NUMERICAL SOLUTIONS OF A CLASS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS IN HERMITE SERIES

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The purpose of this paper is to present a Hermite polynomial approach for solving a high-order ordinary differential equation with nonlinear terms under mixed conditions. The method we used is a matrix method based on collocation points together with truncated Hermite series and reduces the solution of equation to solution of a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Hermite coefficients. In addition, to illustrate the validity and applicability of the method, some numerical examples together with residual error analysis are performed and the obtained results are compared with the existing result in literature.

Key Words: *Hermite polynomials and series, nonlinear ordinary differential equations, matrix and collocation method.*

1. Introduction

Nonlinear ordinary differential equations play an important role in many physical and technical applications and are essential tools for modelling many physical situations [1-3,14,15,19,20] such as chemical reactions, spring-mass systems, bending of beams and so forth. Most of these type equations have no analytical solution and numerical methods are required to obtain approximate solutions[1-3]. Recently, Sezer and coworkers[4,6,7,16,17,18] have presented the matrix and collocation methods for solving linear and nonlinear differential and integral equations in terms of special polynomials. In this study, we develop the mentioned matrix and collocation for solving the mth-order ordinary differential equation with nonlinear terms in the form

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x)y^{(p)}(x)y^{(q)}(x) = g(x) \qquad (1)$$

under the mixed conditions

$$\sum_{k=0}^{m-1} \left(a_{kj}y^{(k)}(a) + b_{kj}y^{(k)}(b) + c_{kj}y^{(k)}(c)\right) = \lambda_j \qquad (2)$$

and apply to find the approximate solution in the truncated Hermite series

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n H_n(x) \qquad , -\infty < a \le x \le b < \infty\,. \qquad (3)$$

Here $P_k(x), Q_{pq}(x)\ and\ g(x)$ are the functions defined on the interval $-\infty < a \le x \le b < \infty$; $a_{kj}, b_{kj}, c_{kj}\ and\ \lambda_j$ are appropriate constants; $a_n\ (n = 0,1, \dots, N \ge m)$ are unknown Hermite coefficients to be determined; $H_n(x)$, $n = 0,1, \dots, N$, are the Hermite polynomials defined by

$$H_n(x) = \sum_{k=0}^{[\![\frac{n}{2}]\!]} \frac{(-1)^k n!\, 2^{n-2k}}{(n-2k)!\, k!} x^{n-2k} \tag{4}$$

These polynomials are orthogonal on $(-\infty, \infty)$ with respect to weight function $w(x) = e^{-x^2}$ and generally defined by Rodrigues

$$H_n(x) = (-1)^n e^{x^2} (e^{-x^2})^{(n)}. \tag{5}$$

The first few Hermite polynomials can be given as, explicit expressions from (4) to (5),

$H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12x, H_4(x) = 16x^4 - 48x^2 + 12.$

Also, the sequence of Hermite polynomials $H_n(x)$ satisfies the recurrence relation

$$\begin{gathered} H_n'(x) = 2nH_{n-1}(x) \\ \text{with } H_0'(x) = 0\ , H_1'(x) = 2H_0(x) \end{gathered} \tag{6}$$

2. Fundamental Matrix Relations

Let us consider the mth-order nonlinear differential Eq.(1) and find the matrix forms of each term in the equation. We firstly convert the solution $y(x)$ by a truncated Hermite series (3) to matrix form

$$y(x) \cong y_N(x) = H(x)A \tag{7}$$

where

$$\begin{gathered} H(x) = [H_0(x) \quad H_1(x) \quad \cdots \quad H_n(x)] \\ A = [a_0 \quad a_1 \quad \cdots \quad a_N]^T . \end{gathered}$$

Also, by using the expression (6), for $n = 0,1, \dots, N$, we find the recurrance relation between the matrix $H(x)$ and its derivative $H^{(k)}(x)$ as

$$H^{(k)}(x) = H(x)M^k, k = 0,1, \dots, m \tag{8}$$

where

$$M = \begin{bmatrix} 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, M^0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

By using (7) and (8), we have the matrix relation

$$y^{(k)}(x) \cong y_N^{(k)}(x) = H^{(k)}(x)A$$
$$= H(x)M^kA \quad , k = 0,1,\dots,m. \tag{9}$$

In addition, we can obtain the following matrix forms of the expressions $\left(y^{(0)}(x)\right)^2$, $y^{(1)}(x)y^{(0)}(x)$, $\left(y^{(1)}(x)\right)^2$, $y^{(2)}(x)y^{(1)}(x)$, $y^{(2)}(x)y^{(0)}(x)$ and $\left(y^{(2)}(x)\right)^2$, by means of similar operations as (7)-(9) [7-8] :

$$\left(y^{(0)}(x)\right)^2 = H(x)\bar{H}(x)\bar{A} \tag{10-1}$$

$$y^{(1)}(x)y^{(0)}(x) = H(x)M\bar{H}(x)\bar{A} \tag{10-2}$$

$$\left(y^{(1)}(x)\right)^2 = H(x)M\bar{H}(x)\bar{M}\bar{A} \tag{10-3}$$

$$y^{(2)}(x)y^{(1)}(x) = H(x)M^2\bar{H}(x)\bar{M}\bar{A} \tag{10-4}$$

$$y^{(2)}(x)y^{(0)}(x) = H(x)M^2\bar{H}(x)\bar{A} \tag{10-5}$$

$$\left(y^{(2)}(x)\right)^2 = H(x)M^2\bar{H}(x)\overline{M^2}\bar{A} \tag{10-6}$$

where

$$y^{(0)}(x) = y(x), \qquad y^{(1)}(x) = y'(x), \qquad y^{(2)}(x) = y''(x)$$
$$\bar{H}(x) = diag[H(x) \quad H(x) \quad \dots \quad H(x)]\ ;\ \bar{H}(x)_{(N+1)x(N+1)^2}$$
$$\bar{M} = diag[M \quad M \quad \dots \quad M]\ ;\ \bar{M}_{(N+1)^2x(N+1)^2}$$
$$\overline{M^2} = diag[M^2 \quad M^2 \quad \dots \quad M^2]\ ;\ \overline{M^2}_{(N+1)^2x(N+1)^2}$$
$$A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}_{(N+1)x1} , \qquad \bar{A} = \begin{bmatrix} a_0A \\ a_1A \\ \vdots \\ a_NA \end{bmatrix}_{(N+1)^2x1} .$$

Now we can define the collocation points as

$$x_i = a + \frac{b-a}{N}i \;\; i = 0,1,\dots,N \;\; ; \; a \le x_0 < x_1 < \dots < x_N = b. \tag{11}$$

By substituting the collocation points (11) into Eq. (1), we get the system of matrix equations, for $i = 0,1,\dots,N$

$$\sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_i)y^{(p)}(x_i)y^{(q)}(x_i) = \; g(x_i)$$

or the compact form

$$\sum_{k=0}^{m} P_k Y^{(k)} + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq} Y^{(p,q)} = G \tag{12}$$

where

$$P_k = diag[P_k(x_0) \quad P_k(x_1) \quad \ldots \quad P_k(x_N)]$$

$$Q_{pq} = diag[Q_{pq}(x_0) \quad Q_{pq}(x_1) \quad \ldots \quad Q_{pq}(x_N)]$$

$$Y^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix}, \quad Y^{(p,q)} = \begin{bmatrix} y^{(p)}(x_0)y^{(q)}(x_0) \\ y^{(p)}(x_1)y^{(q)}(x_1) \\ \vdots \\ y^{(p)}(x_N)y^{(q)}(x_N) \end{bmatrix}, \qquad G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

By putting the collocation points (11) into the matrix relation (9), we obtain the matrix equation, for $i = 0,1,\ldots,N$,

$$y^{(k)}(x_i) = H(x_i)M^k A \;\; \Rightarrow \;\; Y^{(k)} = HM^k A \tag{13}$$

where

$$H = \begin{bmatrix} H(x_0) \\ H(x_1) \\ \vdots \\ H(x_N) \end{bmatrix} = \begin{bmatrix} H_0(x_0) & H_1(x_0) & \ldots & H_N(x_0) \\ H_0(x_1) & H_1(x_1) & \ldots & H_N(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(x_N) & H_1(x_N) & \ldots & H_N(x_N) \end{bmatrix}.$$

On the other hand, we can write the nonlinear part of Eq.(12) as

$$\sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq} Y^{(p,q)} = Q_{00}Y^{(0,0)} + Q_{10}Y^{(1,0)} + Q_{11}Y^{(1,1)} + Q_{20}Y^{(2,0)} + Q_{21}Y^{(2,1)} + Q_{22}Y^{(2,2)}. \tag{14}$$

Besides, by substituting the collocation points (11) into the relations (10-1)-(10-6), the matrices $Y^{(0,0)}$, $Y^{(1,0)}$, $Y^{(1,1)}$, $Y^{(2,0)}$, $Y^{(2,1)}$ and $Y^{(2,2)}$ are obtained as follows:

$$\begin{aligned} &Y^{(0,0)} = H_{0,0}^{*}\bar{A} \qquad Y^{(1,0)} = H_{1,0}^{*}\bar{A} \qquad Y^{(1,1)} = H_{1,1}^{*}\bar{A} \\ &Y^{(2,0)} = H_{2,0}^{*}\bar{A} \qquad Y^{(2,1)} = H_{2,1}^{*}\bar{A} \qquad Y^{(2,2)} = H_{2,2}^{*}\bar{A} \end{aligned} \tag{15}$$

where

$$H_{0,0}^{*} = \begin{bmatrix} H(x_0)\bar{H}(x_0) \\ H(x_1)\bar{H}(x_1) \\ \vdots \\ H(x_N)\bar{H}(x_N) \end{bmatrix}, \qquad H_{1,0}^{*} = \begin{bmatrix} H(x_0)M\bar{H}(x_0) \\ H(x_1)M\bar{H}(x_1) \\ \vdots \\ H(x_N)M\bar{H}(x_N) \end{bmatrix}$$

$$H_{1,1}^{*} = \begin{bmatrix} H(x_0)M\overline{H}(x_0)\overline{M} \\ H(x_1)M\overline{H}(x_1)\overline{M} \\ \vdots \\ H(x_N)M\overline{H}(x_N)\overline{M} \end{bmatrix}, \quad H_{2,0}^{*} = \begin{bmatrix} H(x_0)M^2\overline{H}(x_0) \\ H(x_1)M^2\overline{H}(x_1) \\ \vdots \\ H(x_N)M^2\overline{H}(x_N) \end{bmatrix}$$

$$H_{2,1}^{*} = \begin{bmatrix} H(x_0)M^2H(x_0)\overline{M} \\ H(x_1)M^2\overline{H}(x_1)\overline{M} \\ \vdots \\ H(x_N)M^2\overline{H}(x_N)\overline{M} \end{bmatrix}, \quad H_{2,2}^{*} = \begin{bmatrix} H(x_0)M^2\overline{H}(x_0)\overline{M^2} \\ H(x_1)M^2\overline{H}(x_1)\overline{M^2} \\ \vdots \\ H(x_N)M^2\overline{H}(x_N)\overline{M^2} \end{bmatrix}$$

$$\bar{A} = [a_0A \quad a_1A \quad \ldots \quad a_NA]^T.$$

3. Hermite Matrix Collocation Method

We now are ready to construct the fundamental matrix equation corresponding to Eq.(1). For this purpose, substituting the matrix relations (13)-(15) into Eq.(12), we obtain the fundamental matrix equation

$$\sum_{k=0}^{m} P_k H M^{(k)} A + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq} H_{(p,q)}^{*} \bar{A} = G$$

or the compact form

$$WA + V\bar{A} = G \tag{16}$$

where

$$W = \sum_{k=0}^{m} P_k H M^k = [w_{ij}]; \quad i,j = 0,1,\ldots,N$$

$$V = \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq} H_{p,q}^{*} = [v_{mn}]; \quad m = 0,1,\ldots,N; n = 0,1,\ldots,(N+1)^2 - 1 .$$

Also we can write the matrix equation (16) in the augmented matrix form

$$[W;V:G] \tag{17}$$

or clearly

$$[W;V:G] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N}; & v_{00} & v_{01} & \cdots & v_{0(N+1)^2-1} : & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N}; & v_{10} & v_{11} & \cdots & v_{1(N+1)^2-1} : & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN}; & v_{N0} & v_{N1} & \cdots & v_{N(N+1)^2-1} : & g(x_N) \end{bmatrix}.$$

Besides, by using the matrix relation (9), the matrix relation for the mixed conditions (2) is obtained

$$\sum_{k=0}^{m-1} \big(a_{kj}H(a) + b_{kj}H(b) + c_{kj}H(c)\big) M^k A = [\lambda_j]$$

or briefly

$$UA + O^*\bar{A} = \lambda \quad \Rightarrow \quad [U; O^*: \lambda] \tag{18}$$

or clearly

$$[U; O^*: \lambda] = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} & ; & 0 & 0 & \cdots & 0: & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & 0 & 0 & \cdots & 0: & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & 0 & 0 & \cdots & 0: & \lambda_{m-1} \end{bmatrix}$$

where

$$U = [u_{j0} \quad u_{j1} \quad \cdots \quad u_{jN}]; \quad j = 0,1,\ldots,m-1$$

$$= \sum_{k=0}^{m-1} \left(a_{kj}X(a) + b_{kj}X(b) + c_{kj}X(c)\right)B^k$$

$$\lambda = [\lambda_0 \quad \lambda_1 \quad \cdots \quad \lambda_{m-1}]^T$$

$$O^* = [0 \quad 0 \quad \cdots \quad 0] \ (zero\ matrix)$$

Consequenlty, to find Hermite coefficients a_n $(n = 0,1,\ldots,N)$ related with the approximate solution (3) of the problem (1)-(2), by replacing the m row matrices (18) by the last m rows(or any m rows) of the augmented matrix(17), we obtain the resulting matrix

$$\left[\widetilde{W}; \tilde{V}: \tilde{G}\right] =$$

$$\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N}; & v_{00} & v_{01} & \cdots & v_{0(N+1)^2-1}; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N}; & v_{10} & v_{11} & \cdots & v_{1(N+1)^2-1}; & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N}; & v_{N-m,0} & v_{N-m,1} & \cdots & v_{N-m,(N+1)^2-1}; & g(x_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & 0 & 0 & \cdots & 0 & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & 0 & 0 & \cdots & 0 & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & 0 & 0 & \cdots & 0 & \lambda_{m-1} \end{bmatrix}.$$

From this nonlinear system, that is the matrix equation $\widetilde{W}A + \tilde{V}\bar{A} = G$, the unknown coefficients a_n $(n = 0,1,\ldots,N)$ are determined; therefore the truncated Hermite series solution (3) is obtained as

$$y_N(x) = \sum_{n=0}^{N} a_n H_n(x) \ .$$

4. Accuracy of Solutions and Residual Error Estimation

We can check the accuracy of the obtained solutions as follows [8]. Since the truncated Hermite series (3) is an approximate solution of Eq.(1). When the solution $y_N(x)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is, for

$x = x_l \epsilon [a,b], \ l = 0,1,2,\ldots$

$$R_N(x_l) = \sum_{k=0}^{m} P_k(x_l) y^{(k)}(x_l) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_l) y^{(p)}(x_l) y^{(q)}(x_l) - g(x_l) \cong 0$$

or

$$R_N(x_l) \leq 10^{-k_l}\,, (k_l \text{ is any positive numbers})\,.$$

If $max\, 10^{-k_l} = 10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $R_N(x_l)$ at each of the points becomes smaller than the prescribed 10^{-k}. Therefore, if $R_N(x_l) \to 0$ when N is sufficiently large enough, then the error decreases.

On the other hand, by means of the residual function $R_N(x)$ and the mean value of the function $|R_N(x)|$ on the interval $[a, b]$, the accuracy of the solution can be controlled and the error can be estimated [9].

Thus, the upper bound of the mean error $\bar{R}_n$ as follows:

$$\left|\int_a^b R_N(x)\,dx\right| \leq \int_a^b |R_N(x)|\,dx$$

and

$$\left|\int_a^b R_N(x)dx\right| = (b-a)|R_N(c)|\ , \qquad a \leq c \leq b$$

$$\left|\int_a^b R_N(x)dx\right| = (b-a)|R_N(c)| \leq \int_a^b |R_N(x)|\,dx$$

$$|R_N(c)| \leq \frac{\int_a^b |R_N(x)|\,dx}{b-a} = \bar{R}_n.$$

Moreover we use different error norms for measuring errors. These are defined as follows:

1- $L_2 = \left(\sum_{i=0}^{n}(e_i)^2\right)^{1/2}$

2- $L_\infty = max(e_i),\ 0 \leq i \leq n$

3- $RMS = \sqrt{\frac{\sum_{i=1}^{n}(e_i)^2}{n+1}}$

where $e_i = |y(x_i) - y_N(x_i)|$ also y and y_N are the exact and approximate solutions of the problem, respectively [11].

5. Numerical Example

The method of this study is useful in finding the solutions of a class of nonlinear equations in terms of Hermite polynomials and the accuracy. We illustrate it by the several numerical examples and perform all of them on the computer using a program written separately in MATLAB R2017b.

5.1. Example 1.

Consider the second order nonlinear differential equation

$$y''(x) - 2y'(x) + y(x) + y^2(x) - y''(x)y(x) = g(x) \tag{19}$$

with the initial and boundary conditions

$$y(0) = 3,\ y'(0) = 2$$
$$0 \leq x \leq 1 \quad (20)$$

where $g(x) = 2 + 2e^x$.

While the exact solution is $y(x) = 1 + 2e^x$, the proposed method is applied and the approximate solutions of (19) under the conditions (20) are obtained as $y_2(x) = 3 + 2x + x^2$, $y_3(x) = 3 + 2x + x^2 + 0.3944x^3$, $y_4(x) = 3 + 2x + x^2 + 0.32544x^3 + 0.10704x^4$, $y_5(x) = 3 + 2x + x^2 + 0.336336x^3 + 0.074608x^4 + 0.025772x^5$ for $N = 2 - 5$, respectively.

In Table 1 and Table 2, we see that absolute errors and L_2, L_∞ and RMS errors are calculated for $N = 2, 3, 4, 5$, respectively. In table 3, the exact solution of the problem and the the approximate solutions of the problem obtained with Taylor matrix method [12] and the proposed method on $x \in [\mathbf{0}, \mathbf{1}]$ for $N = 5$, are presented.

Table 1: Absolute error of Example 1 for $N = 2 - 5$ and $h = 0.1$.

		Absolute Errors		
x_l	e_2	e_3	e_4	e_5
0	0	0	0	0
0.1	3.4184e-04	5.2564e-05	5.6922e-06	2.21837e-06
0.2	2.8000e-03	3.4968e-04	3.0732e-05	1.27917e-05
0.3	9.7000e-03	9.3118e-04	6.3711e-05	3.04095e-05
0.4	2.3600e-02	1.6100e-03	8.1011e-05	4.99869e-05
0.5	4.7440e-02	1.9000e-03	7.2541e-05	6.78585e-05
0.6	8.4200e-02	9.5280e-04	7.0177e-05	8.42649e-05
0.7	1.3751e-01	2.2000e-03	1.7919e-04	1.02848e-04
0.8	2.1110e-01	9.1000e-03	6.1299e-04	1.26842e-04
0.9	3.3092e-01	2.1700e-02	1.7000e-03	1.51611e-04
1	4.3664e-01	4.22000e02	4.1000e-03	1.53143e-04

Table 2: L_2 ,L_∞ and RMS errors of Example 1 for $N = 2 - 5$

N	L_2-*error*	L_∞-*error*	RMS-*error*
2	5.997233e-01	4.365636e-01	2.998616e-01
3	4.842227e-02	4.216365e-02	2.421113e-02
4	4.483755e-03	4.083656e-03	2.241877e-03
5	2.973264e-04	1.531430e-04	1.486632e-04

Table 3: Comporison of the solutions of Example 1 for $N = 5$ and $h = 0.1$

	Exact	**Taylor**	**Proposed**

x_l	Solution	Matrix Method	Method
0	3	3	3
0.1	3.21034	3.21034	3.21034
0.2	3.44280	3.44280	3.44281
0.3	3.69971	3.69972	3.69974
0.4	3.98364	3.98365	3.98369
0.5	4.29744	4.29744	4.29751
0.6	4.64423	4.64424	4.64432
0.7	5.02750	5.02751	5.02760
0.8	5.45108	5.45107	5.45120
0.9	5.91920	5.91913	5.91935
1	6.43656	6.43630	6.43671

5.2. Example 2

Consider the second order nonlinear differential equation of the form

$$\frac{d^2y}{d^2x} + \left(\frac{dy}{dx}\right)^2 + y^2(x) = 1 - \sin(x) \tag{21}$$

with the initial conditions

$$y(0) = 0, y'(0) = 1 \text{ [10]}. \tag{22}$$

Similarly, while the exact solution is $y(x) = sinx$, the approximate solutions of (21) under the conditions (22) are obtained as $y_2(x) = x$, $y_3(x) = x - 0.163980335781936x^3$, $y_4(x) = x - 0.1697171792058040x^3 + 0.009876360178284074x^4$, $y_5(x) = x - 0.166728x^3 + 0.000304x^4 + 0.007936x^5$ for $N = 2 - 5$, respectively.

Considering $N = 2 - 5$, the obtained approximate solutions are compared with the exact solution in Fig.1 and the absolute errors are demonstrated in Tab. 4. In Table 5, for $N = 5$, The solutions, exact and obtained with Taylor matrix method and suggested method, are compared [12].

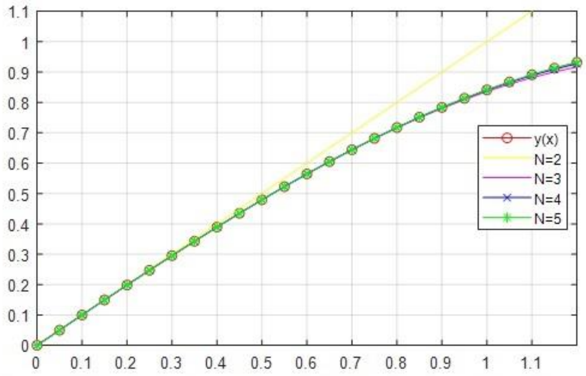

Fig.1 Comporison of the Hermite polynomial solutions and exact solution of Example 2 for $N = 2 - 5$

Table 4: Absolute errors of Example 2 for $N = 2 - 5$ and $h = 0.1$

	Absolute errors			
x_n	e_2	e_3	e_4	e_5
0	0	0	0	0
0.1	1.665833e-04	2.603017e-06	2.146190e-06	3.488682e-08
0.2	1.330669e-03	1.882651e-05	1.126605e-05	1.28875e-07
0.3	4.479793e-03	5.232427e-05	2.257198e-05	1.157813e-07
0.4	1.058165e-02	8.691620e-05	2.740695e-05	1.127313e-07
0.5	2.057446e-02	7.691942e-05	2.291349 e-05	4.613957e-07
0.6	3.535752e-02	6.222592e-05	2.140782 e-05	7.803649e-07
0.7	5.578231e-02	4.629424e-04	5.936562 e-05	1.402682e-06
0.8	8.264390e-02	1.314022e-03	2.059295 e-04	4.15998e-06
0.9	1.166730e-01	2.868574e-03	5.708533 e-04	1.396141e-05
1	1.585290e-01	5.451320e-03	1.311803 e-03	4.101519e-05

Table 5: Comporison of the solutions of Example 2 for $N = 5$ and $h = 0.1$

x_l	Exact Solution	Taylor Matrix Method	Proposed Method
0	0	0	0
0.1	0.0998334	0.0998333	0.0998333
0.2	0.1986693	0.1986691	0.1986692
0.3	0.2955202	0.2955199	0.2955201
0.4	0.3894183	0.3894181	0.3894184
0.5	0.4794255	0.4794253	0.4794260
0.6	0.5646424	0.5646418	0.5646432
0.7	0.6442176	0.6442164	0.6442190
0.8	0.7173560	0.7173556	0.7173602
0.9	0.7833269	0.7833332	0.7833408
1	0.8414709	0.8415000	0.8415120

The upper bound of the mean error ,$\bar{R}_n$, of Example 2 can be calculated as $\bar{R}_2 = 7.93031e - 01$, $\bar{R}_3 = 4.124611e - 02, \bar{R}_4 = 1.305709e - 02, \bar{R}_5 = 2.25636e - 03$ as in the method given and $\bar{R}_n$ of Example 2 is illustrated in Fig.2 for $N = 2 - 5$.

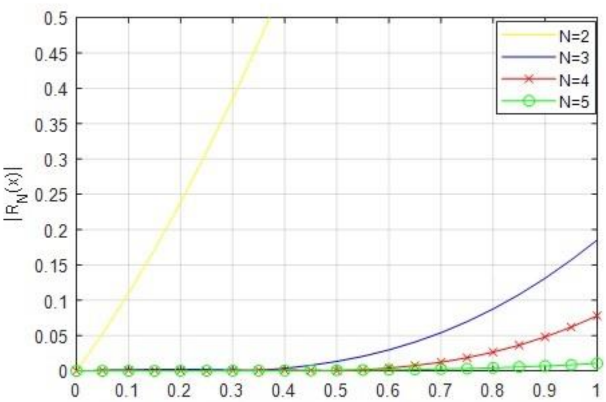

Fig.2 The residuel error functions of Example 2 for $N = 2 - 5$

5.3. Example 3

Consider the following differential equation

$$y'(x) - 2y(x) + y^2(x) = g(x)$$
$$g(x) = \begin{cases} (x-1)^2, & 0 \le x \le 1/2 \\ x^2 - 2\ \ , & 1/2 < x \le 1 \end{cases}. \tag{23}$$

It is solved with the suggested method for the boundary condition

$$y(0) = 0. \tag{24}$$

For $0 \le x \le 1/2$, $P_0(x) = -2$, $P_1(x) = 1$, $Q_{00}(x) = 1$.

For $g_1(x) = (x-1)^2$, $N = 2$ and the ordering points are $x_0 = 0, x_1 = 1/4\,, x_2 = 1/2$ for the interval $0 \le x \le 1/2$.

The solution is investigated in the following form

$$y(x) \cong \sum_{n=0}^{1} a_n x^n.$$

Using our proposed method, the fundamental matrix equation is obtained as

$$P_1 HMA + P_0 HA + Q_{00} H_{00}^* \bar{A} = G.$$

Here,

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ H = \begin{bmatrix} H(0) \\ H(1/4) \\ H(1/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & \frac{1}{2} & -\frac{7}{4} \\ 1 & 1 & -1 \end{bmatrix}$$

$$Q_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ M = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \ P_0 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \ A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$H_{00}^* = \begin{bmatrix} H(0)\bar{H}(0) \\ H(1/4)\bar{H}(1/4) \\ H(1/2)\bar{H}(1/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \\ 1 & 1/2 & -7/4 & 1/2 & 1/4 & -7/8 & -7/4 & -7/8 & 49/16 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

$$\bar{H}(x) = diag[H(x) \quad H(x) \quad H(x)]$$

$$A = [a_0 \quad a_1 \quad a_2]^T, \bar{A} = \ [a_0 A \quad a_1 A \quad a_2 A]^T$$

$$G_1(x) = [g_1(0) \quad g_1(1/4) \quad g_1(1/2)]^T = [1 \quad 9/16 \quad 1/4]^T.$$

The following fundamental matrix equation is obtained as

$$\begin{bmatrix} -2 & 2 & 4 \\ -2 & 1 & \frac{11}{2} \\ 2 & 0 & 6 \end{bmatrix} A + \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \\ 1 & \frac{1}{2} & -\frac{7}{4} & \frac{1}{2} & \frac{1}{4} & -\frac{7}{16} & -\frac{7}{8} & -\frac{7}{16} & \frac{49}{64} \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix} \bar{A} \ =$$

$$\begin{bmatrix} 1 \\ 9/16 \\ 1/4 \end{bmatrix}.$$

$[1 \quad 0 \quad -2]A + [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\bar{A} = 0$ is obtained for the boundary condition $y(0) = 0$ and using the proposed method, the desired matrix equation

$$\begin{bmatrix} -2 & 2 & 4 \\ 1 & 0 & -2 \\ -2 & 0 & 6 \end{bmatrix} A + \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix} \bar{A} = \begin{bmatrix} 1 \\ 0 \\ 1/4 \end{bmatrix}$$

is constructed .

Thus, Hermite's coefficients are found as $a_0 = 0, a_1 = 1/2, a_2 = 0$. For $0 \le x \le 1/2$, the solution

$$y(x) = 1/2x\,, \qquad 0 \le x \le 1/2$$

is obtained.

For $g_1(x) = x^2 - 2$, $N = 2$ and the ordering points are $x_0 = 1/2\,, x_1 = 3/4\,, x_2 = 1$ for the interval $1/2 < x \le 1$.

$y(1/2) = 1/4$ is obtained for $x_0 = 1/2$ by considering $y(x) = 1/2x$. Thus, the following matrix equation is written as follow

$$\begin{bmatrix} -2 & 0 & 6 \\ 1 & 1 & -1 \\ -2 & -2 & 4 \end{bmatrix} A + \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 4 & 4 & 2 & 4 & 4 \end{bmatrix} \bar{A} = \begin{bmatrix} -7/4 \\ 1/4 \\ -1 \end{bmatrix}.$$

Thus, since there are no real Hermite's coefficients, there is no approximate solution for the interval $1/2 < x \le 1$.

The solution of (23) under the conditions (24)

$$y(x) = \begin{cases} 1/2x, & 0 \le x \le 1/2 \\ \emptyset\ , & 1/2 < x \le 1 \end{cases}$$

overlaps with the exact solution of the problem.

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7. Conclusion

Nonlinear differential equations used in engineering, physics, mathematics or in many modelling problems are usually difficult to solve as analytically. In this study, to solve a class of nonlinear differential equations with boundary conditions, we introduce a matrix method depending on Hermite polynomials and collocation points. Also the residual error analysis has been also developed for the accuracy of solutions. The present method and the error analysis procedures are applied to some examples which have been solved by Taylor matrix method in the literature. The results related with examples have been shown in Tables 1-5 and Figures 1-2. As it is seen from the numerical examples, the method provides a better approximation than the other methods such as Taylor matrix method. A significant advantage of the proposed method, the Hermite coefficients of the solution can be found obviously by developing computer programs. The method also can be developed and applied to solve other high order nonlinear differential equations with new strategies.

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