## THE DUAL SPATIAL QUATERNIONIC EXPRESSION OF RULED SURFACES

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*In this paper, the ruled surface which corresponds to a curve on dual unit sphere is rederived with the help of dual spatial quaternions. We extend the term of dual expression of ruled surface using dual spatial quaternionic method. The correspondences in dual space of closed ruled surfaces are quaternionically expressed. As a consequence, the integral invariants of these surfaces and the relationships between these invariants are shown.*

Key Words: *Real quaternion, spatial quaternion, dual spatial quaternion, closed ruled surface, distribution parameter, dual angle of pitch.*

## 1. Introduction

The quaternions was discovered in 1843 by William Rowan Hamilton. Quaternions arose historically from Hamilton's essays in the mid nineteenth century to generalize complex numbers in some way that would be applicable to three-dimensional (3D) space. Quaternions are less intuitive than Euler Angles and the math can be a little more complicated. This application note covers the basic mathematical concepts needed to understand and use the quaternion outputs of Robotics orientation sensors. The technology did not penetrate the computer animation community until the landmark Siggraph 1985 paper of Ken Shoemake [10]. Shoemake's paper is that it took the concept of the orientation frame for moving 3D objects and cameras, and it introduced quaternions to animators as a solution. Many studies have been made on quaternionic and dual quaternionic curves, [1,2,8,11,13].

The Serret-Frenet formulae for a quaternionic curves in $IR^3$ and $IR^4$ are introduced by K. Bharathi and M. Nagaraj, [1]. Let $s \in I = [0,1]$ be the arc parameter along the smooth curve

$$\alpha : [0,1] \to Q$$
$$\alpha(s) = \sum_{n=1}^{3} \alpha_i(s) e_i .$$

This is called a spatial quaternionic curve, [1].

Let $\alpha(s)$ be a curve parametrized by arclength function $s$. Then for the unit speed spatial quaternionic curve $\alpha$ with frame vectors the following Frenet equations are given by [1]

$$t'(s) = k(s)n_1(s), n_1'(s) = -k(s)t(s) + r(s)n_2(s), n_2'(s) = -r(s)n_1(s). \tag{1}$$

A surface is said to be ruled if it is generated by moving a straight line continuously in $\mathbf{E}^3$. A practical application of ruled surfaces is that they are used in civil engineering. A ruled surface in $IR^3$ is a surface containing at least one parameter family of straight lines. Thus a ruled surface has a parametrization in the form

$$\vec{\varphi}(s,v) = \vec{\alpha}(s) + v\,\vec{x}(s). \tag{2}$$

where we call $\alpha$ the anchor curve, $x$ is the generator vector of ruled surface. When the above ruled surface satisfies $\varphi(s+2\pi, v) = \varphi(s,v)$, it is called closed ruled surface, [3]. It is well known from

H.R.Müller [9] that a closed ruled surface generated by oriented line of a rigid body has two real integral invariants; the pitch and the angle of pitch. They are known as the integral invariants of a closed ruled surface, [5, 9]. There have been many studies on ruled surfaces. In some studies, the dual expression of the ruled surface has been investigated. However, the ruled surface was not studied as a quaternionic. In [12], they investigated the ruled surface as quaternionic. They have quaternionally calculated the integral invariants of the ruled surface.

Dual numbers were introduced in the 19th century by W. K. Clifford. The set of dual numbers given by $ID = \{a + \varepsilon a^* : a, a^* \in IR, \varepsilon^2 = 0\}$ is a commutative ring, the set, $ID^3 = \{\vec{A} = \vec{a} + \varepsilon \vec{a}^* \mid \vec{a}, \vec{a}^* \in IR^3, \varepsilon^2 = 0\}$ meets the all real vector space axioms over the ring. The set is module over the ring $ID$ which is named $ID$ - module or dual space. The elements of $ID^3$ call dual vector. According to E. Study, a unit dual vector $X(s)$ corresponds only one oriented line where the real vector $x$ shows the direction of this line and the real vector $x^*$ shows the vectorial moment respect to the origin point. A differentiable closed curve $X(s)$ on the dual unit sphere depending on a real parameter $s$, represents a differentiable family of one parameter straight lines in $IR^3$ which we call closed ruled surface, [4, 6].

The dual vector expression of a ruled surface is

$$\vec{\varphi}(s,u) = \vec{x}(s) \wedge \vec{x}^*(s) + u\,\vec{x}(s), \tag{3}$$

where the $\overleftarrow{x} \wedge \overleftarrow{x^*}$ is the anchor curve. $s$ is not the arc-parameter of this curve. The ruled surface $(X)$ is given by $\vec{X}(s) = \vec{x}(s) + \varepsilon\,\vec{x}^*(s)$.

The dual angle of a closed ruled surface which is constructed by the dual unit vector $X = x + \varepsilon x^*$ is given by

$$\Lambda_X = \langle D, X\rangle \text{ or } \Lambda_X = \lambda_x - \varepsilon L_x \tag{4}$$

where $\lambda_x$ and $L_x$ are respectively the angle of pitch and the pitch of the closed ruled surface, [4].

## 2. Preliminaries

### 2.1 Real and Dual Quaternions

Real quaternion is defined by the $1, e_1, e_2, e_3$. 1 is real number, $e_1, e_2, e_3$ are vectors with the following properties:

$$\begin{aligned} &e_1^{\,2} = e_2^{\,2} = e_3^{\,2} = e_1 \times e_2 \times e_3 = -1, e_1, e_2, e_3 \in IR^3 \\ &e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2. \end{aligned} \tag{5}$$

Real quaternion set can be denoted

$$\mathbf{K} = \{q = d + ae_1 + be_2 + ce_3 \mid d, a, b, c \in IR\}.$$

Let $q_1 = S_{q_1} + V_{q_1} = d_1 + a_1e_1 + b_1e_2 + c_1e_3$ and $q_2 = d_2 + a_2e_1 + b_2e_2 + c_2e_3$ be two quaternions in $\mathbf{K}$ , the quaternion multiplication of $q_1$ and $q_2$ is given by

$$\begin{aligned} q_1 \times q_2 = d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)e_1 \\ + (d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)e_2 + (d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)e_3. \end{aligned}$$

which is equivalent to

$$q_1 \times q_2 = S_{q_1}S_{q_2} - \langle V_{q_1}, V_{q_2}\rangle + S_{q_1}V_{q_2} + S_{q_2}V_{q_1} + V_{q_1} \wedge V_{q_1} \tag{6}$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product on $IR^3$, respectively, [7]. The symmetric real-valued bilineer form of $h$ which is defined as

$$h:\mathbf{K}\times\mathbf{K}\to IR$$
$$(q_1,q_2)\to h(q_1,q_2)=\frac{1}{2}(q_1\times\overline{q}_2+q_2\times\overline{q}_1). \tag{7}$$
It is called quaternion inner product, [1]. Let $q$ be a real quaternion. Its conjugate is $\overline{q}=S_q-V_q.$ The three-dimensional real Euclidean space $IR^3$ is identified with the space of spatial quaternions $\mathbf{Q}=\{q\in\mathbf{K}\mid q+\overline{q}=0\}$ in obvious manner. In this case, the elements of $\mathbf{Q}$ are $q=ae_1+be_2+ce_3.$ As a result, the quaternion multiplication of the two spatial quaternions is [1, 7],
$$q_1\times q_2=-\langle q_1,q_2\rangle+q_1\wedge q_2. \tag{8}$$
Let $q$ and $q*$ be two real quaternions. Dual quaternion set can be denoted $\mathbf{K}_D=\{Q=q+\varepsilon q*\mid q,q*\in\mathbf{K}\}$. Also we can type $Q=D+Ae_1+Be_2+Ce_3$ where $A,B,C,D\in ID$ such that $S_Q=D$ is the scalar part of $Q$ and

$V_Q=Ae_1+Be_2+Ce_3$ is the vector part of $Q$. The multiplication of two dual quaternions $Q$ and $P$ is defined as
$$Q\times P=q\times p+\varepsilon(q\times p^*+q^*\times p). \tag{9}$$
It can be easily seen that
$$Q\times P=S_QS_P-\langle V_Q,V_P\rangle+S_QV_P+S_PV_Q+V_Q\wedge V_P \tag{10}$$
in which $\langle,\rangle$ and $\wedge$ are the inner and cross products on $ID^3$, respectively, [7, 11].
The symmetric dual-valued bilineer form $H$ which is defined as
$$H:\mathbf{K}_D\times\mathbf{K}_D\to ID$$
$$(Q,P)\to H(Q,P)=\frac{1}{2}(Q\times\overline{P}+P\times\overline{Q}) \tag{11}$$
is called dual quaternion inner product. $\mathbf{Q}_D=\{Q\in\mathbf{K}_D\mid Q+\overline{Q}=0\}$ is called the dual spatial quaternions set. The elements of this set are called dual spatial quaternion. The element of $\mathbf{Q}_D$ is $Q=Ae_1+Be_2+Ce_3.$ As a result, the quaternion multiplication of the two spatial dual spatial quaternions is [7, 11]
$$Q\times P=-\langle Q,P\rangle+Q\wedge P, \tag{12}$$

### 2.2 The Spatial Quaternionic Expression of Ruled Surfaces

Parametric expression of the spatial quaternion expression of a ruled surface is
$$\vec{\varphi}:I\times IR\to Q$$
$$(s,v)\to\vec{\varphi}(s,v)=\vec{\alpha}(s)+v\,\vec{x}(s) \tag{13}$$
where $\alpha$ spatial quaternionic curve and $x$ spatial quaternionic vector, [12].
The spatial quaternionic definition of distribution parameter of $\varphi$ is [12]
$$P_x=\frac{h(x\times x',\alpha')}{N(x')^2}=\frac{1}{2}\frac{((x\times x')\times\overline{\alpha'}+\alpha'\times\overline{(x\times x')})}{N(x')^2}. \tag{14}$$

The angle of pitch and the pitch of the closed spatial quaternionic ruled surface are given by [12]
$$\lambda_x=h(\vec{d},\vec{x}),L_x=h(\vec{V},\vec{x}).$$
Let $\varphi$, $x$ and $x^*$ be the spatial quaternionic ruled surface, the directrix of this surface and the vectorial moment of $x$, respectively. Then there exists a point $Z$, such that [12]

$$\vec{x}^* = \vec{z} \times \vec{x}. \tag{15}$$

## 3. The Dual Spatial Quaternionic Expression of Ruled Surface

Let $\alpha$ be spatial quaternionic curve, $\{t, n_1, n_2\}$ be Frenet vectors of $\alpha$, $\{t^*, n_1^*, n_2^*\}$ be vectorial moments of Frenet vectors. $T = t + \varepsilon t^*$, $N_1 = n_1 + \varepsilon n_1^*$ and $N_2 = n_2 + \varepsilon n_2^*$ vectors draw curves on the unit dual sphere. The dual spatial quaternionic expressions of the closed ruled surfaces corresponding to these curves in Euclidean space are given. The relationships between integral invariants of the obtained surfaces are computed as dual spatial quaternionic.

Let us write the dual spatial quaternionic expression of a ruled surface corresponding to the dual curve. Acoording to the (15), the vectorial moment of $\overleftarrow{x}$ is

$$\vec{x}^* = \vec{\alpha} \times \vec{x}, \tag{16}$$

where $\vec{\alpha}$ and $\vec{x}$ are orthogonal. Right-multiplying both sides of (16) by $x$ gives

$$\vec{x}^* \times \vec{x} = (\vec{\alpha} \times \vec{x}) \times \vec{x} \Rightarrow \vec{x}^* \times \vec{x} = -\vec{\alpha}.$$

Taking into consideration (8), $\vec{x} \times \vec{x}^* = -\vec{x}^* \times \vec{x}$ is obtained. From the equation (13), the dual spatial quaternionic expression of ruled surface corresponding to the dual curve is

$$\vec{\varphi}(s,v) = \vec{x}(s) \times \vec{x}^*(s) + v\,\vec{x}(s), \tag{17}$$

in which $\vec{x}(s) \times \vec{x}^*(s)$ is the anchor curve. $s$ is not the arc-parameter of this curve. In the present text, dual spatial quaternionic ruled surface term will be used instead of the dual spatial quaternionic expression of ruled surface corresponding to the dual curve.

The arc-parameter of dual curve is $d\Phi = d\varphi + \varepsilon d\varphi^*$, the we obtain

$$d\Phi^2 = H(dX, dX) = H(X', X')ds^2$$

$$d\varphi^2 + 2\varepsilon d\varphi . d\varphi^* = \frac{1}{2}(dX \times d\overline{X} + dX \times d\overline{X}) = h(dx, dx) + 2\varepsilon h(dx, dx^*).$$

Hence, we can write from the last equation

$$d\varphi^2 = h(dx, dx), d\varphi . d\varphi^* = h(dx, dx^*).$$

**Definition 3.1.** Distribution parameter of dual spatial quaternionic ruled surface is

$$\frac{1}{d} = \frac{h(dx, dx^*)}{h(dx, dx)} = \frac{d\varphi^*}{d\varphi}. \tag{18}$$

**Definition 3.2.** In the dual plane $(V_2, V_3)$ of the moving system, let us chose a unit dual spatial quaternionic vector

$$N_1 = \cos\Phi V_2 + \sin\Phi V_3 \tag{19}$$

which makes a dual angle $\Phi = \varphi + \varepsilon\varphi^*$ with $V_2$ such that during the closed motion when the axis $V_1$ generates the closed spatial quaternionic ruled surface $V_1(s)$, let the unit vector, $N_1$ generate a developable spatial quaternionic ruled surface, along the orthogonal trajectory of the closed spatial quaternionic ruled surface. Then we call the total differential of $\Phi$ as the dual angle of pitch of the closed spatial quaternionic ruled surface $V_1(s)$. Thus, the dual angle of pitch of $V_1(s)$ is

$$\Lambda_{V_1} = -\oint d\Phi. \tag{20}$$

The dual spatial quaternionic Steiner vector is given by

$$\vec{D} = \vec{d} + \varepsilon\vec{d}^* = \oint \vec{W} \cdot \tag{21}$$

**Theorem 3.1.** The dual angle of pitch of dual spatial quaternionic ruled surface is given by

$$\Lambda_X = H(\vec{D}, \overleftarrow{X}). \tag{22}$$

**Proof:** The two orthonormal systems $N = \{\vec{N}_1, \vec{N}_2, \vec{N}_3\}$ and $V = \{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ are right-handed systems which represent the fixed space and the moving space, respectively. Assume the transition matrix is

$$B = \begin{bmatrix} 0 & 0 & 1 \\ \cos\Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos\Phi & 0 \end{bmatrix} \tag{23}$$

Hence, we can write

$$V = BN \tag{24}$$

Here, if we differentiate (24) in terms of $s$, it becomes

$$dV_2 = -d\Phi V_3,\ dV_3 = d\Phi V_2. \tag{25}$$

Solving (25) by using (11), we obtain

$$-d\Phi = H(dV_2, V_3) = -H(V_2, dV_3), \tag{26}$$

where $V_1 = v_1 + \varepsilon v_1^*$, $V_2 = v_2 + \varepsilon v_2^*$ and $V_3 = v_3 + \varepsilon v_3^*$ are dual spatial quaternionic vectors. By taking dual quaternionic inner product and equation $dV_i = \sum_{j=1}^{3} \Psi_{ij} V_j$ into account, we solve

$$\begin{aligned} H(dV_2, V_3) &= \frac{1}{2}(dV_2 \times \overline{V_3} + V_3 \times \overline{dV_2}), \\ &= \frac{1}{2}((-\Psi_3 V_1 + \Psi_1 V_3) \times \overline{V_3} + V_3 \times (\overline{-\Psi_3 V_1 + \Psi_1 V_3})), \\ &= \Psi_1 \end{aligned}$$

and

$$\begin{aligned} H(dV_3, V_2) &= \frac{1}{2}(dV_3 \times \overline{V_2} + V_2 \times \overline{dV_3}), \\ &= \frac{1}{2}((\Psi_2 V_1 - \Psi_1 V_2) \times \overline{V_2}) + V_2 \times (\overline{\Psi_2 V_1 - \Psi_1 V_2})), \\ &= -\Psi_1. \end{aligned}$$

$$H(dV_2, V_3) = -H(dV_3, V_2) = \Psi_1 \tag{27}$$

is obtained for the dual angle of pitch of closed dual spatial quaternionic ruled surface drawn by a dual spatial quaternionic vector $\vec{V}_1 = \vec{v}_1 + \varepsilon \vec{v}_1^*$.

Now let us find the dual angle of pitch of the dual spatial quaternionic ruled surface drawn by a dual quaternionic vector $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ which move strongly on the $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ system.

$$X = CV \tag{28}$$

in which $C$ is an orthogonal matrix. Considering reference [9] and dual quaternion inner product, dual angle of pitch is obtained

$$\Lambda_X = H(\vec{D}, \overleftarrow{X}). \tag{29}$$

**Theorem 3.2.** Let $T = t + \varepsilon t^*$, $N_1 = n_1 + \varepsilon n_1^*$ and $N_2 = n_2 + \varepsilon n_2^*$ be the dual spatial quaternionic vectors on the unit dual sphere. Then the dual spatial quaternionic Darboux vector is given by

$$W = RT + KN_2 = rt + kn_2 + \varepsilon(rt^* + r^* t + kn_2^* + k^* n_2) \tag{30}$$

**Proof:** Let the dual spatial quaternionic Darboux vector be

$$W = a_1 T + a_2 N_1 + a_3 N_2 \tag{31}$$

Right-multiplying both sides of (31) by $T$ gives

$$W \times T = -a_1 - a_2 N_2 + a_3 N_1 . \tag{32}$$

On the other hand, it can be written

$$W \times T = -\langle W, T\rangle + W \wedge T = -a_1 + dT = -a_1 + KN_1 . \tag{33}$$

From the equations (32) and (33), $a_2 = 0$ and $a_3 = K$ are found. Similarly, it can be written

$$W \times N_1 = -a_2 - KT + RN_2 \tag{34}$$

and $a_1 = R$ and $a_3 = K$ are found. If the values found are replaced by (31), then

$$W = RT + KN_2 = rt + kn_2 + \varepsilon(rt^* + r^* t + kn_2^* + k^* n_2) \tag{35}$$

is reached, wherein $K = k + \varepsilon k^*$ and $R = r + \varepsilon r^*$.

The geometric location of $T = t + \varepsilon t^*$, $N_1 = n_1 + \varepsilon n_1^*$ and $N_2 = n_2 + \varepsilon n_2^*$ dual spatial quaternionic vectors draws dual curves on the dual sphere. If these curves are closed, ruled surfaces corresponding to these curves are closed. These closed dual curves are shown as $(T)$, $(N_1)$ and $(N_2)$, respectively. The distribution parameters and dual angles of pitch for closed dual spatial quaternionic ruled surfaces corresponding to $(T)$, $(N_1)$ and $(N_2)$ will be given as follows.

**Theorem 3.3.** The distribution parameters and the dual angles of the pitch of closed dual spatial quaternionic ruled surfaces corresponding to $(T),(N_1),(N_2)$ are

$$i) P_T = 0, \qquad P_{N_1} = \frac{r}{k^2 + r^2}, \qquad P_{N_2} = \frac{1}{r},$$
$$ii) \Lambda_T = \oint r + \varepsilon \oint r^*, \quad \Lambda_{N_1} = 0, \quad \Lambda_{N_2} = \oint k + \varepsilon \oint k^* .$$

**Proof:** i) From the equation (17), the parametric equations of closed dual spatial quaternionic ruled surfaces corresponding to $(T),(N_1),(N_2)$ are

$$\begin{aligned} &\varphi_t(s,v) = \vec{t}(s) \times \vec{t}^*(s) + v\vec{t}(s), \vec{t}^*(s) = \overleftarrow{\alpha}(s) \times \vec{t}(s), \\ &\varphi_{n_1}(s,v) = \vec{n}_1(s) \times \vec{n}_1^*(s) + v\vec{n}_1(s), \vec{n}_1^*(s) = \overleftarrow{\alpha}(s) \times \vec{n}_1(s), \\ &\varphi_{n_2}(s,v) = \vec{n}_2(s) \times \vec{n}_2^*(s) + v\vec{n}_2(s), \vec{n}_2^*(s) = \overleftarrow{\alpha}(s) \times \vec{n}_2(s) \end{aligned} \tag{36}$$

respectively.
Let us calculate distribution parameters of these surfaces:
By formula of the equation (14), we obtain

$$\begin{aligned} P_T &= \frac{h(t \times t', (t \times t^*)')}{N(t')^2} = \frac{h(kn_2, k(n_1 \times t^*)) + h(kn_2, k(t \times n_1^*))}{N(t')^2} \\ &= \frac{\frac{1}{2}(kn_2 \times k\overline{(n_1 \times t^*)} + k(n_1 \times t^*) \times k\overline{n_2}) + \frac{1}{2}(kn_2 \times k\overline{(t \times n_1^*)} + k(t \times n_1^*) \times k\overline{n_2})}{h(kn_1, kn_1)} \\ &= n_2 \times \overline{(n_1 \times t^*)} + (n_1 \times t^*) \times \overline{n_2} + n_2 \times \overline{(t \times n_1^*)} + (t \times n_1^*) \times \overline{n_2}. \end{aligned}$$

Since $t^*$ and $n_1^*$ are vectorial moment, the distribution parameter of closed dual spatial quaternionic ruled surface corresponding to $(T)$ is

$$
\begin{aligned}
P_T &= n_2\times(-\langle n_1,t^*\rangle - n_1\wedge(\alpha\wedge t)) - (-\langle n_1,t^*\rangle + n_1\wedge(\alpha\wedge t))\times n_2 \\
&\quad + n_2\times(-\langle t,n_1^*\rangle - t\wedge(\alpha\wedge n_1)) - (-\langle t,n_1^*\rangle + t\wedge(\alpha\wedge n_1))\times n_2 \\
&= n_2\times(-\langle n_1,t^*\rangle - (\langle n_1,t\rangle\alpha - \langle\alpha,n_1\rangle t)) - (-\langle n_1,t^*\rangle + (\langle n_1,t\rangle\alpha \\
&\quad -\langle\alpha,n_1\rangle t))\times n_2 + n_2\times(-\langle t,n_1^*\rangle - (\langle t,n_1\rangle\alpha - \langle\alpha,t\rangle n_1)) \\
&\quad -(-\langle t,n_1^*\rangle + (\langle t,n_1\rangle\alpha - \langle\alpha,t\rangle n_1))\times n_2 \\
&= 0.
\end{aligned}
$$

Similarly, the distribution parameters of closed dual spatial quaternionic ruled surfaces corresponding to $(N_1)$ and $(N_2)$ are

$$
\begin{aligned}
P_{N_1} &= \frac{h(n_1\times n_1',(n_1\times n_1^*)')}{N(n_1')^2} \\
&= \frac{\frac{1}{2}[(kn_2+rt)\times(-k\langle\alpha,t\rangle n_1 - k\langle\alpha,n_1\rangle t + r\langle\alpha,n_2\rangle n_1 + r\langle\alpha,n_1\rangle n_2 - t) + (k\langle\alpha,t\rangle n_1 + k\langle\alpha,n_1\rangle t - r\langle\alpha,n_2\rangle n_1 - r\langle\alpha,n_1\rangle n_2 + t)\times(-kn_2-rt)]}{k^2+r^2} \\
&= \frac{r}{k^2+r^2},
\end{aligned}
$$

$$
\begin{aligned}
P_{N_2} &= \frac{h(n_2\times n_2',(n_2\times n_2^*)')}{N(n_2')^2} \\
&= \frac{\frac{1}{2}[-r^2t\times(-\langle n_1,n_2^*\rangle + \langle\alpha,n_1\rangle n_2) + r^2(-\langle n_1,n_2^*\rangle - \langle\alpha,n_1\rangle n_2)\times t + 2r - r^2t\times(-\langle n_2,n_1^*\rangle + \langle\alpha,n_2\rangle n_1) + r^2(-\langle n_2,n_1^*\rangle - \langle\alpha,n_2\rangle n_1)\times t]}{r^2} \\
&= \frac{1}{r}.
\end{aligned}
$$

ii) From the equations (21) and (30), the dual spatial quaternionic Steiner vector is

$$
\vec{D} = t\oint r + n_2\oint k + \varepsilon(t^*\oint r + t\oint r^* + n_2^*\oint k + n_2\oint k^*) \tag{37}
$$

Let $\Lambda_T$ be dual angle of pitch of closed dual spatial quaternionic ruled surface corresponding to $(T)$ . Using the equations (22) and (37), we obtain

$$
\Lambda_T = H(D,T) = \frac{1}{2}(D\times\overline{T} + T\times\overline{D})
$$

$$
\begin{aligned}
\Lambda_T &= \frac{1}{2}(2\oint r + 2\varepsilon\oint r^* - \varepsilon(t\times t^*)\oint r - \varepsilon(t\times t^*)\oint r - \varepsilon(n_2\times t^*)\oint k \\
&\quad -\varepsilon(n_2^*\times t)\oint k - \varepsilon(t\times n_2^*)\oint k - \varepsilon(t^*\times t)\oint r - \varepsilon(t^*\times t)\oint r \\
&\quad -\varepsilon(t^*\times n_2)\oint k)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_T &= \frac{1}{2}(2\oint r + 2\varepsilon\oint r^* - \varepsilon(t\wedge t^*)\oint r - \varepsilon(t\wedge t^*)\oint r - \varepsilon(-\langle n_2,t^*\rangle \\
&\quad -\langle\alpha,n_2\rangle t)\oint k - \varepsilon(-\langle n_2^*,t\rangle + \langle\alpha,t\rangle n_2)\oint k - \varepsilon(-\langle t,n_2^*\rangle - \langle\alpha,t\rangle n_2)\oint k
\end{aligned}
$$

$$-\varepsilon(t^* \wedge t)\oint r - \varepsilon(t^* \wedge t)\oint r - \varepsilon(-\langle t^*, n_2\rangle + \langle \alpha, n_2\rangle t)\oint k)$$

$$\Lambda_T = \oint r + \varepsilon \oint r^*.$$

Similarly, the dual angles of pitch of closed dual spatial quaternionic ruled surfaces corresponding to $(N_1)$ and $(N_2)$ are

$$\Lambda_{N_1} = H(D, N_1) = \frac{1}{2}(D \times \overline{N_1} + N_1 \times \overline{D})$$

$$\Lambda_{N_1} = \frac{1}{2}(-\varepsilon(-\langle n_2, n_1^*\rangle - \langle \alpha, n_2\rangle n_1)\oint k - \varepsilon(-\langle t^*, n_1\rangle + \langle \alpha, n_1\rangle t)\oint r$$
$$-\varepsilon(-\langle n_2^*, n_1\rangle + \langle \alpha, n_1\rangle n_2)\oint k - \varepsilon(-\langle t, n_1^*\rangle - \langle \alpha, t\rangle n_1)\oint r$$
$$-\varepsilon(-\langle n_1, t^*\rangle - \langle \alpha, n_1\rangle t)\oint r - \varepsilon(-\langle n_1, n_2^*\rangle - \langle \alpha, n_1\rangle n_2)\oint k$$
$$-\varepsilon(-\langle n_1^*, t\rangle + \langle \alpha, t\rangle n_1)\oint r - \varepsilon(-\langle n_1^*, n_2\rangle + \langle \alpha, n_2\rangle n_1)\oint k)$$

$$\Lambda_{N_1} = 0,$$

$$\Lambda_{N_2} = H(D, N_2) = \frac{1}{2}(D \times \overline{N_2} + N_2 \times \overline{D})$$

$$\Lambda_{N_2} = \frac{1}{2}(2\oint k + 2\varepsilon \oint k^* - 2\varepsilon(n_2 \times n_2^*)\oint k - 2\varepsilon(n_2^* \times n_2)\oint k$$
$$-\varepsilon(-\langle t, n_2^*\rangle - \langle \alpha, t\rangle n_2)\oint r - \varepsilon(-\langle t^*, n_2\rangle + \langle \alpha, n_2\rangle t)\oint r$$
$$-\varepsilon(-\langle n_2, t^*\rangle - \langle \alpha, n_2\rangle t)\oint r - \varepsilon(-\langle n_2^*, t\rangle + \langle \alpha, t\rangle n_2)\oint r)$$

$$\Lambda_{N_2} = \oint k + \varepsilon \oint k^*.$$

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