SOME ANALYTICAL SOLUTIONS BY MAPPING METHODS FOR NONLINEAR
CONFORMABLE TIME-FRACTIONAL PHI-4 EQUATION

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In this paper, the practice of two types of mapping methods are used to solve the time fractional Phi-four equation by means of conformable fractional derivative. The solutions are derived using Jacobi's elliptic functions for two different value of the modulus and are obtained the some soliton solutions. The found solutions are identified bright optical soliton, dark soliton, singular soliton, combo soliton solution and periodic solutions.

Key words: Optical solitons, The time fractional Phi-four equation, Conformable derivative, Mapping methods.

1. Introduction

Homogenous and non-homogenous differential equations have been briefly studied in the literature since they act as a bridge between mathematics and physics [1-12]. Recently, there has been considerable interests and significant theoretical developments in fractional calculus used in many fields and in fractional differential equations and its applications [13-38]. In [13]; M. Ekici et al. used the first integral method by using conformable fractional derivative for getting the optical soliton solutions, in [14]; Tchier et al obtained solutions of the time fractional reaction-diffusion equations with residual power series method, Rezazadeh et al. found travelling wave solution of conformable fractional generalized reaction Duffing model by generalized projective Riccati equation method in [15]. Inc et al investigated approximate solution of some nonlinear equations by using Residual power series method in [16], Eslami obtained the exact traveling wave solutions to the fractional coupled nonlinear Schrodinger equations in [17] and in [18] are studied on numerical solutions for the Caputo-Fabrizio fractional heat-like equation. Many more researches related to fractional derivatives can be saw in [19-38].

In this work, we analyse the time fractional Phi-four equation by means of conformable fractional derivative operator [19,20] to form optical solitons using the various types of mapping methods. Recently, there are a lot of important of this equation in particle and nuclear physics. In [20];The some wave solutions of the Phi-four equation with a nonlinear variant are analysed via the Weierstrass elliptic function process. In [21]; is applied the modified simple equation method to obtain the analytical solutions of the Phi-four equations. In [22]; are investigated the spectral solutions of time dependent nonlinear Phi-four equations by using Jacobi-Gauss-Lobatto collocation process. In [23]; are analysed the Envelope solitons for the generalized Phi-four equations. Exact analytical solutions of where in said equation are obtained using tanh method with fractional complex transform in [24]. In [25] applied the RPS process to find the series solution of this equation. In this work, the found solutions are identified bright optical soliton, dark soliton, singular
soliton, combo soliton solution and periodic solutions. There are several applications for the mapping method and its diversities [26-31]. Two main group of solutions are investigated via mapping methods. These are Jacobian elliptic functions (JEFs) and periodic wave solutions (PWS). When modulus $m \to 1$ or $m \to 0$, the Jacobian elliptic functions re-edited as hyperbolic functions and trigonometric functions.

The time fractional Phi-four equation is presented as follow [19,25]:

$$q_i^{(n)} - q_{xx} + \varepsilon^2 q + \alpha q^3 = 0, \ t \geq 0, \ 0 < \eta \leq 1. \quad (1)$$

where $q_i^{(n)}$ is the conformable derivative operator; $\varepsilon$ and $\sigma$ are real valued constants. In [32]; scientists studied conformable type of fractional derivative in 2014 as a new definition of local fractional operator. It's very easier to work with this fractional derivative. Recently, several studies have been done related to conformable type of fractional calculations [33-38].

The definition of conformable fractional derivative of order $\eta \in (0,1)$ defined as the following expression [32]

$$D_\eta f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon^{1/\eta}) - f(t)}{\varepsilon}, \ f: (0, \infty) \to \mathbb{R}. \quad (2)$$

Some of the features of conformable fractional derivative as follows [32,33].

a) $D_\eta t^\alpha = \alpha t^{\alpha-\eta}, \ \forall \eta \in \mathbb{R},$

b) $D_\eta (fg) = f D_\eta g + g, D_\eta f,$

c) $D_\eta (f/g) = t^{1-\eta} g(t) f'(g(t)),$

d) $D_\eta (f/g) = g D_\eta f - f D_\eta g.$

2. Analysis of mapping method

Assume the general nonlinear partial differential equation,

$$A(q, q_x^{(n)}, q_t, q_{xx}, q_{tt}, \ldots) = 0. \quad (3)$$

where $q$ is an unknown function depending on $x$ and $t$, $A$ is a polynomial in $q = q(x,t)$ and the sub-indices represent the partial conformable fractional derivatives.

• Suppose the traveling wave variable:

$$q(x,t) = u(\phi), \quad \phi = x - Q \frac{t^{\eta}}{\eta}, \quad (4)$$

Then, from Eq. (4), Eq. (3) is turn to an ordinary differential equation for $u(\phi)$:

$$B(u, u_{\phi}, u_{\phi\phi}, u_{\phi\phi\phi}, \ldots) = 0. \quad (5)$$

where the sub-indices represent the ordinary derivatives with respect to $\phi$.

• Consider the solution of Eq. (5),

$$u(\phi) = \sum_{i=0}^{N} a_i G^i(\phi), \quad (6)$$
where \( a_n \neq 0 \) and \( G(\phi) \) can be expressed as follows:
\[
(G'(\phi))^2 = fG^2(\phi) + \frac{1}{2}gG^4(\phi) + h,
\]  
(7)
where \( h, g, f \) are arbitrary constants.

- \( N \) is found by balancing between the nonlinear terms and the highest order derivatives in Eq. (5).
- Replacing Eq. (2.4) together with Eq. (7) into the Eq. (5), then equating each coefficient of the polynomials to zero, gives a set of algebraic equations for \( a_i \ (i = 1, 2, \ldots, N) \), \( f, g, h \) and \( Q \).
- Solving the obtained system, we obtain values for \( a_i \ (i = 1, 2, \ldots, N) \) and \( Q \). Then, the solutions of Eq. (5) are obtained.

It is clear that if \( N \in \mathbb{Z} \), we can easily apply the above steps to find some of JEFs and PWS solutions. If \( N \notin \mathbb{Z} \), the some of solutions are obtained as rational statements including JEFs and PWS solutions.

3. Applications for the time-fractional Phi-Four equation

Mapping method

By placing Eq. (4) into Eq. (1), are obtained nonlinear equation as follows
\[
(Q^2 - 1)u^\prime \prime + \varepsilon^2 u + \sigma u^3 = 0,
\]  
(8)
Assumed the solution of Eq. (8) is demonstrable as a finite series as follows:
\[
u(\phi) = \sum_{j=0}^{N} \alpha_j G^j(\phi),
\]  
(9)
where \( G(\phi) \) satisfies Eq. (7), \( \phi = x - Q \int^{\frac{t}{\eta}} \) and \( \alpha_j \) for \( j = 1, N \) are values to be defined.

By balancing \( u'' \) with \( u^3 \) in Eq. (8), is obtained \( N = 1 \).

We can select the solution of Eq. (9) as following shape:
\[
q(\phi) = \alpha_0 + \alpha_1 G(\phi).
\]  
(10)
Substituting (10) into (8), collecting the coefficients of \( G(\phi) \), and solving the obtaining system, the following groups of some solutions are found:

One of the four groups of values as follows
\[
\alpha_0 = 0, \alpha_1 = \frac{\sqrt{g\varepsilon}}{\sqrt{f\sqrt{\sigma}}}, Q = \frac{\sqrt{f - \varepsilon^2}}{\sqrt{f}}.
\]  
(11)

Type 1. \( G(\phi) = sn[\phi; m] \) or \( G(\phi) = cd[\phi; m] \). So \( f = -(m^2 + 1), \ g = 2m^2 \) and \( h = 1 \). Then, the PWSs of Eq. (8) are,
\[ u(\phi) = \frac{\sqrt{2\varepsilon m}}{\sqrt{-m^2 + 1}} \text{sn}[\phi; m], \]
\[ u(\phi) = \frac{\sqrt{2\varepsilon m}}{\sqrt{-m^2 + 1}} \text{cd}[\phi; m]. \]  \hspace{1cm} (12)

When \( m \to 1 \), Eq. (12) is rewrote in the form of dark soliton solution of Eq. (1);
\[ q(x, t) = \frac{\sqrt{2\varepsilon}}{\sqrt{-2\varepsilon}} \tanh \left( x - \frac{\sqrt{2 - \varepsilon^2}}{\sqrt{-2}} \frac{t^n}{\eta} \right), \]
and in the form of singular soliton solution;
\[ q(x, t) = \frac{\sqrt{2\varepsilon}}{\sqrt{-2\varepsilon}} \cos \left( x - \frac{\sqrt{2 - \varepsilon^2}}{\sqrt{-2}} \frac{t^n}{\eta} \right), \]  \hspace{1cm} (13)

Type 2. \( G(\phi) = \text{cn}[\phi; m] \). So \( f = 2m^2 - 1 \), \( g = -2m^2 \) and \( h = 1 - m^2 \). Then, the PWS of Eq. (8) is,
\[ u(\phi) = \frac{\sqrt{2\varepsilon m}}{\sqrt{2m^2 - 1}} \text{cn}[\phi; m], \]
When \( m \to 1 \), Eq. (15) is rewrote in the form of bright optical soliton solution of Eq. (1);
\[ q(x, t) = \frac{\sqrt{2\varepsilon}}{\sqrt{-2\varepsilon}} \sec h \left( x - \sqrt{1 - \varepsilon^2} \frac{t^n}{\eta} \right), \]  \hspace{1cm} (16)

Type 3. \( G(\phi) = \text{dn}[\phi; m] \). So \( f = 2 - m^2 \), \( g = -2 \) and \( h = m^2 - 1 \). Then, the PWS of Eq. (8) is,
\[ u(\phi) = \frac{\sqrt{2\varepsilon m}}{\sqrt{2 - m^2}} \text{dn}[\phi; m], \]
When \( m \to 1 \), Eq. (3.13) is rewrote in the form of bright optical soliton solution of Eq. (1);
\[ q(x, t) = \frac{\sqrt{2\varepsilon}}{\sqrt{-2\varepsilon}} \sec h \left( x - \sqrt{1 - \varepsilon^2} \frac{t^n}{\eta} \right), \]
This solution is the same as Eq. (16).

Type 4. \( G(\phi) = \text{cs}[\phi; m] \). So \( f = 2 - m^2 \), \( g = 2 \) and \( h = 1 - m^2 \). Then, the PWS of Eq. (8) is,
\[ u(\phi) = \frac{\sqrt{2\varepsilon m}}{\sqrt{2 - m^2}} \text{cs}[\phi; m], \]
When \( m \to 0 \), Eq. (18) is rewrote in the form of singular periodic solution of Eq. (1);
\[ q(x, t) = \frac{\varepsilon}{\sqrt{\sigma}} \cot \left( x - \sqrt{2 - \varepsilon^2} \frac{t^n}{\eta} \right), \]  \hspace{1cm} (19)
and when \( m \to 1 \), Eq. (18) is rewrote in the form of singular soliton solution of Eq. (1);
\[ q(x, t) = \frac{\sqrt{2\varepsilon}}{\sqrt{\sigma}} \csc h \left( x - \sqrt{1 - \varepsilon^2} \frac{t^n}{\eta} \right), \]  \hspace{1cm} (20)
Type 5. $G(\phi) = ns[\phi; m]$ or $G(\phi) = cd[\phi; m]$. So $f = -(1 + m^2)$, $g = 2$ and $h = m^2$. Then, the PWSs of Eq. (8) are,

$$u(\phi) = \frac{\sqrt{2}e}{\sqrt{-(1 + m^2)}}ns[\phi; m],$$

(21)

and

$$u(\phi) = \frac{\sqrt{2}e}{\sqrt{-(1 + m^2)}}dc[\phi; m],$$

(22)

When $m \to 0$, Eqs. (21) and (22) are rewrote respectively in the form of the singular periodic solution of Eq. (1);

$$q(x, t) = \frac{\sqrt{2}e}{\sqrt{-\sigma}} \csc(x - \sqrt{1 + e^2} \tau^n),$$

(23)

and

$$q(x, t) = \frac{\sqrt{2}e}{\sqrt{-\sigma}} \sec(x - \sqrt{1 + e^2} \tau^n),$$

(24)

When $m \to 1$, Eq. (22) are rewrote in the form of the singular soliton solution of Eq. (1);

$$q(x, t) = \frac{\sqrt{2}e}{\sqrt{-2\sigma}} \coth(x - \sqrt{2 + e^2} \tau^n).$$

(25)

Modified mapping method

Consider the solution of Eq. (9) is of the shape;

$$u(\phi) = \alpha_0 + \alpha_1 G(\phi) + \beta(G^{-1}(\phi)),$$

(26)

where $G(\phi)$ satisfies Eq.(7).

Substituting (26) into (9), collecting the coefficients of $G(\phi)$, and solving the obtaining system, the following groups of some solutions are found:

One of the sixteen groups of values as follows

$$\alpha_0 = 0, \alpha_1 = \frac{\sqrt{g(f - 3\sqrt{2gh})e^2}}{\sqrt{f^2 - 18gh}}, \beta_1 = \frac{\sqrt{g(f - 3\sqrt{2gh})e^2}}{\sqrt{g\sigma}},$$

(27)

$$Q = \sqrt{\frac{f^2 - 18gh - fe^2 + 3\sqrt{2gh}e^2}{f^2 - 18gh}}.$$

Type 1. $G(\phi) = sn[\phi; m]$ or $G(\phi) = cd[\phi; m]$. So $f = -(m^2 + 1)$, $g = 2m^2$ and $h = 1$. Then, the PWSs of Eq. (8) are,

$$u(\phi) = \sqrt{\frac{2(-(m^2 + 1) - 6m)e^2}{m^2 + 1} - \frac{36m^2}{m^2 + 1} - \frac{1}{m}ns[\phi; m]}(sn[\phi; m] - \frac{1}{m}ns[\phi; m]),$$

(28)
and

\[
    u(\phi) = \sqrt{\frac{2(-(m^2 + 1) - 6\varepsilon^2 m)}{(m^2 + 1)^2 - 36m^2}} \left(c d[\phi; m] - \frac{1}{m} d c[\phi; m]\right)
\]  

(29)

When \( m \to 1 \), Eq. (28) rearranged in the form of the combo periodic singular solution of Eq. (1);

\[
    q(x,t) = \sqrt{\frac{2}{\varepsilon^2}} (\tanh(x - \sqrt{-4 + \varepsilon^2 \frac{t^\eta}{\eta}}) - \coth(x - \sqrt{-4 + \varepsilon^2 \frac{t^\eta}{\eta}}))
\]  

(30)

Type 2. \( G(\phi) = d n[\phi; m] \). So \( f = 2 - m^2 \), \( g = -2 \) and \( h = m^2 - 1 \). Then, the PWS of Eq. (8) is,

\[
    u(\phi) = \sqrt{\frac{-2(2 - m^2 - 3\sqrt{-4(m^2 - 1)})}{(2 - m^2) + 36(m^2 - 1)}} (d n[\phi; m] - \sqrt{1 - m^2} \ d n[\phi; m])
\]  

(31)

When \( m \to 1 \), Eq. (31) rewrote in the form of bright optical soliton solution of Eq. (1);

\[
    q(x,t) = \sqrt{\frac{-2\varepsilon^2}{\varepsilon^2}} (\sec h(x - \sqrt{1 - \varepsilon^2 \frac{t^\eta}{\eta}}))
\]  

(32)

Type 3. \( G(\phi) = c s[\phi; m] \). So \( f = 2 - m^2 \), \( g = 2 \) and \( h = 1 - m^2 \). Then, the PWS of Eq. (8) is,

\[
    u(\phi) = \sqrt{\frac{2(2 - m^2 - 3\sqrt{4(1 - m^2)})}{(2 - m^2) - 36(1 - m^2)}} (c s[\phi; m] - \sqrt{(1 - m^2)} c s[\phi; m])
\]  

(33)

When \( m \to 0 \), Eq. (33) rearranged in the form of the combo periodic singular solution of Eq. (1);

\[
    q(x,t) = \sqrt{\frac{4}{\varepsilon^2}} (\tanh(x - \sqrt{-8 + \varepsilon^2 \frac{t^\eta}{\eta}}) - \coth(x - \sqrt{-8 + \varepsilon^2 \frac{t^\eta}{\eta}}))
\]  

(34)

4. Graphical representation of the solutions

The surface graphics of the obtained solutions are showed below in the figures by using Mathematica.
Fig 1. The surface graphics for the $|q(x,t)|^2$ analytical solution of the time-fractional Phi-four equation obtained with mapping method. (a) Eq. (14), (b) Eq. (23).

Fig 2. The surface graphics for the $|q(x,t)|^2$ analytical solution of the time-fractional Phi-four equation obtained with modified mapping method. (a) Eq. (32), (b) Eq. (34).

We wrote some of solutions found for the presented time-fractional Phi-four equation via conformable fractional derivative operator. Besides we showed 3D graphics for some of solutions in fig 1 and fig 2. The graphics above were drawn for $\varepsilon = 1$, $\sigma = 2$, $\mu = 0.75$.

5. Conclusions

In this paper, the mapping methods are used to find new soliton solutions of the time-fractional Phi-four equation by using conformable fractional derivative. When modulus $m \to 1$ or $m \to 0$, the Jacobian elliptic functions rearranged as trigonometric functions and hyperbolic functions. The existence of solutions derived from these functions are all guaranteed through constraint conditions that are also listed besides the solutions. We say that the presented method is suitable to examine the many problems located in science and engineering.

References


