ON SOLUTIONS OF LOCAL FRACTIONAL SCHRÖDINGER EQUATION

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In this study, we obtain the solution of a local fractional Schrödinger equation (LFSE). The solution is obtained by the implementation of the Laplace transform (LT) and Fourier transform (FT) in closed form in terms of the Mittag-Leffler function (MLF).

Key words: Local fractional calculus; Laplace transforms; Fourier transform; Schrödinger equations.

Introduction

Fractional analysis has much applications in diverse fields of science and engineering; such as, statistics, control theory, optics, quantum mechanics, traffic flow, etc.

Fractional analysis arises in much problems of physics [1,2], continuum mechanics [3], visco-elasticity [4,5], quantum mechanics [6-8], and other branches of applied mathematics [9-13]. However, these spherical defined fractional derivatives do not usually project the local geometric behaviours for a given function. Attempts have been made recently [14-19] to define a local version of the fractional derivative and integral. Also, local fractional (LF) analysis was extended to LF Fourier series [20,21], Yang-FT [21-25], Yang-LT [20,25,26], discrete Yang-FT [23, 27]. The LF diffusion equation with local fractional PDEs was researched in [28,29]. The LF wave equation and nonlinear fractional wave-like equations was investigated in [30,31].

The LFSE in (1+1)-fractal dimensional space takes the form

\[ i \hbar \frac{\partial^\varepsilon u(x,\tau)}{\partial \tau^\varepsilon} = \frac{\hbar^2}{2\mu} \frac{\partial^{2\varepsilon} u(x,\tau)}{\partial x^{2\varepsilon}} \]  

where the wave function \( u(x,\tau) \) is local fractional continuous (LFC) function, \( \mu \) and \( \hbar \) are constants.

In this article our aim is to investigate the nondifferentiable solutions for LFSE by using the local fractional LT method.

Preliminary and Properties

In this portion, we introduce some mathematical preliminaries calculus theory in fractal space for our next progress [21,22,32].

Let \( \psi : \chi \to \zeta \) be a function defined on a fractal set \( \chi \) of fractal dimension \( \sigma \) \((0<\sigma \leq 1)\). A real valued function \( \psi(r) \) defined on the fractal set \( \chi \) is given by the equality

\[ \psi(r) = r^\sigma, \quad r^\sigma \in \chi. \]

The MLF defined on the fractal set \( \chi \) is given by
\[ E_\sigma \left( r^\sigma \right) = \sum_{n=0}^{\infty} \frac{r^{n\sigma}}{\Gamma(1+n\sigma)}, \quad r \in \mathbb{C}, \ 0 < \sigma \leq 1. \quad (3) \]

**Definition 1.** If there exists

\[ |\psi(r) - \psi(r_0)| < \epsilon^\sigma, \quad (4) \]

with \( |r - r_0| < \delta \), for \( \epsilon, \delta \in \mathbb{R}^+ \), then \( \psi(r) \) is called LFC at \( r = r_0 \) and it is denoted by

\[ \lim_{r \to r_0} \psi(r) = \psi(r_0). \quad (5) \]

Suppose a function \( \psi(r) \) is LFC on the interval \((a, b)\), then, we write it as

\[ \psi(r) \in C_\sigma (a, b). \quad (6) \]

**Definition 2.** Let \( \psi(r) \in C_\sigma (a, b) \). The LF derivative of \( \psi(r) \) of order \( \sigma \) at \( r = r_0 \) is given by

\[ \psi^{(\sigma)}(r_0) = \frac{d^\sigma}{dr^\sigma} \psi(r) \bigg|_{r=r_0} \]

\[ = \lim_{r \to r_0} \frac{\Delta^\sigma [\psi(r) - \psi(r_0)]}{(r-r_0)^\sigma}, \quad (7) \]

where \( \Delta^\sigma [\psi(r) - \psi(r_0)] \equiv \Gamma(1+\sigma)[\psi(r) - \psi(r_0)] \) with the Gamma function

\[ \Gamma(1+\sigma) = \int_0^\infty r^{\sigma-1} \exp(-r) \, dr. \quad (8) \]

The LF partial derivative operator of \( \psi(r, t) \) of order \( \sigma (0 < \sigma \leq 1) \) with respect to \( t \) in the domain \( \chi \) is defined as follows

\[ \psi^{(\sigma)}(r, t_0) = \frac{\partial^\sigma}{\partial t^\sigma} \psi(r, t) \bigg|_{t=t_0} \]

\[ = \lim_{t \to t_0} \frac{\Delta^\sigma [\psi(r, t) - \psi(r, t_0)]}{(t-t_0)^\sigma}, \quad (9) \]

where \( \Delta^\sigma [\psi(r, t) - \psi(r, t_0)] \equiv \Gamma(1+\sigma)[\psi(r, t) - \psi(r, t_0)] \).

LF derivative of kth order is defined as follows

\[ \psi^{(k\sigma)}(r) = \frac{d^\sigma}{dr^\sigma} \cdots \frac{d^\sigma}{dr^\sigma} \psi(r), \quad (10) \]

and LF partial derivative of kth order is defined as follows

\[ \psi^{(k\sigma)}(r, t) = \frac{\partial^\sigma}{\partial t^\sigma} \cdots \frac{\partial^\sigma}{\partial t^\sigma} \psi(r, t). \quad (11) \]

**Definition 3.** Let \( \psi(r) \in C_\sigma (a, b) \). The LF integral of \( \psi(r) \) of order \( \sigma (0 < \sigma \leq 1) \) at is given by
\[ sI_k^{(\sigma)}(r) = \frac{1}{\Gamma(1+\sigma)} \int_0^b \psi(r)(dr)^\sigma \]
\[ = \frac{1}{\Gamma(1+\sigma)} \lim_{\nu \to a} \sum_{k=0}^{N-1} \psi(r_k)(\Delta r_k)^\sigma \] \hfill (12)

where \( \Delta r_k = r_{k+1} - r_k \) with \( r_0 = a < r_1 < \ldots < r_{N-1} < r_N = b \).

**Definition 4.** Setting \( \psi \in L_{1,\sigma} \) and \( \|\psi\|_{\sigma} < \infty \), the local fractional LT operator of \( \psi(r) \) is defined as

\[ \mathcal{L}_{\sigma}[\psi(r)] = \Psi(s) = \frac{1}{\Gamma(1+\sigma)} \int_0^\infty \psi(r)E_\sigma(-r^\sigma s^{\sigma})(dr)^\sigma, \quad 0 < \sigma \leq 1. \] \hfill (13)

The inverse formula of the LT operator of \( \Psi(s) \) is given by

\[ \mathcal{L}_{\sigma}^{-1}[\Psi(s)] = \psi(r) = \frac{1}{(2\pi)^\sigma} \int_{a-i\infty}^{a+i\infty} \Psi(s)E_\sigma(r^\sigma s^{\sigma})(ds)^\sigma, \] \hfill (14)

where \( s^{\sigma} = a^{\sigma} + i^\sigma \) and \( \Re(s^{\sigma}) = \alpha^{\sigma} \).

**Definition 5.** By setting \( \psi \in L_{1,\sigma} \) and \( \|\psi\|_{\sigma} < \infty \), the local fractional FT operator of \( \psi(r) \) is defined by

\[ \mathcal{F}_{\sigma}[\psi(r)] = \Psi(w) = \frac{1}{\Gamma(1+\sigma)} \int_{-\infty}^{\infty} \psi(r)E_\sigma(-i^\sigma r^\sigma w^{\sigma})(dr)^\sigma. \] \hfill (15)

The inverse local fractional FT operator of \( \Psi(w) \) is given by

\[ \mathcal{F}_{\sigma}^{-1}[\Psi(w)] = \psi(r) = \frac{1}{(2\pi)^\sigma} \int_{-\infty}^{\infty} \Psi(w)E_\sigma(i^\sigma r^\sigma w^{\sigma})(dw)^\sigma. \] \hfill (16)

**Theorem 6 (LT).** Suppose that \( \psi(r) \in L_{1,\sigma} \),

\[ \mathcal{L}_{\sigma}[\psi(r)] = \Psi(s) \]

and

\[ \lim_{r \to \infty} \psi(r) = 0, \]

then there is

\[ \mathcal{L}_{\sigma}[\psi^{(\sigma)}(r)] = s^{\sigma} \mathcal{L}[\psi(r)] - \psi(0). \] \hfill (17)

We remark that there is

\[ \mathcal{L}_{\sigma}[\psi^{(n\sigma)}(r)] = s^{n\sigma} \mathcal{L}[\psi(r)] - s^{(n-1)\sigma} \psi(0) - s^{(n-2)\sigma} \psi^{(\sigma)}(0) - \ldots - \psi^{((n-1)\sigma)}(0), \] \hfill (18)

where \( n \in N \).

**Theorem 7 (FT).** Suppose that \( \psi(r), \Psi(w) \in L_{1,\sigma} \),

\[ \mathcal{F}_{\sigma}[\psi(r)] = \Psi(w) \]

and
\[
\lim_{|r| \to \infty} \psi(r) = 0.
\]

Then,
\[
\mathcal{F}_\sigma[\psi^{(\sigma)}(r)] = (iw)^\sigma \mathcal{F}[\psi(r)].
\]

We use the slightly modified form of the above results in our research \[33, 34\] instead of (19).

**Main Results**

In this chapter, we will research the solution of the LFSE (1).

**Theorem 7.** Consider the following one-dimensional LFSE
\[
\frac{\partial^e u(x, \tau)}{\partial \tau^e} = -\frac{\hbar_e}{2i\mu} \frac{\partial^{2e} u(x, \tau)}{\partial x^{2e}}, \quad x \in \mathbb{R}, \quad \tau > 0, \quad 0 < \epsilon \leq 1
\]

with the initial conditions
\[
\begin{align*}
    u(x, 0) &= u_0(x), \quad x \in \mathbb{R} \\
    u(x, \tau) &\to 0 \text{ as } |x| \to \infty, \quad \tau > 0
\end{align*}
\]

where \( \hbar \) is Planck constant divided by \( 2\pi \), \( \mu \) is the mass, and \( u(x, \tau) \) is a wave function of the particle. Then, for the solution (21), under the given condition (22), there holds the formula
\[
    u(x, \tau) = \frac{1}{(2\pi)^e} \int_{-\infty}^{\infty} u_0^*(w)E_{e,\epsilon}(-\kappa|w|^{2e}\epsilon)(dw)^e
\]

where \( E_e \) is the MLF and \( \kappa = -\frac{\hbar_e}{2\epsilon\mu} \).

**Proof.** Applying the LT according to the time variable \( \tau \) on both the sides of (21) and using the initial conditions (22) and the formula (17), we find that
\[
    s^e \tilde{u}(x, s) = u_0(x) = \kappa \frac{\partial^{2e}}{\partial x^{2e}} \tilde{u}(x, s)
\]

where \( \kappa = -\frac{\hbar_e}{2ie\mu} \).

If we apply the FT according to space variable and use the formula (20) we find that
\[
    s^e \tilde{u}^*(w, s) - u_0^*(w) = -\kappa|w|^{2e}\tilde{u}^*(w, s).
\]

Solving for \( \tilde{u}^*(w, s) \), we obtain
\[ \tilde{u}^*(w,s) = \frac{u_0^*(w)}{s^\varepsilon + \kappa |w|^{2\varepsilon}}. \]  

To recover the function \( u(x,\tau) \) from (26) it is convenient first to invert the LT and then the FT. Thus, using the formula

\[ \mathcal{L}^{-1}\left[ \frac{s^{\varepsilon-1}}{s^\varepsilon + \gamma^\sigma}; \tau \right] = \tau^{\alpha-\varepsilon} E_{\alpha,\sigma-\varepsilon+1}(-\gamma \tau^\sigma) \]  

where \( E_{\varepsilon,\sigma}(r) \) is the MLF [32] defined by

\[ E_{\varepsilon,\sigma}(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(n\varepsilon + \sigma)}, \quad (\varepsilon, \sigma \in \mathbb{C}, \text{Re}(\varepsilon) > 0, \text{Re}(\sigma) > 0) \]

we obtain

\[ u^*(w,\tau) = u_0^*(w) E_{\varepsilon,\sigma}(-\kappa |w|^{2\varepsilon} \tau^\varepsilon). \]

Now the inverse local FT leads to,

\[
\begin{align*}
\mathcal{F}^{-1}\left[ u_0^*(w) E_{\varepsilon,\sigma}(-\kappa |w|^{2\varepsilon} \tau^\varepsilon); x \right] &= \frac{1}{(2\pi)^\varepsilon} \int_{-\infty}^{\infty} u_0^*(w) E_{\varepsilon,\sigma}(-\kappa |w|^{2\varepsilon} \tau^\varepsilon)(dw)^\varepsilon.
\end{align*}
\]

**Corollary.** Consider the following LFSE

\[
\frac{\partial u(x,\tau)}{\partial \tau} = -\frac{\hbar}{2i\mu} \frac{\partial^2 u(x,\tau)}{\partial x^2}, \quad x \in \mathbb{R}, \tau > 0,
\]

with the following initial conditions

\[
\begin{align*}
\begin{aligned}
&u(x,0) = u_0(x), \quad x \in \mathbb{R} \\
&u(x,\tau) \to 0 \text{ as } |x| \to \infty, \quad \tau > 0.
\end{aligned}
\end{align*}
\]

Then the solution of (31), under above conditions (32), is given by

\[
\begin{align*}
\begin{aligned}
&u(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_0^*(w) E_{\varepsilon,\varepsilon+1}(-\kappa |w|^{2\varepsilon} \tau^\varepsilon)(dw)^\varepsilon.
\end{aligned}
\end{align*}
\]

where \( \kappa = -\frac{\hbar}{2i\mu} \).

**Conclusions**

In this work, we have introduced a LFSE and established solution for the same. The solution has been advanced in terms of the generalized MLF in a compact form with the help of Laplace and FTs and their inverses. We will obtain solutions of the same type differential equations by using the LF operator in future works.

**References**


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