ON SOLUTIONS OF LOCAL FRACTIONAL SCHRODINGER EQUATION

by

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In this study, we obtain the solution of a local fractional Schrodinger equation. The solution is obtained by the implementation of the Laplace transform and Fourier transform in closed form in terms of the Mittag-Leffler function.

Key words*: local fractional calculus, Laplace transforms, Fourier transform, Schrodinger equations*

Introduction

Fractional analysis has much applications in diverse fields of science and engineering; such as, statistics, control theory, optics, quantum mechanics, traffic flow*, etc.*

Fractional analysis arises in much problems of physics [1, 2], continuum mechanics [3], visco-elasticity [4, 5], quantum mechanics [6-8], and other branches of applied mathematics [9-13]. However, these spherical defined fractional derivatives do not usually project the local geometric behaviours for a given function. Attempts have been made recently [14-19] to define a local version of the fractional derivative and integral. Also, local fractional (LF) analysis was extended to LF Fourier series [20, 21], Yang-Fourier transform (FT) [21-25], Yang-Laplace transform (LT) [20, 25, 26], discrete Yang-FT [23, 27]. The LF diffusion equation with local fractional PDE was researched in [28, 29]. The LF wave equation and non-linear fractional wave-like equations was investigated in [30, 31].

The local fractional Schrodinger equation (LFSE) in (1+1)-fractal dimensional space:

$$i^{\varepsilon}\hbar_{\varepsilon}\frac{\partial^{\varepsilon}u(x,\tau)}{\partial\tau^{\varepsilon}}=-\frac{\hbar_{\varepsilon}^{2}}{2\mu}\frac{\partial^{2\varepsilon}u(x,\tau)}{\partial x^{2\varepsilon}} \tag{1}$$

where the wave function $u(x, \tau)$ is local fractional continuous (LFC) function, μ and h_{ε} are constants.

In this article our aim is to investigate the non-differentiable solutions for LFSE by using the local fractional LT method.

Preliminary and properties

In this portion, we introduce some mathematical preliminaries calculus theory in fractal space for our next progress [21, 22, 32].

Let $\psi : \chi \to \varsigma$ be a function defined on a fractal set χ of fractal dimension $\sigma(0 < \sigma \leq 1)$. A real valued function $\psi(r)$ defined on the fractal set χ is given:

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$$\psi(r) = r^{\sigma}, \quad r^{\sigma} \in \chi \tag{2}$$

The Mittag-Lefler function (MLF) defined on the fractal set χ is given:

$$E_{\sigma}\left(r^{\sigma}\right) = \sum_{n=0}^{\infty} \frac{r^{n\sigma}}{\Gamma(1+n\sigma)}, \quad r \in \mathbb{R}, \; 0 < \sigma \le 1 \tag{3}$$

Definition 1. If there exists:

$$\left|\psi(r) - \psi(r_0)\right| < \varepsilon^{\sigma} \tag{4}$$

with $|r - r_0| < \delta$, for $\varepsilon, \delta \in \mathbb{R}^{+}$, then $\psi(r)$ is called LFC at $r = r_0$ and it is denoted:

$$\lim_{r \to r_0} \psi(r) = \psi(r_0) \tag{5}$$

Suppose a function $\psi(r)$ is LFC on the interval (a, b) then:

$$\psi(r) \in C_{\sigma}(a,b) \tag{6}$$

Definition 2. Let $\psi(r) \in C_{\sigma}(a, b)$. The LF derivative of $\psi(r)$ of order σ at $r = r_0$ is given:

$$\psi^{(\sigma)}(r_0) = \left.\frac{d^{\sigma}}{dr^{\sigma}} \psi(r)\right|_{r=r_0} = \lim_{r \to r_0} \frac{\Delta^{\sigma}\left[\psi(r) - \psi(r_0)\right]}{\left(r - r_0\right)^{\sigma}} \tag{7}$$

where

$$\Delta^{\sigma}\left[\psi(r) - \psi(r_0)\right] \cong \Gamma(1+\sigma)\left[\psi(r) - \psi(r_0)\right]$$

with the Gamma function:

$$\Gamma(1+\sigma) = \int_{0}^{\infty} r^{\sigma-1} \exp(-r)\,\mathrm{d}r \tag{8}$$

The LF partial derivative operator of $\psi(r, t)$ of order $\sigma(0 < \sigma \le 1)$ with respect to t in the domain χ:

$$\psi^{(\sigma)}(r,t_0) = \left.\frac{\partial^{\sigma}}{\partial t^{\sigma}} \psi(r,t)\right|_{t=t_0} = \lim_{t \to t_0} \frac{\Delta^{\sigma}\left[\psi(r,t) - \psi(r,t_0)\right]}{\left(t - t_0\right)^{\sigma}} \tag{9}$$

where

$$\Delta^{\sigma}\left[\psi(r,t) - \psi(r,t_0)\right] \cong \Gamma(1+\sigma)\left[\psi(r,t) - \psi(r,t_0)\right]$$

The LF derivative of k^{th} order:

$$\psi^{(k\sigma)}(r) = \frac{d^{\sigma}}{dr^{\sigma}} \cdots \frac{d^{\sigma}}{dr^{\sigma}} \psi(r) \tag{10}$$

and LF partial derivative of k^{th} order:

$$\psi_{r}^{(k\sigma)}(r,t) = \frac{\partial^{\sigma}}{\partial r^{\sigma}} \cdots \frac{\partial^{\sigma}}{\partial r^{\sigma}} \psi(r,t) \tag{11}$$

Definition 3. Let $\psi(r) \in C_{\sigma}(a, b)$. The LF integral of $\psi(r)$ of order $\sigma(0 < \sigma \le 1)$:

$${}_{a}I_{b}^{(\sigma)} \psi(r) = \frac{1}{\Gamma(1+\sigma)} \int_{a}^{b} \psi(r)(dr)^{\sigma} = \frac{1}{\Gamma(1+\sigma)} \lim_{\Delta r_k \to 0} \sum_{k=0}^{N-1} \psi(r_k)\left(\Delta r_k\right)^{\sigma} \tag{12}$$

where $\Delta r_k = r_{k+1} - r_k$ with $r_0 = a < r_1 < ... < r_{N-1} < r_N = b$.

Definition 4. Setting $\psi \in L_{1,\sigma}[\mathbb{R}]$ and $\|\psi\|_{1,\sigma} < \infty$ the local fractional LT operator of $\psi(r)$:

$$\mathscr{L}_\sigma\left[\psi(r)\right] = \Psi(s) = \frac{1}{\Gamma(1+\sigma)}\int_0^\infty \psi(r)E_\sigma\left(-r^\sigma s^\sigma\right)(\mathrm{d}r)^\sigma, \quad 0 < \sigma \le 1 \tag{13}$$

The inverse equation of the LT operator of $\Psi(s)$: $\mathscr{L}$

$$\mathscr{L}_\sigma^{-1}\left[\Psi(s)\right] = \psi(r) = \frac{1}{(2\pi)^\sigma}\int_{\alpha-i\infty}^{\alpha+i\infty} \Psi(s)E_\sigma\left(r^\sigma s^\sigma\right)(\mathrm{d}s)^\sigma \tag{14}$$

where $s^\sigma = \alpha^\sigma + i^\sigma\infty^\sigma$ and $\mathrm{Re}(s^\sigma) = \alpha^\sigma$.

Definition 5. By setting $\psi \in L_{1,\sigma}[\mathbb{R}]$ and $\|\psi\|_{1,\sigma} < \infty$, the local fractional FT operator of $\psi(r)$:

$$\mathscr{F}_\sigma\left[\psi(r)\right] = \Psi(w) = \frac{1}{\Gamma(1+\sigma)}\int_{-\infty}^\infty \psi(r)E_\sigma\left(-i^\sigma r^\sigma w^\sigma\right)(\mathrm{d}r)^\sigma \tag{15}$$

The inverse local fractional FT operator of $\Psi(w)$:

$$\mathscr{F}_\sigma^{-1}\left[\Psi(w)\right] = \psi(r) = \frac{1}{(2\pi)^\sigma}\int_{-\infty}^\infty \Psi(w)E_\sigma\left(i^\sigma r^\sigma w^\sigma\right)(\mathrm{d}w)^\sigma \tag{16}$$

Theorem 6 (LT). Suppose that $\psi(r) \in L_{1,\sigma}[\mathbb{R}]_+$, $\mathscr{L}_\sigma[\psi(r)] = \Psi(s)$, and $\lim_{r\to\infty}\psi(r) = 0$, then there is:

$$\mathscr{L}_\sigma\left[\psi^{(\sigma)}(r)\right] = s^\sigma\mathscr{L}\left[\psi(r)\right] - \psi(0) \tag{17}$$

We remark that there is:

$$\mathscr{L}_\sigma\left[\psi^{(n\sigma)}(r)\right] = s^{n\sigma}\mathscr{L}\left[\psi(r)\right] - s^{(n-1)\sigma}\psi(0) - s^{(n-2)\sigma}\psi^{(\sigma)}(0) - \ldots - \psi^{[(n-1)\sigma]}(0) \tag{18}$$

where $n \in N$.

Theorem 7 (FT). Suppose that $\psi(r)$, $\Psi(w) \in L_{1,\sigma}[\mathbb{R}]$, $\mathscr{F}_\sigma[\psi(r)] = \Psi(s)$, and $\lim_{|r|\to\infty} \psi(r) = 0$. Then:

$$\mathscr{F}_\sigma\left[\psi^{(\sigma)}(r)\right] = (iw)^\sigma\,\mathscr{F}\left[\psi(r)\right] \tag{19}$$

We use the slightly modified form of the previous results in our research [33, 34]:

$$\mathscr{F}_\sigma\left[\psi^{(\sigma)}(r)\right] = -|w|^\sigma\,\mathscr{F}\left[\psi(r)\right] \tag{20}$$

instead of eq. (19).

Main results

In this chapter, we will research the solution of the LFSE (1).

Theorem 7. Consider the following 1-D LFSE:

$$\frac{\partial^\varepsilon u(x,\tau)}{\partial\tau^\varepsilon} = -\frac{\hbar_\varepsilon}{2i^\varepsilon\mu}\frac{\partial^{2\varepsilon}u(x,\tau)}{\partial x^{2\varepsilon}}, \quad x \in \mathbb{R}, \ \tau > 0, \ 0 < \varepsilon \le 1 \tag{21}$$

with the initial conditions:

$$\begin{gathered} u(x,0) = u_0(x), \quad x \in \mathbb{R} \\ u(x,\tau) \to 0 \text{ as } |x| \to \infty, \quad \tau > 0 \end{gathered} \tag{22}$$

where $\hbar$ is Planck constant divided by 2π, μ – the mass, and $u(x, \tau)$ – the wave function of the particle. Then, for the solution (21), under the given condition in eq. (22):

$$u(x,\tau)=\frac{1}{(2\pi)^{\varepsilon}}\int_{-\infty}^{\infty}u_0^*(w)E_{\varepsilon,\varepsilon}\left(-\kappa|w|^{2\varepsilon}\tau^{\varepsilon}\right)(\mathrm{d}w)^{\varepsilon} \tag{23}$$

where E is the MLF and $\kappa = -\hbar_\varepsilon/2i^\varepsilon\mu$.

Proof. Applying the LT according to the time variable τ on both the sides of (21) and using the initial conditions in eq. (22) and the eq. (17):

$$s^{\varepsilon}\bar{u}(x,s)-u_0(x)=\kappa\frac{\partial^{2\varepsilon}}{\partial x^{2\varepsilon}}\bar{u}(x,s) \tag{24}$$

where $\kappa = -\hbar_\varepsilon/2i^\varepsilon\mu$.

If we apply the FT according to space variable and use the eq. (20):

$$s^{\varepsilon}\bar{u}^*(w,s)-u_0^*(w)=-\kappa|w|^{2\varepsilon}\bar{u}^*(w,s) \tag{25}$$

Solving for $\bar{u}^*(w, s)$:

$$\bar{u}^*(w,s)=\frac{u_0^*(w)}{s^{\varepsilon}+\kappa|w|^{2\varepsilon}} \tag{26}$$

To recover the function $u(x, \tau)$ from (26) it is convenient first to invert the LT and then the FT:

$$\mathscr{L}^{-1}\left[\frac{s^{\varepsilon-1}}{\gamma+s^{\sigma}};\tau\right]=\tau^{\sigma-\varepsilon}E_{\sigma,\sigma-\varepsilon+1}\left(-\gamma\tau^{\sigma}\right) \tag{27}$$

where $E_{\varepsilon,\sigma}(r)$ is the MLF [32]:

$$E_{\varepsilon,\sigma}(r)=\sum_{n=0}^{\infty}\frac{r^n}{\Gamma(n\varepsilon+\sigma)},\quad \left(\varepsilon,\sigma\in \mathrm{C},\ \mathrm{Re}(\varepsilon)>0,\ \mathrm{Re}(\sigma)>0\right) \tag{28}$$

we obtain:

$$u^*(w,\tau)=u_0^*(w)E_{\varepsilon,\varepsilon}\left(-\kappa|w|^{2\varepsilon}\tau^{\varepsilon}\right) \tag{29}$$

Now the inverse local FT leads to:

$$u(x,\tau)=\mathscr{F}^{-1}\left[u_0^*(w)E_{\varepsilon,\varepsilon}\left(-\kappa|w|^{2\varepsilon}\tau^{\varepsilon}\right);x\right]=\frac{1}{(2\pi)^{\varepsilon}}\int_{-\infty}^{\infty}u_0^*(w)E_{\varepsilon,\varepsilon}\left(-\kappa|w|^{2\varepsilon}\tau^{\varepsilon}\right)(dw)^{\varepsilon} \tag{30}$$

Corollary. Consider the following LFSE:

$$\frac{\partial u(x,\tau)}{\partial\tau}=-\frac{\hbar}{2i\mu}\frac{\partial^2 u(x,\tau)}{\partial x^2},\quad x\in\mathbb{R},\ \tau>0 \tag{31}$$

with the following initial conditions:

$$\begin{gathered}u(x,0)=u_0(x),\quad x\in\mathbb{R}\\ u(x,\tau)\to 0 \text{ as } |x|\to\infty,\quad \tau>0\end{gathered} \tag{32}$$

Then the solution of (31), under previous conditions eq. (32):

$$u(x,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u_0^*(w)E_{1,1}\left(-\kappa|w|^2\tau\right)\mathrm{d}w \tag{33}$$

where $\kappa = -\hbar_\varepsilon/2i^\varepsilon\mu$.

Conclusions

In this work, we have introduced a LFSE and established solution for the same. The solution has been advanced in terms of the generalized MLF in a compact form with the help of Laplace and FTs and their inverses. We will obtain solutions of the same type differential equations by using the LF operator in future works.

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