A NEW APPROACH FOR FINDING STANDARD HEAT EQUATION AND A SPECIAL NEWELL-WHITEHEAD EQUATION

by

Xiu-Rong GUO\textsuperscript{a b}∗, Yu-Feng ZHANG\textsuperscript{a}∗, Mei GUO\textsuperscript{c}, Zheng-Tao LIU\textsuperscript{c}

\textsuperscript{a} College of Mathematics, China University of Mining and Technology, XuZhou 221116, China

\textsuperscript{b} Basic Courses, Shandong University of Science and Technology, Tai’an 271000, China

\textsuperscript{c} Shandong Tonghui Architectural Design Co. Ltd., Taian 271000, China

Under a frame of $2\times 2$ matrix Lie algebras, Tu Guizhang and Meng Dazhi once established a united integrable model of the AKNS hierarchy, the D-AKNS hierarchy, the Levi hierarchy and the TD hierarchy. Based on this idea, we introduce two block-matrix Lie algebras to present an isospectral problem, whose compatibility condition gives rise to a type of integrable hierarchy which can be reduced to the Levi hierarchy and the AKNS hierarchy, and so on. A united integrable model obtained by us in the paper is different from that given by Tu and Meng. Specially, the main result in the paper can be reduced to two new various integrable couplings of the Levi hierarchy, from which we again obtain the standard heat equation and a special Newell-Whitehead equation.

Key words: Lie algebra, TAH scheme, DS hierarchy, Heat equation, Newell-Whitehead equation

Introduction

Tu Guizhang [1] proposed by using $2\times 2$ Lie algebras a scheme for generating integrable Hamiltonian hierarchies of evolution equations which was called the Tu scheme[2]. Under the frame of the Tu scheme, many interesting integrable Hamiltonian hierarchies and some corresponding properties were obtained, such as the consequences in [2-8]. Tu and Meng [9] employed the $2\times 2$ Lie algebra

$$l_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, l_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, l_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, l_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the commutative relations

to construct an integrable model which could be reduced to the Levi hierarchy, D-AKNS hierarchy and TD hierarchy.

In the paper, we want to introduce two types of block-matrix Lie algebras for which a united integrable model of the Levi hierarchy and the AKNS hierarchy is obtained. Again, the integrable model is further reduced to two different integrable couplings of the Levi hierarchy, one of them is reduced to the standard heat equation, another one can give a special Newell-Whitehead equation, however, the Newell-Whitehead equation is not integrable, which is an interesting fact.

Two Lie algebras

Tu [1] detailed the Lie algebras for generating the integrable Hamiltonian hierarchies. Based on this, we first present the known simple algebra which consists of the following $2 \times 2$ matrices

$$h_1 = l_1, h_2 = l_3, h_3 = l_4, h_4 = l_2,$$

along with the commutative relations

$$[h_1, h_2] = h_1h_2 - h_2h_1 = h_2[h_1, h_1] = -h_1[h_1, h_1] = 0, [h_2, h_3] = h_1 - h_4,$$

A loop algebra of the above Lie algebra is defined as

$$\tilde{H} = \{h_i(n), i = 1, 2, 3, 4; n \in \mathbb{Z}\}, \quad (1)$$

where $[h_i(m), h_j(n)] = [h_i, h_j]x^{m+n}, m, n \in \mathbb{Z}.$

Denote

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From the above $2 \times 2$ matrices, we introduce the following Lie algebra

$$G_i = \{f_1, \ldots, f_8\},$$

where

$$f_1 = \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix}, f_2 = \begin{pmatrix} h_2 & 0 \\ 0 & h_2 \end{pmatrix}, f_3 = \begin{pmatrix} h_3 & 0 \\ 0 & h_3 \end{pmatrix}, f_4 = \begin{pmatrix} h_4 & 0 \\ 0 & h_4 \end{pmatrix}, f_5 = \begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix},$$

$$f_6 = \begin{pmatrix} 0 & h \\ 0 & h \end{pmatrix}, f_7 = \begin{pmatrix} 0 & e_1 \\ 0 & e_1 \end{pmatrix}, f_8 = \begin{pmatrix} 0 & e_2 \\ 0 & e_2 \end{pmatrix}$$

along with the commutative relations


Define

$$\tilde{G}_i = \{f_i(n), \ldots, f_8(n)\}$$
where
\[ f_i(n) = f_i \lambda^n, \quad [f_i(m), f_j(n)] = [f_i, f_j] \lambda^{m+n}, \quad 1 \leq i, j \leq 8, m, n \in \mathbb{Z}. \]

It is easy to see that \( \widetilde{G}_1 \) is a loop algebra, where \( f_5(n) = \widetilde{G}_1 \) is a pseudo-regular, and satisfies the following properties
(i) \( \widetilde{G}_1 = \ker \text{ad} f_5(n) \oplus \text{im} \text{ad} f_5(n) \),
(ii) \( \ker \text{ad} f_5(n) \) is commutative.

Generally, we have the following proposition [1]:

If \( X \) is a regular element of a semisimple Lie algebra \( G \), then \( R = X \otimes \lambda^n \) is pseudo-regular in \( \widetilde{G} \). In what follows, we establish another Lie algebra with the help of the above \( 2 \times 2 \) matrices:
\[
G_{1} = \{g_1, \ldots, g_7\},
\]
where
\[
g_i = f_i, i = 1, 2, 3, 4; \quad g_5 = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix},
\]
along with the commutative relations as follows
\[
[g_1, g_2] = 0, [g_1, g_3] = g_3, [g_1, g_4] = g_4, [g_2, g_3] = -g_3, [g_2, g_4] = g_4, [g_3, g_4] = g_1 - g_2,
[g_1, g_5] = 0, [g_1, g_6] = g_7, [g_1, g_7] = g_6, [g_2, g_5] = 0, [g_2, g_6] = -g_7, [g_2, g_7] = -g_6,
[g_3, g_5] = -g_6 - g_7, [g_3, g_6] = g_5, [g_3, g_7] = -g_5, [g_4, g_5] = g_6 - g_7, [g_4, g_6] = -g_5,
[g_4, g_7] = -g_5, [g_5, g_6] = 2g_7, [g_5, g_7] = 2g_6, [g_6, g_7] = -2g_5.
\]

If set \( \Delta_1 = \{g_1, g_2, g_3, g_4\}, \Delta_2 = \{g_5, g_6, g_7\} \), we find that
\[
G_2 = \Delta_1 \oplus \Delta_2, [\Delta_1, \Delta_1] \subset \Delta_1, i = 1, 2; [\Delta_1, \Delta_2] \subset \Delta_2.
\]

A corresponding loop algebra is defined as
\[
\widetilde{G}_2 = \{g_i(n), g_2(n), \ldots, g_7(n)\},
\]
where
\[
g_i(n) = g_i \lambda^n, [g_i(m), g_j(n)] = [g_i, g_j] \lambda^{m+n}, 1 \leq i, j \leq 7, m, n \in \mathbb{Z}.
\]
The subalgebras \( \Delta_1 \) and \( \Delta_2 \) are all semisimple.

**An expanding integrable model of the Levi hierarchy and some reductions**

Consider an isospectral Lax pair by the loop algebra \( \widetilde{G}_1 \), \( \varphi_e = U \varphi, \varphi_i = V \varphi \),
\[
U = f_2(1) + (r - q) f_2(0) + q \varphi f_4(0) + u_2 f_4(0) + u_2 f_7(0) + u_2 f_4(0),
\]
\[
V = V_1 f_1(0) + V_2 f_3(0) + V_4 f_4(0) + V_5 f_7(0) + \sum_{i=1}^{8} V_i f_i(0),
\]
where
\[
V_i = \sum_{m=0}^{\infty} V_{im} \lambda^{-m}.
\]

Then the stationary zero curvature equation admits that
The equations (6) are local solvable. If set
\[ V_{1,0} = \alpha = \text{constant}, V_{2,0} = \ldots = V_{k,0} = 0, \]
we can get from (6) that
\[ V_{1,1} = 0, V_{2,1} = -2\alpha q, V_{3,1} = -2\alpha r, V_{2,2} = 2\alpha(q, q(q - r)), V_{3,2} = -2\alpha[r, r(q - r)], \]
\[ V_{1,2} = -2\alpha q, V_{2,3} = 2\alpha[-q_{xx} + (q - r)q_{x} + 2(q - r)q_{x} - (q - r)^2q + 2q^2r], V_{1,3} = -2\alpha u_2, \]
\[ V_{3,1} = -2\alpha u_2, V_{6,1} = 0, V_{7,2} = 2\alpha[u_{3,x} - u_{2}(r - q + 2u_1) + ru_1 - qu_1], \cdots \]
Set
\[ V_{(n)} = \sum_{m=0}^{n} (V_{m_{n}} f_{m}(-m) + V_{2m_{n}} f_{3}(-m) + V_{3m_{n}} f_{4}(-m) + V_{4m_{n}} f_{5}(-m) + \sum_{j=0}^{n} V_{j_{m}} f_{j}(-m))\lambda^n = \lambda^n V - V_{-}^{(n)}, \]
then Eq.(5) can be decomposed into an equivalent equation
\[ -U_{+x} + [U, V_{+}] = V_{-x}^{(n)} - [U, V_{-}]^{(n)}. \] (7)

By following the approach presented in [1], we can find that
\[ -U_{+x} + [U, V_{+}] = V_{2,n+1} f_{3}(0) - V_{3,n+1} f_{4}(0) + V_{8,n+1} f_{1}(0) + V_{7,n+1} f_{0}(0). \]

Take \( V^{(o)} = V_{+}^{(n)} + k_{1} f_{2}(0) + k_{2} f_{0}(0), \) a direct calculation gives
\[ V_{x}^{(o)} - [U, V^{(o)}] = k_{1} f_{2}(0) - (V_{2,n+1} f_{3}(0) + (V_{3,n+1} + r k_{1}) f_{4}(0) - (V_{8,n+1} + u_{3} k_{1} - q k_{2} + r k_{2} - 2u_{2} k_{2}) f_{1}(0) - (V_{7,n+1} + u_{3} k_{1} - 2u_{2} k_{2} - 2u_{2} k_{2} - q k_{2} - r k_{2}) f_{0}(0) + k_{2} f_{0}(0). \]

Thus, the compatibility condition of the Lax pair
\[ \psi_{x} = U\psi, \psi_{t} = V^{(o)}\psi \]
gives rise to the following integrable hierarchy
\[ (r - q)_{t} = k_{1,x}, u_{2,x} = -V_{8,n+1} - u_{3} k_{1} + (q - r + 2u_{3})k_{2}, \]
\[ u_{3,t} = -V_{7,n+1} - u_{3} k_{1} + (2u_{2} + q + r)k_{2}, \]
\[ q_{t} = -V_{2,n+1} - qk_{1} + r k_{1} + u_{2} k_{2} = k_{1,x}, \] (8)

If set \( k_{1} = V_{3,n} - V_{2,n} + 2V_{1,n}, u_{2} = u_{3} = 0, \) Eq.(8) reduces to the well-known Levi hierarchy
\[ \begin{aligned}
q_{t} &= V_{2,n} + rV_{2,n} - qV_{3,n}, \\
r_{t} &= V_{3,n} + qV_{3,n} - rV_{2,n},
\end{aligned} \] (9)

If set \( k_{1} = k_{2} = u_{2} = u_{3} = 0, \) then Eq.(8) reduces to the AKNS hierarchy
\[ \begin{aligned}
q_{t} &= -V_{2,n+1}, \\
r_{t} &= V_{3,n+1},
\end{aligned} \] (10)

Therefore, Eq.(8) is a united integrable model of the Levi hierarchy and the
AKNS hierarchy, and it is different from the united integrable model given by Tu and Meng[9].

If take
\[ k_1 = V_{3n} - V_{2n} + 2V_{1n}, \quad k_2 = -V_{6n}, \]
Eq.(8) becomes
\[ q_1 = -V_{3n} + q(V_{3n} - V_{2n} + 2V_{1n}), \quad r_1 = V_{3n} + r(V_{3n} - V_{2n} + 2V_{1n}), \]
\[ u_{2j} = -V_{8,2j} - u_{1j}(V_{3n} - V_{2n} + 2V_{1n}) + (r - q - 2u_{3j})V_{6n}, \]
\[ u_{3j} = -V_{7,3j} - u_{2j}(V_{3n} - V_{2n} + 2V_{1n}) - (q + r + 2u_{6j})V_{6n}, \]
\[ u_{4j} = -V_{6,4j}. \]

**Cases 1:** \( u_i = V_{6n} = 0 \).

Eq.(11) presents that
\[ q_i = V_{2n,x} + rV_{2n} - qV_{3n} = V_{2n,x} - V_{1n,x}, \quad r_i = V_{3n,x} + qV_{3n} - rV_{2n} = V_{3n,x} + V_{1n,x}, \]
\[ u_{2j} = V_{7n,x} + (r - q - 2u_i)V_{6n} = (u_i + u_j)V_{3n} + (u_j - u_i)V_{2n}, \]
\[ u_{3j} = V_{8n,x} + (q - r - 2u_i)V_{7n} + (u_i - u_j)V_{3n} + (u_j - u_i)V_{2n}. \]

According to the theory on integrable couplings [7,8], Eq.(12) is an integrable coupling of the Levi hierarchy (9). When \( n = 2 \), we can get the coupled part of the Levi equation
\[ u_{2j} = 2\alpha u_{3j} - 2\alpha (u_i + u_j), \quad 2\alpha (u_i + u_j)\]
\[ + 2\alpha (r + q - r^2), \quad 2\alpha (q_i - q^2 + q), \]
\[ u_{3j} = 2\alpha u_{2j} - 2\alpha (u_i q - u_i r), \quad 2\alpha (q_i r - r^2), \]
\[ + 2\alpha (u_i q + qr - r^2) + 2\alpha u_{2j} (q_i - q^2 + qr). \]

Specially, if set \( q = r = 0 \), Eq.(13) reduces to
\[ u_{2j} = 2\alpha u_{3j}, \quad u_{3j} = 2\alpha u_{2j}. \]

Take \( u_2 = u_3 = \nu \), Eq.(14) is just right the well-known linear heat equation
\[ v_t = 2\alpha \nu_{xx}. \]

**Case 2:** \( u_i \neq 0 \).

Eq.(11) becomes that
\[ q_i = V_{2n,x} - V_{1n,x}, \quad r_i = V_{3n,x} + V_{1n,x}, \]
\[ u_{2j} = V_{7n,x} - (q - r + 2u_i)V_{8n} - (u_i + u_j)V_{3n} + (u_j - u_i)V_{2n}, \]
\[ u_{3j} = V_{8n,x} + (q - r - 2u_i)V_{7n} + (u_i - u_j)V_{3n} + (u_j - u_i)V_{2n}. \]

When \( n = 2 \), Eq.(16) reduces to
\[ q_i = 2\alpha q_{3i} - 2\alpha (q_i^2 - 2qr) + r, \quad r_i = -2\alpha r_{3i} - 2\alpha (2qr - r^2), \]
\[ u_{1j} = -2\alpha (qu_{3j} - ru_{2j} - qu_{u_j} - r_{2j}), \]
\[ u_{2j} = 2\alpha (ru_{3j} - ru_{2j} + 2u_{3j} - ru_{4j} + qu_{u_j}), \]
\[ - 2\alpha (q_i r + ru_{3j} - ru_{2j} + 2u_{3j} - ru_{4j} + qu_{u_j}), \]
\[ + 2\alpha (u_i + u_j)(r_i + qr - r^2) - 2\alpha (u_i + u_j)(q_i - q^2 + qr), \]
\[ u_{3j} = 2\alpha (u_i + u_j)(r_i + qr - r^2) - 2\alpha (u_i + u_j)(q_i - q^2 + qr). \]

If set \( q = r = 0 \), Eq.(17) gives that
\begin{align}
\begin{cases}
u_{1,i} = 4\alpha u_{2,i} u_{2,j} + u_{2,j} = 2\alpha u_{3,xx} - 4\alpha (u_{i} u_{i} )_{x} + 8\alpha u_{1}^{2} u_{3}, \\
u_{2,j} = 2\alpha u_{2,xx} - 4\alpha (u_{i} u_{i} )_{x} - 8\alpha u_{1}^{2} u_{2},
\end{cases}
\end{align}

(18)

The first equation in (18) is a conserved form, and later two are linear with respect to the variables \( u_{2} \) and \( u_{3} \), respectively.

If set \( u_{2} + u_{3} = u, u_{2} - u_{3} = v, \alpha = \frac{1}{2} \), Eq.(18) becomes

\begin{align}
\begin{cases}
u_{1,i} = u_{1,i} - 2(u_{i} u_{i} )_{x} - 4u_{1}^{2} v, \\
u_{2,j} = -v_{x} - 2(u_{i} v_{i} )_{x} + 4u_{1}^{2} u_{2},
\end{cases}
\end{align}

where the variable \( u_{1} \) satisfies that \( u_{1,j} = \frac{1}{2}(u + v)(u_{i} + v_{i}). \)

**Case3:** \( k_{1} = k_{2} = 0 \). Eq.(8) just reduces to an integrable coupling of the AKNS hierarchy:

\begin{align}
\begin{cases}
u_{1,i} = -V_{2,xx+1}, r_{i} = V_{3,xxx+1}, \\
u_{2,j} = -V_{6,xxx+1}, u_{3,j} = -V_{7,xxx+1}.
\end{cases}
\end{align}

(19)

Eq.(19) can be written as

\begin{align}
\begin{pmatrix}
u_{1,i} \\
u_{2,j}
\end{pmatrix} =
\begin{pmatrix}
u_{2,xxx} \\
u_{6,xxx}
\end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = J \begin{pmatrix} V_{3,xxx} \\
V_{7,xxx}
\end{pmatrix} = JL \begin{pmatrix} V_{3,xxx} \\
V_{7,xxx}
\end{pmatrix},
\end{align}

(20)

where

\begin{align}
J &= \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad L = \begin{pmatrix}
\partial - 2r\partial^{-1}q & 2r\partial^{-1}r & 0 & 0 \\
-2q\partial^{-1}q & -\partial + 2q\partial^{-1}r & 0 & 0 \\
-2u_{1}\partial^{-1}q & 2u_{1}\partial^{-1}r & 0 & 0 \\
2u_{1}\partial^{-1}q & -2u_{1}\partial^{-1}r & 0 & 0
\end{pmatrix},
\end{align}

when \( u_{2} = u_{3} = 0 \), the above \( J \) and \( L \) reduce to the case of the AKNS hierarchy.

Another expanding integrable model of the Levi hierarchy and some reductions

Consider an isospectral problem by the loop algebra \( \tilde{G}_{2} \) as follows

\[ \psi_{x} = U\psi, \psi_{t} = V\psi, \]

(21)

where

\begin{align}
U &= g_{2}(1) + (r-q)g_{2}(0) + qg_{3}(0) + rg_{4}(0) + s_{i}g_{0}(0) + s_{j}g_{1}(0), \\
V &= V_{1}g_{1}(0) + V_{2}g_{2}(0) + V_{3}g_{3}(0) + V_{4}g_{4}(0) + V_{5}g_{5}(0) + V_{6}g_{6}(0) + V_{7}g_{7}(0),
\end{align}

(22)\hspace{1cm}(23)

Where

\[ V_{i} = \sum_{m=0}^{n} V_{im} \lambda^{-m}, i = 1, 2, \ldots, 7. \]

Similar to the previous discussion, we can get a recursion relation among \( V_{im} \) as follows
\[ V_{1m} = qV_{2m} - rV_{3m}, \quad V_{2m} = -(r-q)V_{2m} - 2qV_{1m}, \]
\[ V_{3m} = V_{3m} - (r-q)V_{3m} - 2rV_{1m}, \]
\[ V_{5m} = -(q+r+2s)V_{7m} + (q-r+2s)V_{5m} + (s_1 + s_2)V_{5m} + (s_2 - s_1)V_{2m}, \]
\[ V_{7m} = -V_{6m} + (q-r)V_{7m} + (r-q)V_{5m} - 2s_2V_{3m} - 2s_1V_{1m}, \]
\[ V_{6m} = -V_{7m} + (q-r)V_{6m} - (q+r)V_{5m} - 2s_1V_{3m} - 2s_2V_{1m}. \]

If denote
\[
V^{(n)} = \sum_{m=0}^{n} \left( V_{1m}g_1(n-m) + V_{2m}g_2(n-m) + V_{3m}g_3(n-m) + V_{4m}g_4(n-m) + \sum_{i=5}^{7} V_{5m}g_i(n-m) \right) + (V_{3n} - 2V_{2n} + 2V_{1n})g_2(0)
\]

we can obtain that
\[
V^{(n)} = -\left[ U, V^{(n)} \right] = \left\{ \left[ V_{2n+1} + q(V_{3n} - V_{2n} + 2V_{1n}) \right] g_3(0) + \left[ V_{3n+1} + r(V_{3n} - V_{2n} + 2V_{1n}) \right] g_4(0) + \left( V_{3n} - 2V_{2n} + 2V_{1n} \right) g_2(0) - \left[ V_{6n+1} + s(V_{3n} - V_{2n} + 2V_{1n}) \right] g_5(0) - \left[ V_{7n+1} + s_2(V_{3n} - V_{2n} + 2V_{1n}) \right] g_6(0) \right\}
\]

Therefore, the zero curvature equation \[ U_i - V^{(n)} = 0 \] admits that
\[
\begin{align*}
q_i &= -V_{2n+1} - q(V_{3n} - V_{2n} + 2V_{1n}), \quad r_i = V_{3n+1} + r(V_{3n} - V_{2n} + 2V_{1n}), \\
q_{s_1} &= -V_{6n+1} + s(V_{3n} - V_{2n} + 2V_{1n}) + (q+r+s)V_{5n} + s_1V_{3n} + s_2V_{2n}, \\
q_{s_2} &= -V_{7n+1} - s_2(V_{3n} - V_{2n} + 2V_{1n}) = V_{6n} + (q-r)V_{7n} + (r-q)V_{5n} - 2s_2V_{3n} - 2s_1V_{2n}.
\end{align*}
\]

When \( s_1 = s_2 = 0 \), Eq.(25) reduces to the Levi hierarchy. According to the theory on integrable couplings, (25) is one integrable coupling of the Levi hierarchy, which is different from the first integrable coupling of the Levi hierarchy (11). We can see this point from their reductions.

Set \( V_{10} = \alpha, V_{20} = \ldots = V_{70} = 0 \), we have from (24) that
\[
\begin{align*}
V_{10} &= -2\alpha s_2, \quad V_{20} = -2\alpha s_1, \quad V_{30} = 0, \quad V_{40} = 2\alpha s_2 - 2\alpha(q-r)s_1, \\
V_{50} &= 2\alpha s_1, \quad V_{60} = 2\alpha(q-r)s_2, \quad V_{70} = 2\alpha(-qs_1 - rs_1 - rs_2 + qs_2 - s_1^2 + s_2^2), \ldots
\end{align*}
\]

When \( n = 2 \), Eq.(25) reduces to
\[
\begin{align*}
q_i &= 2\alpha q - 2\alpha(q^2 - 2qr), \quad r_i = -2\alpha r - 2\alpha(2qr - r^2), \\
s_{i,1} &= 2\alpha s_{i,1} - 2\alpha(s_i(q-r)) + 2\alpha(r-q)s_{i,1} + 2\alpha s_{i,2} + 2\alpha s_{i,3} + 2\alpha(r-q)^2s_i, \\
s_{i,2} &= 2\alpha s_{i,2} - 2\alpha(s_i(q-r)), \quad s_{i,3} = 2\alpha r - 2\alpha(q-r)s_{i,3} + 2\alpha s_{i,4} + 2\alpha r - 2\alpha s_{i,5} + 2\alpha(r-q)^2s_i,
\end{align*}
\]

Eq.(26) is different from Eq.(17). Specially, when set \( q = r = 0 \), Eq.(26) gives that
\[
\begin{align*}
s_{i,1} &= 2\alpha s_{i,1} - 4\alpha s_i s_1^2 - s_1^2, \\
s_{i,2} &= 2\alpha s_{i,2} - 4\alpha s_i s_2^2 - s_2^2,
\end{align*}
\]

which is various from Eq.(16) which was reduced from the integrable coupling (11). Hence, the integrable coupling of the Levi hierarchy (25) is really different from the integrable
We see that when $s_1 = s_2$, Eq.(27) reduces to Eq.(16). When set $s_1 = is_2, s_2 = v$, Eq.(27) gives

$$v_t = v_{xx} + 4v^3,$$

(28)

which is a special Newell-Whitehead equation.

This is an integrable equation, but the Newell-Whitehead equation:

$$u_t = u_{xx} + u - u^3,$$

(29)

It is not integrable. Therefore, Eq.(28) could possess the similar travelling wave solutions.

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