another function

Let $g^{(1)}(t) > 0$, $-\infty \le a < t < b \le +\infty$. The derivatives and integrals with respect to another function are presented.

The derivative with respect to another function is defined [9]:

$$\mathrm{D}_g^{(1)}\Lambda(t)=\left[\frac{1}{\dfrac{\mathrm{d}g(t)}{\mathrm{d}t}}\right]\left[\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t}\right]=\left[\frac{\mathrm{d}t}{\mathrm{d}g(t)}\right]\left[\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t}\right]=\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}g(t)}=\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t}\mathrm{d}\Lambda \tag{5}$$

The integral with respect to another function is defined [9]:

$${}_aI_{t,g}^{(1)}M(\tau)=\int_a^t M(t)g^{(1)}(t)\mathrm{d}t \tag{6}$$

For $\alpha = 0$ and $\alpha = -\infty$, eq. (6) can be re-written:

$${}_0I_{t,g}^{(1)}M(\tau)=\int_{-\infty}^t M(t)g^{(1)}(t)\mathrm{d}t \tag{7}$$

and

$${}_{-\infty}I_{t,g}^{(1)}M(\tau)=\int_{-\infty}^t M(t)g^{(1)}(t)\mathrm{d}t \tag{8}$$

respectively.

The derivative of higher order with respect to another function is defined [9]:

$$\mathrm{D}_g^{(n)}\Lambda(t)=\left(\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\Lambda(t) \tag{9}$$

Their relationships bettwen eqs. (5) and (6) can be given [9]:

$$\Lambda(t)=\left[\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_a^t\Lambda(t)g^{(1)}(t)\mathrm{d}t=\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t\Lambda(t)g^{(1)}(t)\mathrm{d}t \tag{10}$$

and

$$\Lambda(t)=\int_a^\tau\left\{\left[\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\Lambda(t)\right\}g^{(1)}(t)\mathrm{d}\tau+\Lambda(a)=\int_a^t\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t)\mathrm{d}\tau+\Lambda(a) \tag{11}$$

Thus, we may get the following formula:

$$\Lambda(t)-\Lambda(a)=\int_a^\tau\left\{\left[\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\Lambda(t)\right\}g^{(1)}(t)\mathrm{d}\tau \tag{12}$$

Taking $g(t) = -t^{-\alpha}$, where $0 < \alpha$, eqs. (5) and (6) can be written:

$$\mathrm{D}_{-t^{-\alpha}}^{(1)}\Lambda(t)=\frac{t^{\alpha+1}}{a}\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} \tag{13}$$

and

$${}_aI_{t,-t^{-\alpha}}^{(1)}M(\tau)=a\int_a^t M(t)\frac{\mathrm{d}t}{t^{\alpha+1}} \tag{14}$$

respectively.

For $g(t) = \ln(t-1)$, we may present:

$$\mathrm{D}_{\ln(t-1)}^{(1)}\Lambda(t)=(t-1)\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} \tag{15}$$

$${}_aI_{t,\ln(t-1)}^{(1)}M(\tau)=\int_a^t\frac{M(t)}{t-1}\mathrm{d}t \tag{16}$$

Similarly, for $g(t) = \mathrm{e}^{\lambda t}-1$ with $\lambda\in\mathbb{R}_+$, we may give:

$$\mathrm{D}_{e^{\lambda t}-1}^{(1)}\Lambda(t)=\lambda\mathrm{e}^{-\lambda t}\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} \tag{17}$$

and

$${}_aI_{t,e^{\lambda t}-1}^{(1)}M(\tau)=\lambda\int_a^t M(t)\mathrm{e}^{\lambda t}\mathrm{d}t \tag{18}$$

When $\alpha = 0$ we may get:

$$\Lambda(t)=\left[\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_0^t\Lambda(t)g^{(1)}(t)\mathrm{d}t=\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\Lambda(t)g^{(1)}(t)\mathrm{d}t \tag{19}$$

$$\Lambda(t)-\Lambda(0)=\int_0^\tau\left\{\left[\frac{1}{g^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\Lambda(t)\right\}g^{(1)}(t)\mathrm{d}\tau \tag{20}$$

$$ {}_{0}I^{(1)}_{t,-t^{-\alpha}}M(\tau)=a\int_{0}^{t}M(t)\frac{\mathrm{d}t}{t^{\alpha+1}} \tag{21}$$

$$ {}_{0}I^{(1)}_{t,\ln(t-1)}M(\tau)=\int_{0}^{t}\frac{M(t)}{t-1}\mathrm{d}t \tag{22}$$

and

$$ {}_{0}I^{(1)}_{t,g}M(\tau)=\lambda\int_{0}^{t}M(t)\mathrm{e}^{\lambda t}\mathrm{d}t \tag{23}$$

When $\alpha = 0$, we may get:

$$ {}_{-\infty}I^{(1)}_{t,-t^{-\alpha}}M(\tau)=a\int_{-\infty}^{t}M(t)\frac{\mathrm{d}t}{t^{\alpha+1}} \tag{24}$$

$$ {}_{-\infty}I^{(1)}_{t,\ln(t-1)}M(\tau)=\int_{-\infty}^{t}\frac{M(t)}{t-1}\mathrm{d}t \tag{25}$$

and

$$ {}_{-\infty}I^{(1)}_{t,e^{\lambda t}-1}M(\tau)=\lambda\int_{-\infty}^{t}M(t)\mathrm{e}^{\lambda t}\mathrm{d}t \tag{26}$$

For $g(t) = 1 - \mathrm{e}^{-\lambda t}$ with $\lambda \in \mathbb{R}$, we may give:

$$ \mathrm{D}^{(1)}_{1-e^{-\lambda t}}\Lambda(t)=\lambda e^{\lambda t}\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} \tag{27}$$

$$ {}_{a}I^{(1)}_{t,1-e^{-\lambda t}}M(\tau)=\lambda\int_{a}^{t}M(t)\mathrm{e}^{-\lambda t}\mathrm{d}t \tag{28}$$

$$ {}_{0}I^{(1)}_{t,1-e^{-\lambda t}}M(\tau)=\lambda\int_{0}^{t}M(t)\mathrm{e}^{-\lambda t}\mathrm{d}t \tag{29}$$

and

$$ {}_{-\infty}I^{(1)}_{t,1-e^{-\lambda t}}M(\tau)=\lambda\int_{-\infty}^{t}M(t)\mathrm{e}^{-\lambda t}\mathrm{d}t \tag{30}$$

For more details of the general calculus with respect to another function and related tasks, see [9-11].

Mathematical models in anomalous viscoelasticity

The Newton-like element containing the new derivative with respect to another function

The Newton-like element containing the new derivative with respect to another function is given:

$$ \sigma(t)=\frac{\gamma}{g^{(1)}(t)}\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}=\gamma\mathrm{D}^{(1)}_{g}\varepsilon(t) \tag{31}$$

where γ is the viscosity of the material.

The Maxwell-like model containing the new derivative with respect to another function

The constitutive equation for the Maxwell-like model containing the new derivative with respect to another function can be given:

$$\mathbb{D}_g^{(1)}\varepsilon(t)=\frac{\sigma(t)}{\gamma}+\frac{\mathrm{D}_g^{(1)}\sigma(t)}{\zeta} \tag{32}$$

where γ is the viscosity of the material and ζ is the elastic modulus of the material.

The Kelvin-Voigt-like model containing the new derivative with respect to another function

The constitutive equation for the Kelvin-Voigt-like model containing the new derivative with respect to another function can be presented:

$$\sigma(t)=\gamma\varepsilon(t)+\zeta\mathrm{D}_g^{(1)}\varepsilon(t) \tag{33}$$

where γ is the viscosity of the material and ζ is the elastic modulus of the material.

The Burgers-like model containing the new derivative with respect to another function

The constitutive equation for the Burgers-like model containing the derivative with respect to another function can be represented in the form:

$$\sigma(t)+a\mathrm{D}_g^{(1)}\sigma(t)+b\mathrm{D}_g^{(2)}\sigma(t)=c\mathrm{D}_g^{(1)}\varepsilon(t)+d\mathrm{D}_g^{(2)}\varepsilon(t) \tag{34}$$

where a, b, c, and d are the material constants.

The Zener-like model containing the new derivative with respect to another function

The constitutive equation for the Zener-like model containing the new derivative with respect to another function can be presented:

$$\sigma(t)+a\mathrm{D}_g^{(1)}\sigma(t)=b\varepsilon(t)+c\mathrm{D}_g^{(1)}\varepsilon(t) \tag{35}$$

where a, b, and c are the material constants.

Applications in the complex materials

In this section, we consider the dashpot, Maxwell-like, Kelvin-Voigt-like, Burgers-like, and Zener-like models involving the scale behaviors of the complex materials with the negative power law function, given as $h(\tau) = -\tau^{-\alpha}$.

The anomalous dashpot element containing the new derivative with respect to another function can be written:

$$\sigma(t)=\gamma\mathrm{D}_{-t^{-\alpha}}^{(1)}\varepsilon(t) \tag{36}$$

where γ is the viscosity of the material.

The constitutive equation for the anomalous Maxwell-like model containing the new derivative with respect to another function can be suggested:

$$\mathrm{D}_{-t^{-\alpha}}^{(1)}\varepsilon(t)=\frac{\sigma(t)}{\gamma}+\frac{1}{\zeta}\mathrm{D}_{-t^{-\alpha}}^{(1)}\sigma(t) \tag{37}$$

where γ is the viscosity of the material and ζ is the elastic modulus of the material.

The constitutive equation for the anomalous Kelvin-Voigt-like model containing the new derivative with respect to another function can be given:

$$\sigma(t) = \gamma\varepsilon(t) + \zeta D^{(1)}_{-t^{-\alpha}}\sigma(t) \tag{38}$$

where γ is the viscosity of the material and ζ is the elastic modulus of the material.

The constitutive equation for the anomalous Burgers-like model containing the derivative with respect to another function can be represented in the form:

$$\sigma(t) + aD^{(1)}_{-t^{-\alpha}}\sigma(t) + bD^{(2)}_{-t^{-\alpha}}\sigma(t) = cD^{(1)}_{-t^{-\alpha}}\varepsilon(t) + dD^{(2)}_{-t^{-\alpha}}\varepsilon(t) \tag{39}$$

where a, b, c, and d are the material constants.

The constitutive equation for the anomalous Zener-like model containing the new derivative with respect to another function can be presented as:

$$\sigma(t) + aD^{(1)}_{-t^{-\alpha}}\sigma(t) = b\varepsilon(t) + cD^{(1)}_{-t^{-\alpha}}\varepsilon(t) \tag{40}$$

where a, b, and c are the material constants.

For more tasks for the classical models in viscoelasticity, see [9-17].

Conclusion

In this present work, we investigated the calculus with respect to another function. We proposed the Newton-like, Maxwell-like, Kelvin-Voigt-like, Burgers-like, and Zener-like models via the new derivatives with respect to another functions. Moreover, we present the dashpot, Maxwell-like, Kelvin-Voigt-like, Burgers-like, and Zener-like models involving the scale behaviors of the complex materials with the negative power law function, given as $h(\tau) = -\tau^{-\alpha}$. The results are proposed to give the better accuracy and efficiency in the descriptions of the complex behaviors of the materials.

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Nomenclature

t – time, [s]

Greek symbols

γ – viscosity of the material, [Pa·s]

$\varepsilon(t)$ – strain, [–]
$\sigma(t)$ – stress, [Pa]

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