A NEW APPROACH FOR FINDING STANDARD HEAT EQUATION AND A SPECIAL NEWELL-WHITEHEAD EQUATION

by

Xiu-Rong GUO [a,b*], ***Yu-Feng ZHANG*** [a*], ***Mei GUO*** [c], ***and Zheng-Tao LIU*** [c]

[a] College of Mathematics, China University of Mining and Technology, XuZhou, China
[b] Basic Courses, Shandong University of Science and Technology, Tai'an, China
[c] Shandong Tonghui Architectural Design Co. Ltd.,Taian, China

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Under a frame of 2×2 matrix Lie algebras, Tu and Meng [9] once established a united integrable model of the Ablowitz-Kaup-Newel-Segur (AKNS) hierarchy, the D-AKNS hierarchy, the Levi hierarchy and the TD hierarchy. Based on this idea, we introduce two block-matrix Lie algebras to present an isospectral problem, whose compatibility condition gives rise to a type of integrable hierarchy which can be reduced to the Levi hierarchy and the AKNS hierarchy, and so on. A united integrable model obtained by us in the paper is different from that given by Tu and Meng. Specially, the main result in the paper can be reduced to two new various integrable couplings of the Levi hierarchy, from which we again obtain the standard heat equation and a special Newell-Whitehead equation.

Key words: *Lie algebra, TAH scheme, DS hierarchy, heat equation, Newell-Whitehead equation*

Introduction

Tu [1] proposed by using 2 × 2 Lie algebras a scheme for generating integrable Hamiltonian hierarchies of evolution equations which was called the Tu scheme [2]. Under the frame of the Tu scheme, many interesting integrable Hamiltonian hierarchies and some corresponding properties were obtained, such as the consequences in [2-8]. Tu and Meng [9] employed the 2 × 2 Lie algebra:

$$l_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ l_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ l_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ l_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the commutative relations

$$[l_1, l_2] = 0,\ [l_1, l_3] = l_3,\ [l_2, l_3] = -l_3,\ [l_1, l_4] = -l_4,\ [l_2, l_4] = l_4,\ [l_3, l_4] = l_1 - l_2$$

to construct an integrable model which could be reduced to the Levi hierarchy, D-AKNS hierarchy and TD hierarchy.

In the paper, we want to introduce two types of block-matrix Lie algebras for which a united integrable model of the Levi hierarchy and the AKNS hierarchy is obtained. Again, the integrable model is further reduced to two different integrable couplings of the Levi hierarchy, one of them is reduced to the standard heat equation, another one can give a special

* Corresponding author, e-mail: springing007@126.com; zyfxz@cumt.edu.cn

Newell-Whitehead equation, however, the Newell-Whitehead equation is not integrable, which is an interesting fact.

Two Lie algebras

Tu [1] detailed the Lie algebras for generating the integrable Hamiltonian hierarchies. Based on this, we first present the known simple algebra which consists of the following 2×2 matrices:

$$h_1 = l_1, \quad h_2 = l_3, \quad h_3 = l_4, \quad h_4 = l_2$$

along with the commutative relations:

$$[h_1, h_2] = h_1 h_2 - h_2 h_1 = h_2,\ [h_1, h_3] = -h_3,\ [h_1, h_4] = 0,\ [h_2, h_3] = h_1 - h_4$$

A loop algebra of the Lie algebra is defined:

$$\tilde{H} \equiv \{h_i(n),\ i = 1,2,3,4;\ n \in \mathbf{Z}\} \tag{1}$$

where $[h_i(m), h_j(n)] = [h_i, h_j]\lambda^{m+n}$, $m, n \in \mathbf{Z}$.

Denote:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

From this 2×2 matrices, we introduce the following Lie algebra:

$$G_1 = \{f_1, \ldots, f_8\}$$

where

$$f_1 = \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix},\ f_2 = \begin{pmatrix} h_4 & 0 \\ 0 & h_4 \end{pmatrix},\ f_3 = \begin{pmatrix} h_2 & 0 \\ 0 & h_2 \end{pmatrix},\ f_4 = \begin{pmatrix} h_3 & 0 \\ 0 & h_3 \end{pmatrix},\ f_5 = \begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix}$$

$$f_6 = \begin{pmatrix} 0 & h \\ 0 & h \end{pmatrix},\ f_7 = \begin{pmatrix} 0 & e_1 \\ 0 & e_1 \end{pmatrix},\ f_8 = \begin{pmatrix} 0 & e_2 \\ 0 & e_2 \end{pmatrix}$$

along with the commutative relations:

$$[f_1, f_2] = 0,\ [f_1, f_3] = f_3,\ [f_1, f_4] = -f_4,\ [f_2, f_3] = -f_3,\ [f_2, f_4] = f_4,\ [f_3, f_4] = f_1 - f_2$$

$$[f_i, f_5] = 0,\ i = 1,2,3,4,6,7,8; [f_1, f_6] = 0,\ [f_1, f_7] = f_8,\ [f_1, f_8] = f_7, [f_2, f_6] = 0$$

$$[f_2, f_7] = -f_8,\ [f_2, f_8] = -f_7,\ [f_3, f_6] = -f_7 - f_8,\ [f_3, f_7] = f_6,\ [f_3, f_8] = -f_6$$

$$[f_4, f_6] = f_7 - f_8,\ [f_4, f_7] = -f_6,\ [f_4, f_8] = -f_6,\ [f_6, f_7] = -2f_8,\ [f_6, f_8] = 2f_7,\ [f_7, f_8] = -2f_6$$

Define:

$$\tilde{G}_1 = \{f_1(n), \ldots, f_8(n)\}$$

where

$$f_i(n) = f_i\lambda^n,\ [f_i(m),\ f_j(n)] = [f_i, f_j]\lambda^{m+n},\ 1 \le i, j \le 8;\ m, n \in \mathbf{Z}$$

It is easy to see that $\tilde{G}_1$ is a loop algebra, where $f_5(_n) = \tilde{G}_1$ – a pseudo-regular, and satisfies the following properties:

- $\tilde{G}_1 = \ker \operatorname{ad} f_5(n) \oplus \operatorname{im} \operatorname{ad} f_5(n)$ and
- $\ker \operatorname{ad} f_5(n)$ is commutative.

Generally, we have the following proposition [1]:

If X is a regular element of a semisimple Lie algebra G, then $R = X \otimes \lambda^n$ is pseudo-regular in $\tilde{G}$. In what follows, we establish another Lie algebra with the help of the above 2×2 matrices:

$$G_1 = \{g_1,...,g_7\}$$

where

$$g_i = f_i,\ i = 1,2,3,4;\ g_5 = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix},\ g_6 = \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix},\ g_7 = \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix}$$

along with the commutative relations:

$$[g_1,g_2]=0,\ [g_1,g_3]=g_3,\ [g_1,g_4]=-g_4,\ [g_2,g_3]=-g_3,\ [g_2,g_4]=g_4,\ [g_3,g_4]=g_1-g_2$$
$$[g_1,g_5]=0,\ [g_1,g_6]=g_7,\ [g_1,g_7]=g_6,\ [g_2,g_5]=0,\ [g_2,g_6]=-g_7,\ [g_2,g_7]=-g_6$$
$$[g_3,g_5]=-g_6-g_7,\ [g_3,g_6]=g_5, [g_3,g_7]=-g_5,\ [g_4,g_5]=g_6-g_7,\ [g_4,g_6]=-g_5$$
$$[g_4,g_7]=-g_5,\ [g_5,g_6]=2g_7,\ [g_5,g_7]=2g_6,\ [g_6,g_7]=-2g_5$$

If set $\Delta_1 = \{g_1, g_2, g_3, g_4\}$, $\Delta_2 = \{g_5, g_6, g_7\}$, we find that:

$$G_2 = \Delta_1 \oplus \Delta_2,\ [\Delta_i,\Delta_i] \subset \Delta_i, i=1,2;\ [\Delta_1,\Delta_2] \subset \Delta_2 \tag{2}$$

A corresponding loop algebra is defined:

$$\tilde{G}_2 = \{g_1(n),\ g_2(n),\ldots, g_7(n)\} \tag{3}$$

where

$$g_i(n) = g_i\lambda^n,\ [g_i(m), g_j(n)] = [g_i, g_j]\lambda^{m+n},\ 1 \le i,j \le 7;\ m,n \in \mathbf{Z}$$

The subalgebras Δ_1 and Δ_2 are all semisimple.

An expanding integrable model of the Levi hierarchy and some reductions

Consider an isospectral Lax pair by the loop algebra $\tilde{G}_1/\varphi_x = U\varphi,\ \varphi t = V\varphi,$ where:

$$U = f_2(1) + (r-q)f_2(0) + qf_3(0) + rf_4(0) + u_1 f_6(0) + u_2 f_7(0) + u_3 f_8(0) \tag{4}$$

$$V = V_1 f_1(0) + V_2 f_3(0) + V_3 f_4(0) + V_4 f_2(0) + \sum_{i=6}^{8} V_i f_i(0) \tag{5}$$

where

$$V_i = \sum_{m\ge 0} V_{im}\lambda^{-m}$$

Then the stationary zero curvature equation admits that:

$$\begin{cases} (V_{1m})_x = qV_{3m} - rV_{2m}, V_{2,m+1} = -(V_{2m})_x - (r-q)V_{2m} - 2qV_{1m} \\ V_{3,m+1} = (V_{3m})_x - (r-q)V_{3m} - 2rV_{1m}, (V_{4m})_x = -(V_{1m})_x \\ V_{7,m+1} = -(V_{8m})_x + (r-q+2u_1)V_{7m} - (q+r+2u_2)V_{6m} - u_1V_{3m} + u_1V_{2m} - 2u_2V_{1m} \\ V_{8,m+1} = -(V_{7m})_x + (q-r+2u_1)V_{8m} - (q-r+2u_3)V_{6m} + u_1V_{3m} + u_1V_{2m} - 2u_3V_{1m} \\ (V_{6m})_x = (q-r+2u_3)V_{7m} - (q+r+2u_2)V_{8m} + (u_3-u_2)V_{2m} + (u_2+u_3)V_{3m} \end{cases} \tag{6}$$

The eq. (6) are local solvable. If set:

$$V_{1,0} = \alpha = \text{constant},\ V_{2,0} = \ldots = V_{8,0}$$

we can get from eq. (6) that:

$$V_{1,1}=0,\ V_{2,1}=-2\alpha q,\ V_{3,1}=-2\alpha r,\ V_{2,2}=2\alpha[q_x-q(q-r)],\ V_{3,2}=-2\alpha[r_x+r(q-r)]$$

$$V_{1,2}=-2\alpha qr,\ V_{2,3}=2\alpha[-q_{xx}+(q-r)_x q+2(q-r)q_x-(q-r)^2q+2q^2r],\ V_{7,1}=-2\alpha u_2$$

$$V_{8,1}=-2\alpha u_3,\ V_{6,1}=0, V_{7,2}=2\alpha[u_{3,x}-u_2(r-q+2u_1)+ru_1-qu_1]$$

Set:

$$V_+^{(n)}=\sum_{m=0}^{n}\left[V_{1m}f_1(-m)+V_{2m}f_3(-m)+V_{3m}f_4(-m)+V_{4m}f_2(-m)+\sum_{i=6}^{8}V_{im}f_i(-m)\right]\lambda^n=\lambda^nV-V_-^{(n)}$$

then eq. (5) can be decomposed into an equivalent equation:

$$-V_{+,x}^{(n)}+[U,V_+^{(n)}]=V_{-,x}^{(n)}-[U,V_-^{(n)}] \tag{7}$$

By following the approach presented in [1], we can find that:

$$-V_{+,x}^{(n)}+[U,V_+^{(n)}]=V_{2,n+1}f_3(0)-V_{3,n+1}f_4(0)+V_{8,n+1}f_7(0)+V_{7,n+1}f_8(0)$$

Take $V^{(n)}=V_+^{(n)}+k_1f_2(0)+k_2f_6(0)$ a direct calculation gives:

$$V_x^{(n)}-[U,V^{(n)}]=k_{1,x}f_2(0)-(V_{2,n+1}+qk_1)f_3(0)+(V_{3,n+1}+rk_1)f_4(0)-$$
$$-(V_{8,n+1}+u_3k_1-qk_2+rk_2-2u_3k_2)f_7(0)-(V_{7,n+1}+u_2k_1-2u_2k_2-qk_2-rk_2)f_8(0)+k_2f_6(0)$$

Thus, the compatibility condition of the Lax pair:

$$\psi_x=U\psi,\ \psi_t=V^{(n)}\psi$$

gives rise to the following integrable hierarchy:

$$\begin{cases}(r-q)_t=k_{1,x}, u_{2,t}=-V_{8,n+1}-u_3k_1+(q-r+2u_3)k_2\\ u_{3,t}=-V_{7,n+1}-u_2k_1+(2u_2+q+r)k_2\\ q_t=-V_{2,n+1}-qk_1, r_t=V_{3,n+1}+rk_1, u_{1,t}=k_{2,x}\end{cases} \tag{8}$$

If set $k_1=V_{3n}-V_{2n}+2V_{1n}$, $u_2=u_3=0$, eq. (8) reduces to the well-known Levi hierarchy:

$$\begin{cases}q_t=V_{2n,x}+rV_{2n}-qV_{3n}\\ r_t=V_{3n,x}+qV_{3n}-rV_{2n}\end{cases} \tag{9}$$

If set $k_1=k_2=u_2=u_3=0$, then eq. (8) reduces to the AKNS hierarchy:

$$\begin{cases}q_t=-V_{2,n+1}\\ r_t=V_{3,n+1}\end{cases} \tag{10}$$

Therefore, eq. (8) is a united integrable model of the Levi hierarchy and the AKNS hierarchy, and it is different from the united integrable model given by Tu and Meng [9].

If take $k_1=V_{3n}-V_{2n}+2V_{1n}$, $k_2=-V_{6n}$, eq. (8) becomes:

$$\begin{cases}q_t=-V_{2,n+1}-q(V_{3n}-V_{2n}+2V_{1n}), r_t=V_{3,n+1}+r(V_{3n}-V_{2n}+2V_{1n})\\ u_{2,t}=-V_{8,n+1}-u_3(V_{3n}-V_{2n}+2V_{1n})+(r-q-2u_3)V_{6n}\\ u_{3,t}=-V_{7,n+1}-u_2(V_{3n}-V_{2n}+2V_{1n})-(q+r+2u_2)V_{6n}\\ u_{1,t}=-V_{6n,x}\end{cases} \tag{11}$$

Cases 1: $u_1 = V_{6n} = 0$. Equation (11) presents that:

$$\begin{cases} q_t = V_{2n,x} + rV_{2n} - qV_{3n} = V_{2n,x} - V_{1n,x}, r_t = V_{3n,x} + qV_{3n} - rV_{2n} = V_{3n,x} + V_{1n,x} \\ u_{2,t} = V_{7n,x} + (r - q - 2u_1)V_{8n} - (u_1 + u_3)V_{3n} + (u_3 - u_1)V_{2n} \\ u_{3,t} = V_{8n,x} + (q - r - 2u_1)V_{7n} + (u_1 - u_2)V_{3n} + (u_2 - u_1)V_{2n} \end{cases} \tag{12}$$

According to the theory on integrable couplings [7, 8], eq. (12) is an integrable coupling of the Levi hierarchy (9). When $n = 2$, we can get the coupled part of the Levi equation:

$$\begin{cases} u_{2,t} = 2\alpha u_{3,xx} - 2\alpha(u_2 r - qu_2)_x + 2\alpha(r - q)(qu_3 - ru_2 - ru_3 - qu_2 - u_2^2) + \\ \qquad + 2\alpha u_3(r_x + rq - r^2)_x + 2\alpha u_3(q_x - q^2 + qr)_x \\ u_{3,t} = 2\alpha u_{2,xx} - 2\alpha(u_3 q - u_3 r)_x + 2\alpha(q - r)(u_{3,x} - u_2 r + u_2 q + ru_1 - qu_1) + \\ \qquad + 2\alpha u_2(r_x + qr - r^2) + 2\alpha u_2(q_x - q^2 + qr) \end{cases} \tag{13}$$

Specially, if set $q = r = 0$, eq. (13) reduces to:

$$\begin{cases} u_{2,t} = 2\alpha u_{3,xx} \\ u_{3,t} = 2\alpha u_{2,xx} \end{cases} \tag{14}$$

Take $u_2 = u_3 = v$, eq. (14) is just right the well-known linear heat equation:

$$v_t = 2\alpha v_{xx} \tag{15}$$

Case 2: $u_1 \neq 0$. Equation (11) becomes:

$$\begin{cases} q_t = V_{2n,x} - V_{1n,x}, r_t = V_{3n,x} + V_{1n,x}, u_{1,t} = -V_{6n,x} \\ u_{2,t} = V_{7n,x} - (q - r + 2u_1)V_{8n} - (u_1 + u_3)V_{3n} + (u_3 - u_1)V_{2n} \\ u_{3,t} = V_{8n,x} + (q - r - 2u_1)V_{7n} + (u_1 - u_2)V_{3n} + (u_2 - u_1)V_{2n} \end{cases} \tag{16}$$

When $n = 2$, eq. (16) reduces to:

$$\begin{cases} q_t = 2\alpha q_{xx} - 2\alpha(q^2 - 2qr)_x, r_t = -2\alpha r_{xx} - 2\alpha(2qr - r^2)_x \\ u_{1,t} = -2\alpha(qu_3 - ru_3 - ru_2 - qu_2 - u_2^2)_x \\ u_{2,t} = 2\alpha u_{3,xx} - 2\alpha(ru_2 - qu_2 + 2u_1u_2 - ru_1 + qu_1)_x - \\ \qquad - 2\alpha(q - r + 2u_1)(u_{2,x} - qu_3 + ru_3 - 2u_1u_3 - ru_1 - qu_1) + \\ \qquad + 2\alpha(u_1 + u_3)(r_x + qr - r^2) - 2\alpha(u_1 - u_3)(q_x - q^2 + qr)_x \\ u_{3,t} = 2\alpha u_{2,xx} - 2\alpha(u_3 q - u_3 r + 2u_1u_3 - ru_1 - qu_1)_x + \\ \qquad + 2\alpha(q - r - 2u_1)(u_{3,x} - u_2 r + u_2 q + 2u_1u_2 + ru_1 - qu_1) - \\ \qquad - 2\alpha(u_1 - u_2)(r_x + qr - r^2) - 2\alpha(u_2 - u_1)(q_x - q^2 + qr) \end{cases} \tag{17}$$

If set $q = r = 0$, eq. (17) gives:

$$\begin{cases} u_{1,t} = 4\alpha u_2 u_{2,x}, u_{2,t} = 2\alpha u_{3,xx} - 4\alpha(u_1u_2)_x + 8\alpha u_1^2 u_3 \\ u_{3,t} = 2\alpha u_{2,xx} - 4\alpha(u_1u_3)_x - 8\alpha u_1^2 u_2 \end{cases} \tag{18}$$

The first equation in eq. (18) is a conserved form, and later two are linear with respect to the variables u_2 and u_3, respectively:

If set $u_2 + u_3 = u$, $u_2 - u_3 = v$, $\alpha = 1/2$, eq. (18) becomes:

$$\begin{cases} u_t = u_{xx} - 2(u_1 u)_x - 4u_1^2 v \\ v_t = -v_{xx} - 2(u_1 v)_x + 4u_1^2 u \end{cases}$$

where the variable u_1 satisfies that $u_{1,t} = 1/2(u + v)(u_x + v_x)$.

Case 3: $k_1 = k_2 = 0$. Equation (8) just reduces to an integrable coupling of the AKNS hierarchy:

$$\begin{cases} q_t = -V_{2,n+1}, r_t = V_{3,n+1} \\ u_{2,t} = -V_{8,n+1}, u_{3,t} = -V_{7,n+1} \end{cases} \tag{19}$$

Equation (19) can be written:

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -V_{2,n+1} \\ V_{3,n+1} \\ -V_{8,n+1} \\ -V_{7,n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} V_{3,n+1} \\ V_{2,n+1} \\ V_{7,n+1} \\ -V_{8,n+1} \end{pmatrix} \equiv J \begin{pmatrix} V_{3,n+1} \\ V_{2,n+1} \\ V_{7,n+1} \\ -V_{8,n+1} \end{pmatrix} = JL \begin{pmatrix} V_{3n} \\ V_{2n} \\ V_{7n} \\ -V_{8n} \end{pmatrix} \tag{20}$$

where

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; L = \begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r & 0 & 0 \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r & 0 & 0 \\ -2u_2\partial^{-1}q & 2u_2\partial^{-1}r & 0 & \partial \\ 2u_3\partial^{-1}q & -2u_3\partial^{-1}r & \partial & 0 \end{pmatrix}$$

when $u_2 = u_3 = 0$, the above J and L reduce to the case of the AKNS hierarchy.

Another expanding integrable model of the Levi hierarchy and some reductions

Consider an isospectral problem by the loop algebra $\tilde{G}_2$:

$$\psi_x = U\psi, \; \psi_t = V\psi \tag{21}$$

where

$$U = g_2(1) + (r - q)g_2(0) + qg_3(0) + rg_4(0) + s_1 g_6(0) + s_2 g_7(0) \tag{22}$$

$$V = V_1 g_1(0) + V_2 g_3(0) + V_3 g_4(0) + V_4 g_2(0) + V_5 g_5(0) + V_6 g_6(0) + V_7 g_7(0) \tag{23}$$

where

$$V_i = \sum_{m=0}^{\infty} V_{im}\lambda^{-m}, \; i = 1, 2, \ldots, 7$$

Similar to the previous discussion, we can get a recursion relation among V_{im}:

$$\begin{cases} V_{1m,x} = qV_{3m} - rV_{2m} = -V_{4m,x}, \; V_{2,m+1} = -V_{2m,x} - (r - q)V_{2m} - 2qV_{1m} \\ V_{3,m+1} = V_{3m,x} - (r - q)V_{3m} - 2rV_{1m} \\ V_{5m,x} = -(q + r + 2s_1)V_{7m} + (q - r + 2s_2)V_{6m} + (s_1 + s_2)V_{3m} + (s_2 - s_1)V_{2m} \\ V_{7,m+1} = -V_{6m,x} + (q - r)V_{7m} + (r - q)V_{5m} - 2s_2 V_{5m} - 2s_2 V_{1m} \\ V_{6,m+1} = -V_{7m,x} + (q - r)V_{6m} - (q + r)V_{5m} - 2s_1 V_{5m} - 2s_1 V_{1m} \end{cases} \tag{24}$$

If denote:

$$V^{(n)}=\sum_{m=o}^{n}\left[V_1 m g_1(n-m)+V_{2m}g_3(n-m)+V_{3m}g_4(n-m)+V_{4m}g_2(n-m)+\sum_{i=5}^{7}V_{im}g_i(n-m)\right]+$$
$$+(V_{3n}-V_{2n}+2V_{1n})g_2(0)$$

we can obtain:

$$V_x^{(n)}-[U,V^{(n)}]=[V_{2,n+1}+q(V_{3n}-V_{2n}+2V_{1n})]g_3(0)+[V_{3,n+1}+r(V_{3n}-V_{2n}+2V_{1n})]g_4(0)+$$
$$+(V_{3n}-V_{2n}+2V_{1n})_x g_2(0)-[V_{6,n+1}+s_1(V_{3n}-V_{2n}+2V_{1n})]g_7(0)-$$
$$-[V_{7,n+1}+s_2(V_{3n}-V_{2n}+2V_{1n})]g_6(0)$$

Therefore, the zero curvature equation $U_t - V_x^{(n)} + [U, V^{(n)}] = 0$ admits:

$$\begin{cases} q_t=-V_{2,n+1}-q(V_{3n}-V_{2n}+2V_{1n}),\ r_t=V_{3,n+1}+r(V_{3n}-V_{2n}+2V_{1n}) \\ s_{1,t}=-V_{6,n+1}-s_1(V_{3n}-V_{2n}+2V_{1n})=V_{7n,x}+(r-q)V_{6n}+(q+r+2s_1)V_{5n}-s_1V_{3n}+s_1V_{2n} \\ s_{2,t}=-V_{7,n+1}-s_2(V_{3n}-V_{2n}+2V_{1n})=V_{6n,x}+(r-q)V_{7n}+(q-r+2s_2)V_{5n}-s_2V_{3n}+s_2V_{2n} \end{cases} \tag{25}$$

When $s_1 = s_2 = 0$, eq. (25) reduces to the Levi hierarchy. According to the theory on integrable couplings, eq. (25) is one integrable coupling of the Levi hierarchy, which is different from the first integrable coupling of the Levi hierarchy eq. (11). We can see this point from their reductions.

Set $V_{1,0} = \alpha, V_{2,0} = ...\ V_{7,0} = 0$, we have from eq. (24):

$$V_{7,1}=-2\alpha s_2,\ V_{6,1}=-2\alpha s_1,\ V_{5,1}=0,\ V_{6,2}=2\alpha s_{2,x}-2\alpha(q-r)s_1$$
$$V_{7,2}=2\alpha s_{1,x}-2\alpha(q-r)s_2, V_{5,2}=2\alpha(-qs_1-rs_1-rs_2+qs_2-s_1^2+s_2^2),\ldots$$

When $n = 2$, eq. (25) reduces to:

$$\begin{cases} q_t=2\alpha q_{xx}-2\alpha(q^2-2qr)_x, r_t=-2\alpha r_{xx}-2\alpha(2qr-r^2)_x \\ s_{1,t}=2\alpha s_{1,xx}-2\alpha(s_2(q-r))_x+2\alpha(r-q)s_{2,x}+2\alpha s_1 r_x+2\alpha s_1 q_x+2\alpha(r-q)^2 s_1+ \\ \quad +2\alpha s_1 r(q-r)-2\alpha s_1 q(q-r)-2\alpha(q+r+2s_1)(qs_1+rs_1-qs_2+rs_2+s_1^2-s_2^2) \\ s_{2,t}=2\alpha s_{2,xx}-2\alpha[s_1(q-r)]_x+2\alpha(r-q)s_{1,x}+2\alpha s_2 r_x+2\alpha q s_2+2\alpha(r-q)^2 s_2+ \\ \quad +2\alpha s_2 r(q-r)-2\alpha s_2 q(q-r)-2\alpha(q-r+2s_2)(qs_1+rs_1+rs_2-qs_2+s_1^2-s_2^2) \end{cases} \tag{26}$$

Equation (26) is different from eq. (17). Specially, when set $q = r = 0$, eq. (26) gives:

$$\begin{cases} s_{1,t}=2\alpha s_{1,xx}-4\alpha s_1(s_1^2-s_2^2) \\ s_{2,t}=2\alpha s_{2,xx}-4\alpha s_2(s_1^2-s_2^2) \end{cases} \tag{27}$$

which is various from eq. (16) which was reduced from the integrable coupling eq. (11). Hence, the integrable coupling of the Levi hierarchy eq. (25) is really different from the integrable coupling eq. (19).

We see that when $s_1 = s_2$, eq. (27) reduces to eq. (16). When set $s_1 = is_2$, $s_2 = v$, eq. (27) gives:

$$v_t = v_{xx} + 4v^3 \tag{28}$$

which is a special Newell-Whitehead equation.

This is an integrable equation, but the Newell-Whitehead equation:

$$u_t = u_{xx} + u - u^3 \tag{29}$$

It is not integrable. Therefore, eq. (28) could possesses the similar travelling wave solutions.

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