THE DUAL SPATIAL QUATERNIONIC EXPRESSION OF RULED SURFACES

by

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In this paper, the ruled surface which corresponds to a curve on dual unit sphere is rederived with the help of dual spatial quaternions. We extend the term of dual expression of ruled surface using dual spatial quaternionic method. The correspondences in dual space of closed ruled surfaces are quaternionically expressed. As a consequence, the integral invariants of these surfaces and the relationships between these invariants are shown.

Key words: *real quaternion, spatial quaternion, dual spatial quaternion, closed ruled surface, distribution parameter, dual angle of pitch*

Introduction

The quaternions were discovered in 1843 by William Rowan Hamilton. Quaternions arose historically from Hamilton's essays in the mid 19th century to generalize complex numbers in some way that would be applicable to 3-D space. They are less intuitive than Euler Angles and, therefore, the math can be a little more complicated. This application note covers the basic mathematical concepts needed to understand and use the quaternion outputs of Robotics orientation sensors. The technology did not penetrate the computer animation community until the landmark Siggraph 1985 paper of Shoemake [1]. Shoemake's paper is that it took the concept of the orientation frame for moving 3-D objects and cameras, and the introduced quaternions to animators as a solution. Many studies have been made on quaternionic and dual quaternionic curves, such as [2-6].

The Serret-Frenet formulae for a quaternionic curves in IR^3 and IR^4 are introduced by Bharathi and Nagaraj, [2]. Let $s \in I = [0,1]$ be the arc parameter along the smooth curve:

$$\alpha : [0,1] \to Q$$

$$\alpha(s) = \sum_{n=1}^{3} \alpha_i(s) e_i$$

This is called a spatial quaternionic curve, [2].

Let $\alpha(s)$ be a curve parametrized by arclength function, s. Then for the unit speed spatial quaternionic curve α with frame vectors the following Frenet equations are given [2]:

$$t'(s) = k(s) n_1(s), \quad n_1'(s) = -k(s) t(s) + r(s) n_2(s), \quad n_2'(s) = -r(s) n_1(s) \tag{1}$$

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A surface is said to be ruled if it is generated by moving a straight line continuously in $\mathbf{E}^3$. A practical application of ruled surfaces is that they are used in civil engineering. A ruled surface in IR^3 is a surface containing at least one parameter family of straight lines. Thus a ruled surface has a parametrization in the form:

$$\overrightarrow{\varphi}(s,v) = \overrightarrow{\alpha}(s) + v\,\overrightarrow{x}(s) \tag{2}$$

where α is the anchor curve and x – the generator vector of ruled surface. When the previous ruled surface satisfies $\varphi(s+2\pi, v) = \varphi(s,v)$, it is called closed ruled surface, [7]. It is well known from Muller [8] that a closed ruled surface generated by oriented line of a rigid body has two real integral invariants, the pitch and the angle of pitch. They are known as the integral invariants of a closed ruled surface, [8, 9]. There have been many studies on ruled surfaces. In some studies, the dual expression of the ruled surface has been investigated. However, the ruled surface was not studied as a quaternionic. Altough in [10], Senyurt and Caliskan investigated the ruled surfaceas quaternoic, the rules surface has not studied as a quaternionic. They have quaternionally calculated the integral invariants of the ruled surface.

Dual numbers were introduced in the 19th century by W. K. Clifford. The set of dual numbers given by $ID = \{a + \varepsilon a^* : a, a^* \in IR, \varepsilon^2 = 0\}$ is a commutative ring, the set, $ID^3 = \{\vec{A} = \vec{a} + \varepsilon \vec{a}^* \mid \vec{a}, \vec{a}^* \in IR^3, \varepsilon^2 = 0\}$ meets the all real vector space axioms over the ring. The set is module over the ring ID which is named ID- module or dual space. The elements of ID^3 call dual vector. According to E. Study, a unit dual vector $X(s)$ corresponds only one oriented line where the real vector x shows the direction of this line and the real vector x^* shows the vectorial moment respect to the origin point. A differentiable closed curve $X(s)$ on the dual unit sphere depending on a real parameter s, represents a differentiable family of one parameter straight lines in IR^3 which we call closed ruled surface, [11, 12].

The dual vector expression of a ruled surface is:

$$\overrightarrow{\varphi}(s,u) = \overrightarrow{x}(s) \wedge \overrightarrow{x}^*(s) + u\,\overrightarrow{x}(s) \tag{3}$$

where $\vec{x} \wedge \overrightarrow{x^*}$ is the anchor curve and s is not the arc-parameter of this curve. The ruled surface (X) is given by $\overrightarrow{X}(s) = \overrightarrow{x}(s) + \varepsilon\,\overrightarrow{x}^*(s)$.

The dual angle of a closed ruled surface which is constructed by the dual unit vector $X = x + \varepsilon x^*$ is given:

$$\Lambda_X = \langle D, X \rangle \quad \text{or} \quad \Lambda_X = \lambda_x - \varepsilon L_x \tag{4}$$

where λ_x and L_x are, respectively, the angle of pitch and the pitch of the closed ruled surface, [4].

Preliminaries

Real and dual quaternions

Real quaternion is defined by the $1, e_1, e_2, e_3$. The 1 is real number, e_1, e_2, e_3 are vectors with the following properties:

$$\begin{gathered} e_1{}^2 = e_2{}^2 = e_3{}^2 = e_1 \times e_2 \times e_3 = -1, \quad e_1, e_2, e_3 \in IR^3 \\ e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2 \end{gathered} \tag{5}$$

Real quaternion set can be denoted:

$$\mathbf{K} = \{q = d + ae_1 + be_2 + ce_3 \mid d, a, b, c \in IR\}$$

Let $q_1 = S_{q_1} + V_{q_1} = d_1 + a_1 e_1 + b_1 e_2 + c_1 e_3$ and $q_2 = d_2 + a_2 e_1 + b_2 e_2 + c_2 e_3$ be two quaternions in $\mathbf{K}$, the quaternion multiplication of q_1 and q_2 is given:

$$q_1 \times q_2 = d_1 d_2 - (a_1 a_2 + b_1 b_2 + c_1 c_2) + (d_1 a_2 + a_1 d_2 + b_1 c_2 - c_1 b_2) e_1 + \\ +(d_1 b_2 + b_1 d_2 + b_1 a_2 - a_1 b_2) e_2 + (d_1 c_2 + c_1 d_2 + a_1 b_2 - b_1 a_2) e_3$$

which is equivalent to:

$$q_1 \times q_2 = S_{q_1} S_{q_2} - \langle V_{q_1}, V_{q_2} \rangle + S_{q_1} V_{q_2} + S_{q_2} V_{q_1} + V_{q_1} \wedge V_{q_1} \tag{6}$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product on IR^3 respectively, [13]. The symmetric real-valued bilineer form of h which is defined:

$$h : \mathbf{K} \times \mathbf{K} \to IR$$

$$(q_1, q_2) \to h(q_1, q_2) = \frac{1}{2}(q_1 \times \overline{q}_2 + q_2 \times \overline{q}_1) \tag{7}$$

It is called quaternion inner product, [2]. Let q be a real quaternion. Its conjugate is $\overline{q} = S_q - V_q$. The 3-D real Euclidean space IR^3 is identified with the space of spatial quaternions $\mathbf{Q} = \{q \in \mathbf{K} \mid q + \overline{q} = 0\}$ in obvious manner. In this case, the elements of $\mathbf{Q}$ are $q = ae_1 + be_2 + ce_3$. As a result, the quaternion multiplication of the two spatial quaternions is [2, 13]:

$$q_1 \times q_2 = -\langle q_1, q_2 \rangle + q_1 \wedge q_2 \tag{8}$$

Let q and q^* be two real quaternions. Dual quaternion set can be denoted $\mathbf{K}_D = \{Q = q + \varepsilon q^* \mid q, q^* \in \mathbf{K}\}$. Also we can type $Q = D + Ae_1 + Be_2 + Ce_3$ where $A, B, C, D \in ID$ such that $S_Q = D$ is the scalar part of Q and $V_Q = Ae_1 + Be_2 + Ce_3$ is the vector part of Q. The multiplication of two dual quaternions Q and P is defined:

$$Q \times P = q \times p + \varepsilon(q \times p^* + q^* \times p) \tag{9}$$

It can be easily seen that:

$$Q \times P = S_Q S_P - \langle V_Q, V_P \rangle + S_Q V_P + S_P V_Q + V_Q \wedge V_P \tag{10}$$

in which $\langle , \rangle$ and $\wedge$ are the inner and cross products on ID^3, respectively, [5, 13].

The symmetric dual-valued bilineer form H which is defined:

$$H : \mathbf{K}_D \times \mathbf{K}_D \to ID$$

$$(Q, P) \to H(Q, P) = \frac{1}{2}(Q \times \overline{P} + P \times \overline{Q}) \tag{11}$$

is called dual quaternion inner product. The $\mathbf{Q}_D = \{Q \in \mathbf{K}_D \mid Q + \overline{Q} = 0\}$ is called the dual spatial quaternions set. The elements of this set are called dual spatial quaternion. The element of $\mathbf{Q}_D$ is $Q = Ae_1 + Be_2 + Ce_3$. As a result, the quaternion multiplication of the two spatial dual spatial quaternions is [5, 13]:

$$Q \times P = -\langle Q, P \rangle + Q \wedge P \tag{12}$$

The spatial quaternionic expression of ruled surfaces

Parametric expression of the spatial quaternion expression of a ruled surface is:

$$\vec{\varphi}: I \times IR \to Q$$
$$(s,v) \to \vec{\varphi}(s,v) = \vec{\alpha}(s) + v\,\vec{x}(s) \tag{13}$$

where α spatial quaternionic curve and x spatial quaternionic vector, [10].

The spatial quaternionic definition of distribution parameter of φ is [10]:

$$P_x = \frac{h(x \times x', \alpha')}{N(x')^2} = \frac{1}{2}\frac{(x \times x') \times \overline{\alpha'} + \alpha' \times \overline{(x \times x')}}{N(x')^2} \tag{14}$$

The angle of pitch and the pitch of the closed spatial quaternionic ruled surface are given [10]:

$$\lambda_x = h(\vec{d}, \vec{x}), L_x = h(\vec{V}, \vec{x})$$

Let φ, x and x^* be the spatial quaternionic ruled surface, the directrix of this surface and the vectorial moment of x, respectively. Then there exists a point Z, such that [10]:

$$\vec{x}^* = \vec{z} \times \vec{x} \tag{15}$$

The dual spatial quaternionic expression of ruled surface

Let α be spatial quaternionic curve, $\{t, n_1, n_2\}$ be Frenet vectors of α, $\{t^*, n_1^*, n_2^*\}$ be vectorial moments of Frenet vectors. The $T = t + \varepsilon t^*$, $N_1 = n_1 + \varepsilon n_1^*$, and $N_2 = n_2 + \varepsilon n_2^*$ vectors draw curves on the unit dual sphere. The dual spatial quaternionic expressions of the closed ruled surfaces corresponding to these curves in Euclidean space are given. The relationships between integral invariants of the obtained surfaces are computed as dual spatial quaternionic.

Let us write the dual spatial quaternionic expression of a ruled surface corresponding to the dual curve. According to the eq. (15), the vectorial moment of $\vec{x}$ is:

$$\vec{x}^* = \vec{\alpha} \times \vec{x} \tag{16}$$

where $\vec{\alpha}$ and $\vec{x}$ are orthogonal. Right-multiplying both sides of eq. (16) by x gives:

$$\vec{x}^* \times \vec{x} = (\vec{\alpha} \times \vec{x}) \times \vec{x} \Rightarrow \vec{x}^* \times \vec{x} = -\vec{\alpha}$$

Taking into consideration (8), $\vec{x} \times \vec{x}^* = -\vec{x}^* \times \vec{x}$ is obtained. From the eq. (13), the dual spatial quaternionic expression of ruled surface corresponding to the dual curve is:

$$\vec{\varphi}(s,v) = \vec{x}(s) \times \vec{x}^*(s) + v\,\vec{x}(s) \tag{17}$$

in which $\vec{x}(s) \times \vec{x}^*(s)$ is the anchor curve and s is not the arc parameter of this curve. In the present text, dual spatial quaternionic ruled surface term will be used instead of the dual spatial quaternionic expression of ruled surface corresponding to dual curve.

The arc-parameter of dual curve is $\mathrm{d}\Phi = \mathrm{d}\varphi + \varepsilon \mathrm{d}\varphi^*$, the we obtain:

$$\mathrm{d}\Phi^2 = H(\mathrm{d}X, \mathrm{d}X) = H(X, X)\mathrm{d}s^2$$

$$\mathrm{d}\varphi^2 + 2\varepsilon \mathrm{d}\varphi \mathrm{d}\varphi^* = \frac{1}{2}(\mathrm{d}X \times \mathrm{d}\overline{X} + \mathrm{d}X \times \mathrm{d}\overline{X}) = h(\mathrm{d}x, \mathrm{d}x) + 2\varepsilon h(\mathrm{d}x, \mathrm{d}x^*)$$

Hence, we can write from the last equation:

$$\mathrm{d}\varphi^2 = h(\mathrm{d}x, \mathrm{d}x), \mathrm{d}\varphi\mathrm{d}\varphi^* = h(\mathrm{d}x, \mathrm{d}x^*)$$

Definition 1. Distribution parameter of dual spatial quaternionic ruled surface is:

$$\frac{1}{d} = \frac{h(\mathrm{d}x, \mathrm{d}x^*)}{h(\mathrm{d}x, \mathrm{d}x)} = \frac{\mathrm{d}\varphi^*}{\mathrm{d}\varphi} \tag{18}$$

Definition 2. In the dual plane (V_2, V_3) of the moving system, let us chose a unit dual spatial quaternionic vector:

$$N_1 = \cos\Phi V_2 + \sin\Phi V_3 \tag{19}$$

which makes a dual angle $\Phi = \varphi + \varepsilon\varphi^*$ with V_2 such that during the closed motion when the axis V_1 generates the closed spatial quaternionic ruled surface $V_1(s)$, let the unit vector, N_1 generate a developable spatial quaternionic ruled surface, along the orthogonal trajectory of the closed spatial quaternionic ruled surface. Then we call the total differential of Φ as the dual angle of pitch of the closed spatial quaternionic ruled surface $V_1(s)$. Thus, the dual angle of pitch of $V_1(s)$:

$$\Lambda_{V_1} = -\oint \mathrm{d}\Phi \tag{20}$$

The dual spatial quaternionic Steiner vector is given:

$$\vec{D} = \vec{d} + \varepsilon\vec{d}^* = \oint \vec{W} \tag{21}$$

Theorem 3. The dual angle of pitch of dual spatial quaternionic ruled surface is given:

$$\Lambda_X = H(\vec{D}, \overline{X}) \tag{22}$$

Proof. The two orthonormal systems $N = \{\vec{N}_1, \vec{N}_2, \vec{N}_3\}$ and $V = \{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ are right-handed systems which represent the fixed space and the moving space, respectively. Assume the transition matrix is:

$$B = \begin{bmatrix} 0 & 0 & 1 \\ \cos\Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos\Phi & 0 \end{bmatrix} \tag{23}$$

Hence, we can write:

$$V = BN \tag{24}$$

Here, if we differentiate eq. (24) in terms of s, it becomes:

$$\mathrm{d}V_2 = -\mathrm{d}\Phi V_3,\ \mathrm{d}V_3 = \mathrm{d}\Phi V_2 \tag{25}$$

Solving eq. (25) by using eq. (11), we obtain:

$$-\mathrm{d}\Phi = H(\mathrm{d}V_2, V_3) = -H(V_2, \mathrm{d}V_3) \tag{26}$$

where $V_1 = v_1 + \varepsilon v_1^*$, $V_2 = v_2 + \varepsilon v_2^*$, and $V_3 = v_3 + \varepsilon v_3^*$ are dual spatial quaternionic vectors. By taking dual quaternionic inner product and equation $\mathrm{d}V_i = \sum_{j=1}^{3} \Psi_{ij} V_j$ into account, we solve:

$$H(\mathrm{d}V_2, V_3) = \frac{1}{2}(\mathrm{d}V_2 \times \overline{V_3} + V_3 \times \mathrm{d}\overline{V_2}) =$$

$$= \frac{1}{2}\left[(-\Psi_3 V_1 + \Psi_1 V_3) \times \overline{V_3} + V_3 \times (\overline{-\Psi_3 V_1 + \Psi_1 V_3})\right] = \Psi_1$$

and

$$H(\mathrm{d}V_3, V_2) = \frac{1}{2}(\mathrm{d}V_3 \times \overline{V_2} + V_2 \times \overline{\mathrm{d}V_3}) =$$

$$= \frac{1}{2}[(\Psi_2 V_1 - \Psi_1 V_2) \times \overline{V_2}) + V_2 \times (\overline{\Psi_2 V_1 - \Psi_1 V_2})] = -\Psi_1$$

$$H(\mathrm{d}V_2, V_3) = -H(\mathrm{d}V_3, V_2) = \Psi_1 \tag{27}$$

is obtained for the dual angle of pitch of closed dual spatial quaternionic ruled surface drawn by a dual spatial quaternionic vector $\vec{V_1} = \vec{v}_1 + \varepsilon \vec{v}_1^*$.

Now let us find the dual angle of pitch of the dual spatial quaternionic ruled surface drawn by a dual quaternionic vector $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ which moves strongly on the $\{\vec{V_1}, \vec{V_2}, \vec{V_3}\}$ system.

$$X = CV \tag{28}$$

in which C is an orthogonal matrix. Considering reference [8] and dual quaternion inner product, dual angle of pitch is obtained:

$$\Lambda_X = H(\vec{D}, \overline{X}) \tag{29}$$

Theorem 4. Let $T = t + \varepsilon t^*$, $N_1 = n_1 + \varepsilon n_1^*$ and $N_2 = n_2 + \varepsilon n_2^*$ be the dual spatial quaternionic vectors on the unit dual sphere. Then the dual spatial quaternionic Darboux vector is given:

$$W = RT + KN_2 = rt + kn_2 + \varepsilon(rt^* + r^* t + kn_2^* + k^* n_2) \tag{30}$$

Proof. Let the dual spatial quaternionic Darboux vector be:

$$W = a_1 T + a_2 N_1 + a_3 N_2 \tag{31}$$

Right-multiplying both sides of eq. (31) by T gives:

$$W \times T = -a_1 - a_2 N_2 + a_3 N_1 \tag{32}$$

On the other hand, it can be written:

$$W \times T = -\langle W, T\rangle + W \wedge T = -a_1 + \mathrm{d}T = -a_1 + KN_1 \tag{33}$$

From the eqs. (32) and (33), $a_2 = 0$ and $a_3 = K$ are found. Similarly, it can be written:

$$W \times N_1 = -a_2 - KT + RN_2 \tag{34}$$

and $a_1 = R$ and $a_3 = K$ are found. If the values found are replaced by eq. (31), then:

$$W = RT + KN_2 = rt + kn_2 + \varepsilon(rt^* + r^* t + kn_2^* + k^* n_2) \tag{35}$$

is reached, wherein $K = k + \varepsilon k^*$ and $R = r + \varepsilon r^*$.

The geometric location of $T = t + \varepsilon t^*$, $N_1 = n_1 + \varepsilon n_1^*$ and $N_2 = n_2 + \varepsilon n_2^*$ dual spatial quaternionic vectors draws dual curves on the dual sphere. If these curves are closed, ruled surfaces corresponding to these curves are closed. These closed dual curves are shown as (T), (N_1), and (N_2), respectively. The distribution parameters and dual angles of pitch for closed dual spatial quaternionic ruled surfaces corresponding to (T), (N_1), and (N_2) will be given.

Theorem 5. The distribution parameters and the dual angles of the pitch of closed dual spatial quaternionic ruled surfaces corresponding to $(T), (N_1), (N_2)$ are:

– (i) $$P_T = 0, \quad P_{N_1} = \frac{r}{k^2 + r^2}, \quad P_{N_2} = \frac{1}{r}$$

– (ii) $$\Lambda_T = \oint r + \varepsilon \oint r^*, \quad \Lambda_{N_1} = 0, \quad \Lambda_{N_2} = \oint k + \varepsilon \oint k^*$$

Proof. (i) From the eq. (17), the parametric equations of closed dual spatial quaternionic ruled surfaces corresponding to $(T),(N_1),(N_2)$ are:

$$\begin{aligned}
\varphi_t(s,v) &= \vec{t}(s) \times \vec{t}^*(s) + v\vec{t}(s), \vec{t}^*(s) = \vec{\alpha}(s) \times \vec{t}(s) \\
\varphi_{n_1}(s,v) &= \vec{n}_1(s) \times \vec{n}_1^*(s) + v\vec{n}_1(s), \vec{n}_1^*(s) = \vec{\alpha}(s) \times \vec{n}_1(s) \\
\varphi_{n_2}(s,v) &= \vec{n}_2(s) \times \vec{n}_2^*(s) + v\vec{n}_2(s), \vec{n}_2^*(s) = \vec{\alpha}(s) \times \vec{n}_2(s)
\end{aligned} \tag{36}$$

respectively.

Let us calculate distribution parameters of these surfaces:

By formula of the eq. (14), we obtain:

$$P_T = \frac{h[t \times t', (t \times t^*)']}{N(t')^2} = \frac{h[kn_2, k(n_1 \times t^*)] + h[kn_2, k(t \times n_1^*)]}{N(t')^2} =$$

$$= \frac{\frac{1}{2}(kn_2 \times k\overline{(n_1 \times t^*)} + k(n_1 \times t^*) \times k\overline{n_2}) + \frac{1}{2}(kn_2 \times k\overline{(t \times n_1^*)} + k(t \times n_1^*) \times k\overline{n_2})}{h(kn_1, kn_1)} =$$

$$= n_2 \times \overline{(n_1 \times t^*)} + (n_1 \times t^*) \times \overline{n_2} + n_2 \times \overline{(t \times n_1^*)} + (t \times n_1^*) \times \overline{n_2}$$

Since t^* and n_1^* are vectorial moment, the distribution parameter of closed dual spatial quaternionic ruled surface corresponding to (T) is:

$$\begin{aligned}
P_T &= n_2 \times [-\langle n_1, t^* \rangle - n_1 \wedge (\alpha \wedge t)] - [-\langle n_1, t^* \rangle + n_1 \wedge (\alpha \wedge t)] \times n_2 + \\
&+ n_2 \times [-\langle t, n_1^* \rangle - t \wedge (\alpha \wedge n_1)] - [-\langle t, n_1^* \rangle + t \wedge (\alpha \wedge n_1)] \times n_2 = \\
&= n_2 \times [-\langle n_1, t^* \rangle - (\langle n_1, t \rangle \alpha - \langle \alpha, n_1 \rangle t)] - [-\langle n_1, t^* \rangle + (\langle n_1, t \rangle \alpha - \\
&- \langle \alpha, n_1 \rangle t)] \times n_2 + n_2 \times [-\langle t, n_1^* \rangle - (\langle t, n_1 \rangle \alpha - \langle \alpha, t \rangle n_1)] - \\
&- [-\langle t, n_1^* \rangle + (\langle t, n_1 \rangle \alpha - \langle \alpha, t \rangle n_1)] \times n_2 = 0
\end{aligned}$$

Similarly, the distribution parameters of closed dual spatial quaternionic ruled surfaces corresponding to (N_1) and (N_2) are:

$$P_{N_1} = \frac{h[n_1 \times n_1', (n_1 \times n_1^*)']}{N(n_1')^2} =$$

$$= \frac{\left\{\begin{array}{l} \frac{1}{2}[(kn_2 + rt) \times (-k\langle \alpha, t \rangle n_1 - k\langle \alpha, n_1 \rangle t + r\langle \alpha, n_2 \rangle n_1 + r\langle \alpha, n_1 \rangle n_2 - t) + \\ + (k\langle \alpha, t \rangle n_1 + k\langle \alpha, n_1 \rangle t - r\langle \alpha, n_2 \rangle n_1 - r\langle \alpha, n_1 \rangle n_2 + t) \times (-kn_2 - rt)] \end{array}\right\}}{k^2 + r^2} = \frac{r}{k^2 + r^2}$$

$$P_{N_2} = \frac{h[n_2 \times n_2', (n_2 \times n_2^*)']}{N(n_2')^2} =$$

$$=\frac{\begin{Bmatrix}\frac{1}{2}[-r^2 t\times(-\langle n_1,n_2^*\rangle+\langle\alpha,n_1\rangle n_2)+r^2(-\langle n_1,n_2^*\rangle-\langle\alpha,n_1\rangle n_2)\times t+2r-\\ -r^2 t\times(-\langle n_2,n_1^*\rangle+\langle\alpha,n_2\rangle n_1)+r^2(-\langle n_2,n_1^*\rangle-\langle\alpha,n_2\rangle n_1)\times t]\end{Bmatrix}}{r^2}=\frac{1}{r}$$

(ii) From the eqs. (21) and (30), the dual spatial quaternionic Steiner vector is:

$$\vec{D}=t\oint r+n_2\oint k+\varepsilon\left(t^*\oint r+t\oint r^*+n_2^*\oint k+n_2\oint k^*\right) \tag{37}$$

Let Λ_T be dual angle of pitch of closed dual spatial quaternionic ruled surface corresponding to (T). Using the eqs. (22) and (37), we obtain:

$$\Lambda_T=H(D,T)=\frac{1}{2}(D\times\overline{T}+T\times\overline{D})$$

$$\Lambda_T=\frac{1}{2}\Big[2\oint r+2\varepsilon\oint r^*-\varepsilon(t\times t^*)\oint r-\varepsilon(t\times t^*)\oint r-\varepsilon(n_2\times t^*)\oint k-$$
$$-\varepsilon(n_2^*\times t)\oint k-\varepsilon(t\times n_2^*)\oint k-\varepsilon(t^*\times t)\oint r-\varepsilon(t^*\times t)\oint r-\varepsilon(t^*\times n_2)\oint k)\Big]$$

$$\Lambda_T=\frac{1}{2}(2\oint r+2\varepsilon\oint r^*-\varepsilon(t\wedge t^*)\oint r-\varepsilon(t\wedge t^*)\oint r-\varepsilon(-\langle n_2,t^*\rangle-$$
$$-\langle\alpha,n_2\rangle t)\oint k-\varepsilon(-\langle n_2^*,t\rangle+\langle\alpha,t\rangle n_2)\oint k-\varepsilon(-\langle t,n_2^*\rangle-\langle\alpha,t\rangle n_2)\oint k-$$
$$-\varepsilon(t^*\wedge t)\oint r-\varepsilon(t^*\wedge t)\oint r-\varepsilon(-\langle t^*,n_2\rangle+\langle\alpha,n_2\rangle t)\oint k)$$

$$\Lambda_T=\oint r+\varepsilon\oint r^*$$

Similarly, the dual angles of pitch of closed dual spatial quaternionic ruled surfaces corresponding to (N_1) and (N_2) are:

$$\Lambda_{N_1}=H(D,N_1)=\frac{1}{2}(D\times\overline{N_1}+N_1\times\overline{D})=$$
$$=\frac{1}{2}(-\varepsilon(-\langle n_2,n_1^*\rangle-\langle\alpha,n_2\rangle n_1)\oint k-\varepsilon(-\langle t^*,n_1\rangle+\langle\alpha,n_1\rangle t)\oint r-$$
$$-\varepsilon(-\langle n_2^*,n_1\rangle+\langle\alpha,n_1\rangle n_2)\oint k-\varepsilon(-\langle t,n_1^*\rangle-\langle\alpha,t\rangle n_1)\oint r-$$
$$-\varepsilon(-\langle n_1,t^*\rangle-\langle\alpha,n_1\rangle t)\oint r-\varepsilon(-\langle n_1,n_2^*\rangle-\langle\alpha,n_1\rangle n_2)\oint k-$$
$$-\varepsilon(-\langle n_1^*,t\rangle+\langle\alpha,t\rangle n_1)\oint r-\varepsilon(-\langle n_1^*,n_2\rangle+\langle\alpha,n_2\rangle n_1)\oint k)=0$$

$$\Lambda_{N_2}=H(D,N_2)=\frac{1}{2}(D\times\overline{N_2}+N_2\times\overline{D})=$$
$$=\frac{1}{2}(2\oint k+2\varepsilon\oint k^*-2\varepsilon(n_2\times n_2^*)\oint k-2\varepsilon(n_2^*\times n_2)\oint k-$$
$$-\varepsilon(-\langle t,n_2^*\rangle-\langle\alpha,t\rangle n_2)\oint r-\varepsilon(-\langle t^*,n_2\rangle+\langle\alpha,n_2\rangle t)\oint r-$$
$$-\varepsilon(-\langle n_2,t^*\rangle-\langle\alpha,n_2\rangle t)\oint r-\varepsilon(-\langle n_2^*,t\rangle+\langle\alpha,t\rangle n_2)\oint r)$$

$$\Lambda_{N_2}=\oint k+\varepsilon\oint k^*.$$

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