A COMPUTATIONAL METHOD FOR SOLVING DIFFERENTIAL EQUATIONS WITH QUADRATIC NONLINEARITY BY USING BERNOULLI POLYNOMIALS

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In this paper, a matrix method is developed to solve quadratic nonlinear differential equations. It is assumed that the approximate solutions of main problem which we handle primarily, is in terms of Bernoulli polynomials. And both the approximate solution and the main problem are written in matrix form to obtain the solution. The absolute errors are applied to numeric examples to demonstrate efficiency and accuracy of this technique. The obtained tables and figures in the numeric examples show that this method is very sufficient and reliable for solution of nonlinear equations. Also, a formula is utilized based on residual functions and mean value theorem to seek error bounds.

Key Words*: Bernoulli polynomials, Approximate Solutions, Numerical Methods, Nonlinear Differential Equations.*

1. Introduction

Nonlinear differential equations are of great importance as a model of many physical, chemical and technical problems such as fluid dynamics, optimal control, diffusion problems, electrical networks, nuclear physics, financial mathematics and gas dynamics [1-3]. Researchers have developed a number of approximate solution methods since it is difficult to find the exact solutions of such problems in recent years [4-18]. In this study, by using Bernoulli polynomials and collocation points [19-23], an approximate method has devoloped for the solution of mth order nonlinear ordinary differential equations.

Consider the mth order differential equations with quadratic nonlinearity of the form

$$\sum_{k=0}^{m} P_k(x)\,y^{(k)}(x)+\sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x)y^{(p)}(x)\,y^{(q)}(x)=g(x) \qquad (1)$$

with the initial-boundary conditions

$$\sum_{k=0}^{m-1}\left(a_{kj}y^{(k)}(a)+b_{kj}y^{(k)}(b)+c_{kj}y^{(k)}(c)\right)=\lambda_i, \qquad j=0,1,2,...,m-1 \qquad (2)$$

where $P_k(x),Q_{pq}(x)$ and $g(x)$ are continuous on the interval $-\infty<a\le x,t\le b<\infty$.

Our aim is to obtain the approximate solutions of the problem (1)-(2) using Bernoulli polynomials.

Define the Bernoulli polynomials [24]

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n$$

or

$$B_n(x)=\sum_{r=0}^{n}\binom{n}{r}b_r x^{n-r}\ ,\ \ b_r=B_r(0)\text{ (Bernoulli numbers).} \qquad (3)$$

And also derivative of Bernoulli polynomials can be indicated as

$$\frac{d}{dx}B_n(x)=\sum_{k=0}^{n-1}\binom{n}{k}(n-k)B_k x^{n-k-1}=nB_{n-1}(x). \qquad (4)$$

To explain this method, let us consider the following approximate solution in the truncated Bernoulli series form:

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_n B_n(x) \qquad (5)$$

where a_n, $n=0,1,2,...,N$ are unknown Bernoulli coefficients .

2. Main Matrix Relations and Method of Solution

The matrix form of Eq. (5) is written as

$$y(x)\cong y_N(x)=\mathbf{B}(x)\mathbf{A}; \qquad \mathbf{A}=[a_0 \quad a_1 \quad \dots \quad a_N]^{\mathrm{T}}. \qquad (6)$$

On the other hand, by using Eq. (4) for $n=0,1,2,...,N$, the kth derivative of the $\mathbf{B}(x)$ is given as

$$\mathbf{B}^{(k)}(\mathrm{x})=\mathbf{B}(x)(\mathbf{E})^k, \qquad k=0,1,...,m \qquad (7)$$

where

$$\mathbf{E}=\begin{bmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \mathrm{N}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

And then we obtain the matrix form of $y^{(k)}(x)$ with the help of relations (6) and (7)

$$y^{(k)}(x)\cong {y_N}^{(k)}(x)=\mathbf{B}(x)(\mathbf{E})^k\,\mathbf{A}. \qquad (8)$$

Also matrix forms of nonlinear terms of Eq (1) are

$$\left(y^{(0)}(x)\right)^2=\mathbf{B}(x)\bar{\mathbf{B}}(x)\bar{\mathbf{A}} \qquad (9\text{-}1)$$

$$y^{(1)}(x)y^{(0)}(x)=\mathbf{B}(x)(\mathbf{E})\bar{\mathbf{B}}(x)\bar{\mathbf{A}} \qquad (9\text{-}2)$$

$$\left(y^{(1)}(x)\right)^2=\mathbf{B}(x)\mathbf{E}\bar{\mathbf{B}}(x)\bar{\mathbf{E}}\bar{\mathbf{A}} \qquad (9\text{-}3)$$

$$y^{(2)}(x)y^{(1)}(x)=\mathbf{B}(x)\mathbf{E}^2\bar{\mathbf{B}}(x)\bar{\mathbf{E}}\bar{\mathbf{A}} \qquad (9\text{-}4)$$

$$y^{(2)}(x)y^{(0)}(x)=\mathbf{B}(x)\mathbf{E}^2\bar{\mathbf{B}}(x)\bar{\mathbf{A}} \qquad (9\text{-}5)$$

$$\left(y^{(2)}(x)\right)^2=\mathbf{B}(x)\mathbf{E}^2\bar{\mathbf{B}}(x)\bar{\mathbf{E}}^2\bar{\mathbf{A}} \qquad (9\text{-}6)$$

where

$\bar{\mathbf{B}}(x)=diag\left[\mathbf{B}(x) \quad \mathbf{B}(x) \quad \dots \quad \mathbf{B}(x)\right]_{(N+1)\times(N+1)^2}$, $\bar{\mathbf{E}}=diag\left[\mathbf{E} \quad \mathbf{E} \quad \dots \quad \mathbf{E}\right]_{(N+1)^2\times(N+1)^2}$,

$\bar{\mathbf{E}}^2=diag\left[\mathbf{E}^2 \quad \mathbf{E}^2 \quad \dots \quad \mathbf{E}^2\right]_{(N+1)^2\times(N+1)^2}$, $\bar{\mathbf{A}}=[a_0\mathbf{A} \quad a_1\mathbf{A} \quad \dots \quad a_N\mathbf{A}]^{\mathrm{T}}$.

Let define collocation points x_i

$$x_i=a+\frac{b-a}{N}i,\ i=0,1,...,N \quad . \qquad (10)$$

For the linear part of Eq(1), using collocation points in Eq. (8) we have

$$y^{(k)}(x_i)=\mathbf{B}(x_i)(\mathbf{E})^k\,\mathbf{A}\Rightarrow\mathbf{Y}^{(k)}=\mathbf{B}(\mathbf{E})^k\,\mathbf{A} \tag{11}$$

where

$$\mathbf{B}=\begin{bmatrix}\mathbf{B}(x_0)\\ \mathbf{B}(x_1)\\ \vdots\\ \mathbf{B}(x_N)\end{bmatrix}=\begin{bmatrix}B_0(x_0) & B_1(x_0) & \dots & B_N(x_0)\\ B_0(x_1) & B_1(x_1) & \dots & B_1(x_1)\\ \vdots & \vdots & \cdots & \vdots\\ B_0(x_N) & B_1(x_N) & \dots & B_1(x_N)\end{bmatrix}_{(N+1)\times(N+1)}.$$

Also, matrix form of variable coefficients $P_k(x)$ with collocation points is obtained as:

$$\mathbf{P_k}=Diag\left[P_k(x_0)\quad P_k(x_1)\quad\dots\quad P_k(x_N)\right]_{(N+1)\times(N+1)}. \tag{12}$$

So matrix form of linear part is

$$\sum_{k=0}^{m}P_k(x)\,y^{(k)}(x)=\sum_{k=0}^{m}\mathbf{P}_k\mathbf{B}\mathbf{E}^k\mathbf{A}. \tag{13}$$

On the other hand, for the nonlinear part of Eq. (2), by substituting the collocation points, we get

$$y^{(p)}(x_i)\,y^{(q)}(x_i)=\mathbf{Y}^{(p,q)}=\begin{bmatrix}y^{(p)}(x_0)\,y^{(q)}(x_0)\\ y^{(p)}(x_1)\,y^{(q)}(x_1)\\ \vdots\\ y^{(p)}(x_N)\,y^{(q)}(x_N)\end{bmatrix}. \tag{14}$$

It is clear that from Eqs. (9-1)-(9-6) and (14)

$$\begin{gathered}\mathbf{Y}^{(0,0)}=\mathbf{B}^{*}{}_{0,0}\bar{\mathbf{A}}\;,\;\mathbf{Y}^{(1,0)}=\mathbf{B}^{*}{}_{1,0}\bar{\mathbf{A}}\;,\;\mathbf{Y}^{(1,1)}=\mathbf{B}^{*}{}_{1,1}\bar{\mathbf{A}}\;,\\ \mathbf{Y}^{(2,0)}=\mathbf{B}^{*}{}_{2,0}\bar{\mathbf{A}}\;,\;\mathbf{Y}^{(2,1)}=\mathbf{B}^{*}{}_{2,1}\bar{\mathbf{A}}\;,\;\mathbf{Y}^{(2,2)}=\mathbf{B}^{*}{}_{2,2}\bar{\mathbf{A}}\end{gathered} \tag{15}$$

where

$$\mathbf{B}^{*}{}_{0,0}=\begin{bmatrix}B(x_o)\bar{B}(x_0)\\ B(x_1)\bar{B}(x_1)\\ \vdots\\ B(x_N)\bar{B}(x_N)\end{bmatrix},\ \mathbf{B}^{*}{}_{1,0}=\begin{bmatrix}B(x_o)E\bar{B}(x_0)\\ B(x_1)E\bar{B}(x_1)\\ \vdots\\ B(x_N)E\bar{B}(x_N)\end{bmatrix},\ \mathbf{B}^{*}{}_{1,1}=\begin{bmatrix}B(x_o)E\bar{B}(x_0)\bar{E}\\ B(x_1)E\bar{B}(x_1)\bar{E}\\ \vdots\\ B(x_N)E\bar{B}(x_N)\bar{E}\end{bmatrix}$$

$$\mathbf{B}^{*}{}_{2,0}=\begin{bmatrix}B(x_o)E^2\bar{B}(x_0)\\ B(x_1)E^2\bar{B}(x_1)\\ \vdots\\ B(x_N)E^2\bar{B}(x_N)\end{bmatrix},\ \mathbf{B}^{*}{}_{2,1}=\begin{bmatrix}B(x_o)E^2\bar{B}(x_0)\bar{E}\\ B(x_1)E^2\bar{B}(x_1)\bar{E}\\ \vdots\\ B(x_N)E^2\bar{B}(x_N)\bar{E}\end{bmatrix},\ \mathbf{B}^{*}{}_{2,2}=\begin{bmatrix}B(x_o)E^2\bar{B}(x_0)\bar{E}^2\\ B(x_1)E^2\bar{B}(x_1)\bar{E}^2\\ \vdots\\ B(x_N)E^2\bar{B}(x_N)\bar{E}^2\end{bmatrix}$$

$p,q=0,1,2$.

Therefore, the nonlinear part of the Eq. (1) is written in the matrix form

$$\sum_{p=0}^{2}\sum_{q=0}^{p}Q_{pq}(x_i)\,y^{(p)}(x_i)y^{(q)}(x_i)=\sum_{p=0}^{2}\sum_{q=0}^{p}\mathbf{Q_{pq}}\mathbf{B}^{*}{}_{p,q}\bar{\mathbf{A}}\quad. \tag{16}$$

Using Eqs. (13) and (16) we construct the fundamental matrix equation

$$\sum_{k=0}^{m}\mathbf{P}_k\mathbf{B}\mathbf{E}^k\mathbf{A}+\sum_{p=0}^{2}\sum_{q=0}^{p}\mathbf{Q}_{\mathbf{pq}}\mathbf{B}^{*}{}_{p,q}\bar{\mathbf{A}}=\mathbf{G} \tag{17}$$

where

$$\mathbf{G}=\left[g(x_0)\quad g(x_1)\quad \dots \quad g(x_N)\right]^T .$$

Compact form of Eq. (17) is

$$\mathbf{W}\mathbf{A}+\mathbf{V}\bar{\mathbf{A}}=\mathbf{G}$$

where

$$\mathbf{W}=\left[w_{ij}\right]=\sum_{k=0}^{m}\mathbf{P}_k\mathbf{B}\mathbf{E}^k; i,\, j=0,1,...,N$$

$$\mathbf{V}=\left[v_{mn}\right]=\sum_{p=0}^{2}\sum_{q=0}^{p}\mathbf{Q}_{\mathbf{pq}}\mathbf{B}^{*}{}_{p,q}; m=0,1,...,N, n=0,1,...,(N+1)^2-1 .$$

Also, the augmented form of Eq (17) can be written as

$$\left[\mathbf{W};\mathbf{V}:\mathbf{G}\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & v_{00} & v_{01} & \cdots & v_{0,(N+1)^2-1} & : & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & v_{10} & v_{11} & \cdots & v_{1,(N+1)^2-1} & : & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & v_{N0} & v_{N1} & \cdots & v_{N,(N+1)^2-1} & : & g(x_N) \end{bmatrix} . \tag{18}$$

Also, for the matrix relation of the mixed conditions (2) the following matrix form is obtained by using Eq (11) :

$$\sum_{k=0}^{m-1}\left[a_{kj}\mathbf{B}(\mathrm{a})+b_{kj}\mathbf{B}(\mathrm{b})+c_{kj}\mathbf{B}(\mathrm{c})\right](\mathbf{E})^k\,\mathbf{A}=\lambda_j \Rightarrow \mathbf{U}\mathbf{A}+\mathbf{0}^{*}\bar{\mathbf{A}}=\lambda$$

or clearly

$$\left[\mathbf{U};\mathbf{0}^{*}:\boldsymbol{\lambda}\right]=\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} & ; & 0 & 0 & \cdots & 0 & : & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & 0 & 0 & \cdots & 0 & : & \lambda_1 \\ \cdots & \cdots & \ddots & \vdots & ; & \cdots & \cdots & \ddots & \cdots & : & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & 0 & 0 & \cdots & 0 & : & \lambda_{m-1} \end{bmatrix} \tag{19}$$

where

$$\mathbf{U}=\left[u_{j0}\quad u_{j1}\quad \cdots \quad u_{jN}\right],\ j=0,1,2,...,m-1$$
$$\boldsymbol{\lambda}=\left[\lambda_0\quad \lambda_1\quad \dots\quad \lambda_{m-1}\right]^T \text{ and } \mathbf{0}^{*}=\left[0\quad 0\quad \dots\quad 0\right] .$$

Finally, by replacing m rows of matrix (19) by any m rows of matrices (18) ,we have the new matrix and by solving this matrix we obtain the unknown Bernoulli coefficients a_n ($n=0,1,...,N$).
Then by substituting these coefficients in Eq. (3), the truncated Bernoulli series solution $y_N(x)$ is gained.

3. Accuracy of solution

Accuracy of this solution can be checked easily as follows. It can be written that

$$E(x_r)=\left|\sum_{k=0}^{m}P_k(x_r)\,y^{(k)}(x_r)+\sum_{p=0}^{2}\sum_{q=0}^{p}Q_{pq}(x_r)y^{(p)}(x_r)\,y^{(q)}(x_r)-g(x_r)\right|\cong 0$$

or

$E(x_r) \le 10^{-k_r}$ (k_r is any positive integer) for $x_r \in [a,b], r = 0,1,2,..$. Because the solution obtained by the Bernoulli collocation method must be satisfied the Eq. (1) approximately. If max $10^{-k_r} = 10^{-k}$ (k is any positive integer) is already defined, then the truncation limit N is increased until the variation of $E(x_r)$ at each point becomes smaller than the defined max 10^{-k} [13-14].

On the other hand the upper bound error $\tilde{R}_N$ can be calculated by means of the Bernoulli Residual functions and the mean value theorem. For this problem the Residual functions in terms of Bernoulli approximate solutions $y_N(x)$ is obtained as follows:

$$R_N(x) = L[y_N(x)] - g(x)$$

where

$$L[y_N(x)] = \sum_{k=0}^{m} P_k(x) y_N^{(k)}(x) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x) y_N^{(p)}(x) y_N^{(q)}(x).$$

Applying the mean value theorem to obtained Residual functions on $[a,b]$, we determine the upper bound error $\tilde{R}_N$ as follows [25]:

$$\left|\int_a^b R_N(x)\,dx\right| \le \int_a^b |R_N(x)|\,dx$$

$$\int_a^b R_N(x)\,dx = (b-a) R_N(x_0) \Rightarrow \left|\int_a^b R_N(x)\,dx\right| = (b-a)|R_N(x_0)|$$

hence

$$(b-a)|R_N(x_0)| \le \int_a^b |R_N(x)|\,dx \Rightarrow |R_N(x)| \le \frac{\int_a^b |R_N(x)|\,dx}{(b-a)} = \tilde{R}_N, \quad x_0 \in (a,b).$$

4. Numerical Examples

In this section, the discussed method in previous sections is applied for different nonlinear differential equations.

Example 1.

Consider the nonlinear oscillator differential equation [16].

$$\frac{d^2u}{dt^2} - u + u^2 + \left(\frac{du}{dt}\right)^2 - 1 = 0 \tag{20}$$

with the initial conditions $u(0) = 2$, $u'(0) = 0$.

The exact solution of the problem is $u(t) = 1 + \cos t$. In Table 1 the absolute error function obtained by Bernoulli collocation method for $N = 5,8,12$, Variational iteration method and Homotopy perturbation method and [16] are compared. It is obvious in Table 1 that the obtained results by using present method are more convergent than obtained by the other methods. And also it is clear that for the present method when N increases, the results are more convergent. In Fig. 1, we compare the exact solution $u(t)$ and the Bernoulli solutions $u_N(t)$ for $N=5,8,12$. The upper error bounds are obtained as $\{\tilde{R}_5,\tilde{R}_8,\tilde{R}_{12}\}=\{7.23e-04, 9.04e-08, 2.34e-13\}$.

Table 1. The comparison of absolute error functions Eq. (20).

t	Error of HPM [16]	Error of VIM [16]	Error of Present Method N=5	N=8	N=12
0.0	0	0	0	0	0
0.1	8.3350e−6	8.3330e−6	2.0176e-07	1.0939e-11	1.1102e-16
0.2	1.3342e−4	1.3333e−4	9.2470e-07	3.2216e-11	0
0.3	6.7601e−4	6.7500e−4	1.6771e-06	4.9182e-11	1.1102e-16
0.4	2.1390e−3	2.1333e−3	2.1190e-06	6.8051e-11	1.1102e-16
0.5	5.2299e−3	5.2085e−3	2.5525e-06	8.4936e-11	1.1102e-16
0.6	1.0864e−2	1.0800e−2	3.4593e-06	1.0180e-10	1.1102e-16
0.7	2.0170e−2	2.0011e−2	4.1200e-06	1.1926e-10	1.1102e-16
0.8	3.4493e−2	3.4142e−2	3.5686e-07	1.2142e-10	2.2204e-16
0.9	5.5402e−2	5.4696e−2	1.8546e-05	5.8650e-10	3.3307e-16
1	8.4698e−2	8.3383e−2	7.3684e-05	5.0965e-09	7.5495e-15

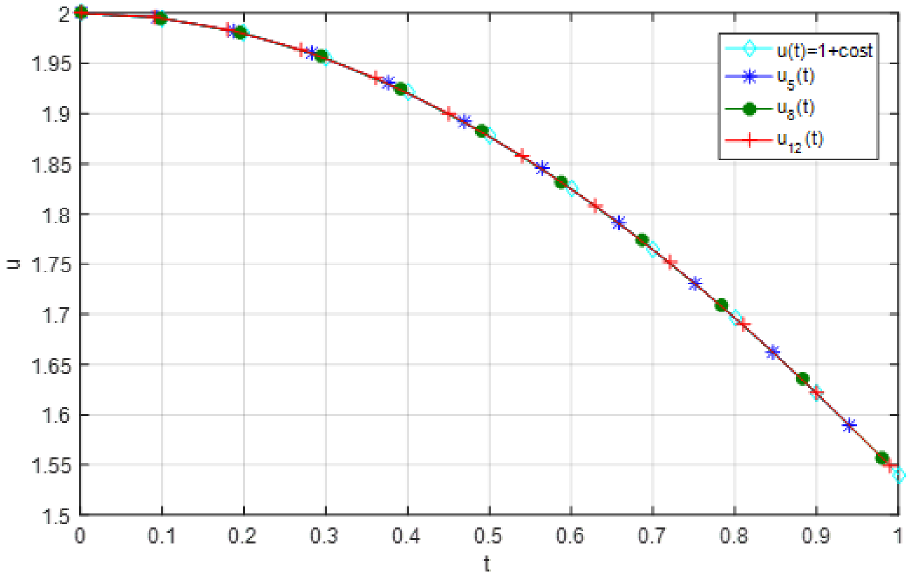

Figure 1. The comparison of the exact solution $u(t)=1+\cos t$, approximate solutions $u_N(t)$ for $N=5,8,12$.

Example 2:

We pay attention the following quadratic Riccati differential equation [12]

$$u'(t)=e^t-e^{3t}+2e^{2t}u(t)-e^t u^2(t) \qquad 0\le t\le 1 \tag{21}$$

with the condition $u(0)=1$ where the exact solution of this equation is $u(t)=e^t$.

Here $P_0(t) = -2e^{2t}$, $P_1(t) = 1$ and $Q_{00}(t) = e^t$. The main matrix equation of this problem with respect to given method as follow:

$$(P_0B + P_1BE)A + (Q_{00}B\bar{B})\bar{A} = G$$

The approximate solutions of this problem have been obtained for $N = 8,10,14$. The comparison of the exact solution and the approximate solutions is given in Fig.2, also errors of present method are compared in Fig.3. for $N = 8,10,14$. Also we obtained the error bounds as $\{\tilde{R}_8, \tilde{R}_{10}, \tilde{R}_{14}\} = \{3.45e-09, 5.82e-11, 0\}$.

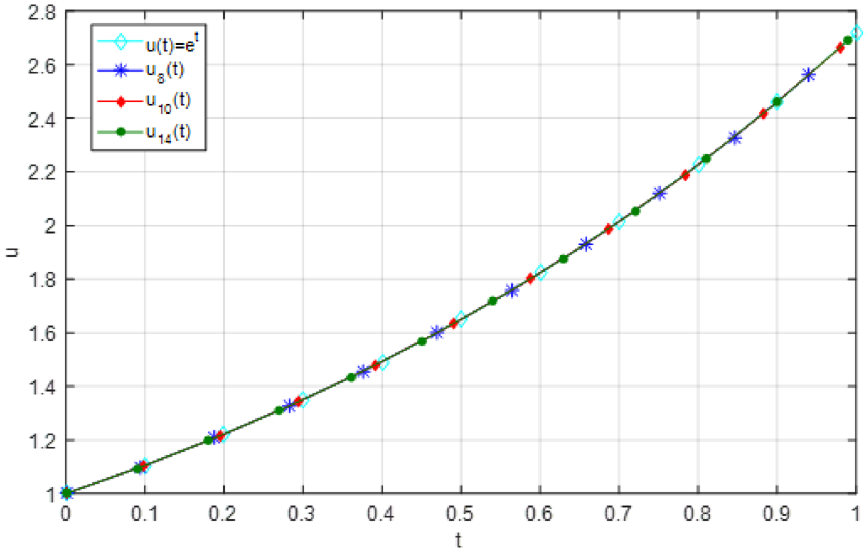

Figure 2. The comparison of the exact solution $u(t) = e^t$, approximate solutions $u_N(t)$ for $N = 8,10,14$

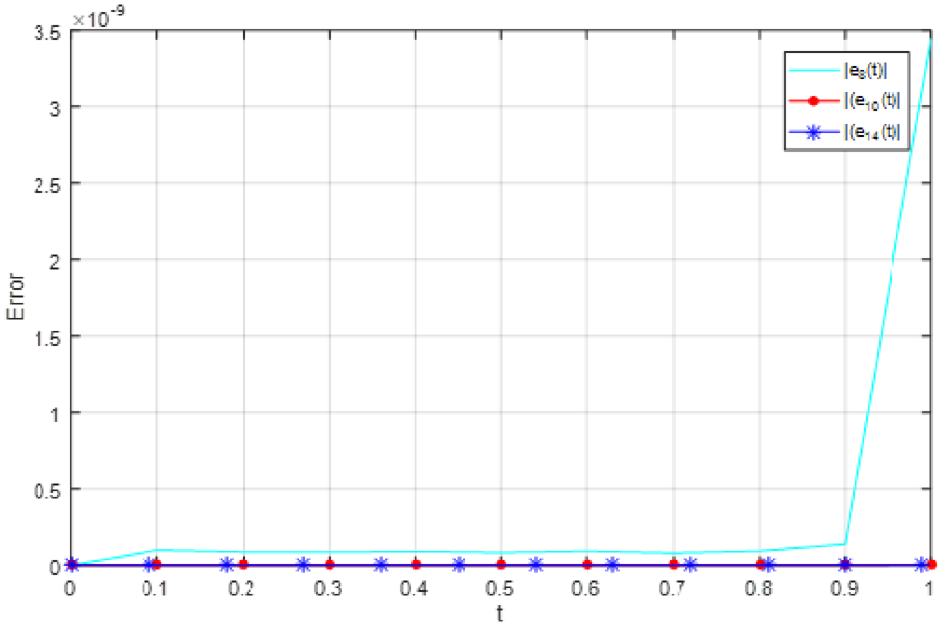

Figure 3. The Comparison of absolute error function $|e_N|$ for $N = 8,10,14$.

Example 3.

Consider the third order nonlinear differential equation [17]

$$u'''(t)+2e^{-3u(t)}=4(1+t)^{-3} \tag{22}$$

with the initial conditions $u(0)=0, u'(0)=1, u''(0)=-1$.

Exact solution of this problem is $u(t)=\ln(1+t)$. The comparisons of approximate solutions and the absolute error functions are in Fig. 4 and Fig. 5, respectively.

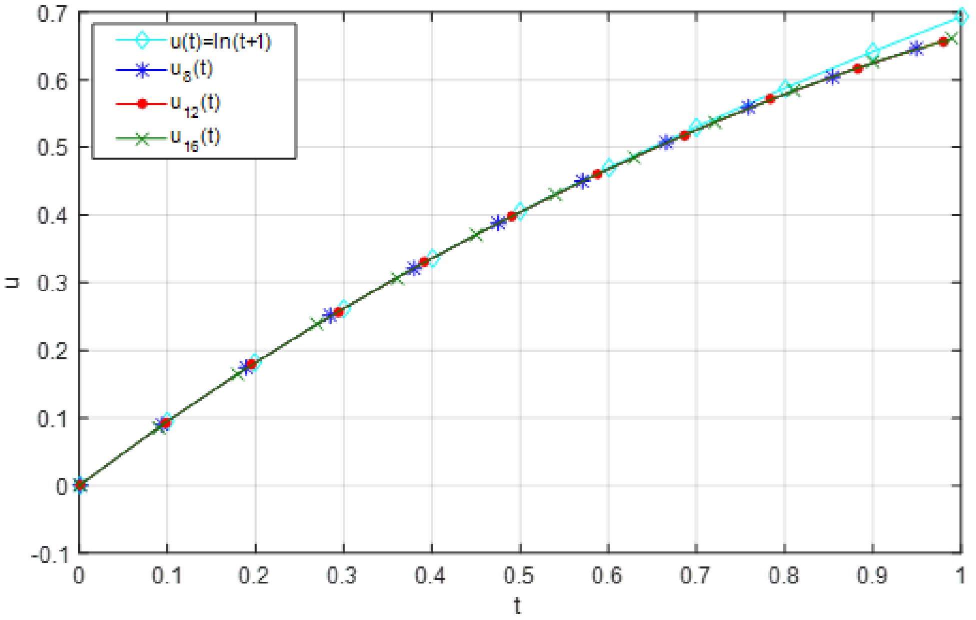

Figure 4. The comparison of the exact solution $u(t)=\ln(1+t)$, approximate solutions $u_N(t)$ for $N=8,12,16$.

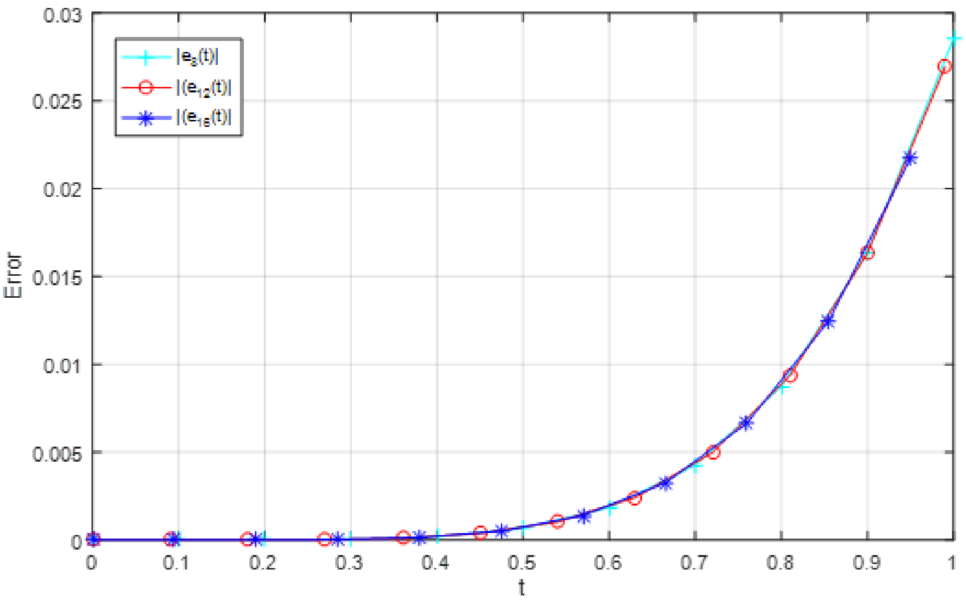

Figure 5. The Comparison of absolute error function $|e_N|$ for $N=8,12,16$.

5. Conclusions

In this article, a matrix method based on Bernoulli polynomials and collocation points has been developed for the solution of nonlinear differential equations. It is clear from the examples which has been handled that the method is successful solving on quadratic non-linear equations. The applicability of the method is very advantageous as it is based on easy software code. The examples in this article have been solved using matlab application in a timely manner. The outcome of matlab program shows a strong correlation between value of N and the convergency of the solution. And also comparison with the other methods and the exact solution shows that this method is very effective.

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