N-BISHOP DARBOUX VECTOR OF THE SPACELIKE CURVE WITH SPACELIKE BINORMAL

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In this article, the N-Bishop frame in Minkowski space is investigated for spacelike curves with a spacelike binormial. Some features of the normal expansion are proven via for the spacelike curve. Then, a new Darboux frame called by the N-Bishop Darboux frame is introduced at first time. Furthermore, some geometrical properties of the N-Bishop Darboux frame are proven. As a result, the N-Bishop Darboux axis and momentum rotation vector are calculated.

Key Words: spacelike curve, N-Bishop frame, N-Bishop Darboux frame

Introduction

A Serret-Frenet frame is a standard tool used to analyze the properties of a curve that does not change under Euclidean movements and whose geometric properties are independent of any motion. This frame is a very useful method for calculating some differential geometrical properties such as curvature and torsion at every point of the curve. Recently, different types of frames have begun to be examined in order to examine the invariants of the curve. Developed by L. Bishop [1] in 1975, the Bishop frame succeeded in building a more practical alternative parallel frame without using the second derivative on the curve. Since the discovery of the Bishop frame, many studies have been made on curves and surfaces. The Bishop frame is investigated for various special curves such as involute-evolute curves, Bertrand curves, helix, slant helix. B. Bükcü and his colleagues [2,3] examined the Bishop Darboux axis of a special Bishop movement and a space curriculum. The slant helix according to the Bishop frame was studied by Bükcü [4] in 2009. In the same year M. Karacan [5] calculated the Bishop Darboux rotation axes of spacelike curves. The spacelike curves with spacelike binormal and the timelike curve in Minkowski space were studied according to the Bishop frame, [6-7]. Cetin and his colleagues [8] examined Smarandache curves in 2014 with this alternative frame. In 2010, S. Yılmaz and his colleagues [9] created a new Bishop alternative frame by introducing a different approach to the Bishop frame, which is called the Type-2 Bishop frame. Many new works have been started with the new type-2 Bishop frame. S. Yılmaz [10] investigated new spherical indicators according to Type-2 Bishop frame. Furthermore, S. Yılmaz and his colleagues [11] examined the characterizations of spacelike curves in Minkowski 3-space according to type-2 Bishop in 2015. Much work has been done on the Bishop frame as defined by L.Bishop and S. Yılmaz. Moreover, a different frame $\{N, C, N \times C = W\}$ was given in [12]. Y.Yaylı and O. Keskin [13] defined N-Bishop frame by providing a different approach to the alternative Bishop frame using the frame $\{N, C, W\}$ in 2017.

Yücesan [14] worked on the Darboux rotation axes in Minkowski space in 2004. On the other hand there are some basics of Minkowski 3-space in [15-18].

In this study, spacelike curves with spacelike binormal in Minkowski space were investigated according to N-Bishop frame in Minkowski 3-space for the first time. Firstly, the N-Bishop alternative formulas for the spacelike curves with spacelike binormal in the Minkowski space of the alternate frame are calculated. Then N-Bishop frame and the transition matrix of the N-Bishop frame were studied in Minkowski 3-space. Later, the properties of these curves related to normal expansions were examined. For the first time in the last part of the work the N-Bishop Darboux frame was defined and some theoretical features of this frame were investigated.

Preliminaries

H. Minkowski developed the Minkowski space-time model using Einstein's special relativity theory. In 1907, he showed that there could be a 4-component space-time that combined time with three spatial dimensions. In this work, we consider Minkowski space $\mathbb{R}_1^3 = \left(\mathbb{R}^3, g\left(,\right)\right)$, which is defined by the following metric $g\left(\xi,\zeta\right) = \left\langle \xi,\zeta \right\rangle_L = -\xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3$ for $\xi = \left(\xi_1,\xi_2,\xi_3\right)$, $\zeta = \left(\zeta_1,\zeta_2,\zeta_3\right)$. Metric $g\left(,\right)$ is called the Lorentzian metric, which is a non-degenerate metric of index 1. In Minkowski space, if $g\left(\xi,\xi\right) > 0$ or $\xi = 0$, a vector $\xi \in \mathbb{R}_1^3$ is called spacelike; if $g\left(\xi,\xi\right) < 0$, it is called timelike; if $g\left(\xi,\xi\right) = 0$ and $\xi \neq 0$, it is called a lightlike vector. The exterior product in Minkowski space is denoted by $\times_L$. Furthermore, the norm in Minkowski space is described by $\left\|\xi\right\|_L = \sqrt{\left|g\left(\xi,\xi\right)\right|}$. The vectoral product of spacelike curves with a spacelike binormal is defined by $\xi \times_L \zeta = \left(\xi_3\zeta_2 - \xi_2\zeta_3, \xi_3\zeta_1 - \xi_1\zeta_3, \xi_1\zeta_2 - \xi_2\zeta_1\right)$ for vectors $\xi,\zeta \in \mathbb{R}_1^3$. If vectors e_1,e_2,e_3 are timelike, spacelike, and spacelike, respectively, then the vectoral product holds

$$e_1 \times_L e_2 = e_3, e_2 \times_L e_3 = -e_1, e_3 \times_L e_1 = e_2.$$

As we know, the derivative formula of Serret-Frenet frame is as follows

$$T'(s) = \kappa N(s), N'(s) = -\kappa T(s) + \tau B(s), B'(s) = -\tau B(s).$$

Bishop frame is a new alternative frame designed in 1975. The alternative Bishop frame is relatively parallel to the unit tangent field, and its derivative matrix is

$$T'(s) = k_1 N_1(s) + k_2 N_2(s), N_1'(s) = -k_1 T(s), N_2'(s) = -k_2 T(s)$$

where k_1 and k_2 are the curvatures of the Bishop frame, [1]. From another viewpoint, Yılmaz et al. defined a new Bishop frame in 2010, which is parallel to the binormial vector field, and its derivatives are identified by $N_1'(s) = -k_1 B(s), N_2'(s) = -k_2 B(s), B'(s) = k_1 N_1(s) + k_2 N_2(s)$. This new Bishop frame is called Type-2 Bishop frame [9-11]. The alternative moving frame along a curve in a Euclidean 3-space is defined by $\left\{N, C, N \times C = W\right\}$, where $N, C = \frac{N'}{\left\|N'\right\|}$ and $\boldsymbol{W} = \frac{\boldsymbol{\tau T} + \boldsymbol{\kappa B}}{\left\|\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2\right\|}$. The derivatives of the alternative moving frame are

$$N'(s) = f(s)C(s),\ C'(s) = -f(s)N(s) + g(s)W(s),\ W'(s) = -g(s)C(s)$$

where $f = \sqrt{\kappa^2 + \tau^2}$ and $g = \frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{\kappa^2 + \tau^2} = \sigma f$ are the differentiable functions, [12]. Keskin and Yaylı first introduced the N-Bishop frame for a normal direction curve, which was defined as an integral curve of the principal normal of a curve. The derivative matrix of this new N-Bishop frame is

$$N'(s) = k_1 N_1(s) + k_2 N_2(s),\ N_1'(s) = -k_1 N(s),\ N_2'(s) = -k_2 N(s),\ [13].$$

Main Results

In this section, we will define the N-Bishop frame and N-Bishop Darboux vector for a spacelike curve with a spacelike binormal in Minkowski 3-space. If a curve α is a spacelike curve with a spacelike binormal, then T and N are spacelike, whereas N is timelike. Thus, the inner products of the Frenet frame vectors are $\langle T,T\rangle_L=1, \langle N,N\rangle_L=-1, \langle B,B\rangle_L=1$. For the adapted N-Bishop frame of the spacelike curve with a spacelike binormal, the principal normal N is timelike, and N_1, N_2 are spacelike. Furthermore, the vectors satisfy the equations

$$\langle N,N\rangle_L=-1, \langle N_1,N_1\rangle_L=1, \langle N_2,N_2\rangle_L=1 \text{ and } N\times_L N_1=N_2,\ N_1\times_L N_2=-N,\ N_2\times_L N=N_1.$$

Theorem 3.1. Let α be a spacelike curve with a spacelike binormal with unit speed. The derivative matrix of the adapted frame $\{N,N_1,N_2\}$ is defined by

$$\begin{bmatrix} N' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & p_{01} & p_{02} \\ p_{01} & 0 & p_{12} \\ p_{02} & -p_{12} & 0 \end{bmatrix} \begin{bmatrix} N \\ N_1 \\ N_2 \end{bmatrix}$$

for functions $p_{ij}, i, j=0,1,2.$

Proof. As we know, the orthonormal frame can be written as

$$N' = p_{00}N + p_{01}N_1 + p_{02}N_2$$
$$N_1' = p_{10}N + p_{11}N_1 + p_{12}N_2$$
$$N_2' = p_{20}N + p_{21}N_1 + p_{22}N_2$$

for functions $p_{ij}, i, j=0,1,2.$ If curve α is a spacelike curve with a spacelike binormal, then principal normal N is timelike, whereas N_1 and N_2 are spacelike. Then, we have the following equations:

$$-p_{00}=\langle N',N\rangle_L=0, \quad -p_{10}=\langle N_1',N\rangle_L, \quad -p_{20}=\langle N_2',N\rangle_L,$$
$$p_{01}=\langle N',N_1\rangle_L, \quad p_{11}=\langle N_1',N_1\rangle_L=0, \quad p_{21}=\langle N_2',N_1\rangle_L,$$
$$p_{02}=\langle N',N_2\rangle_L, \quad p_{12}=\langle N_1',N_2\rangle_L, \quad p_{22}=\langle N_2',N_2\rangle_L=0.$$

First, taking the derivative of $\langle N,N_1\rangle_L=0$, we have $p_{01}=p_{10}$. Second, if we take the derivative of $\langle N,N_2\rangle_L=0$, we obtain $p_{02}=p_{20}$. Third, taking the derivative of $\langle N_1,N_2\rangle_L=0$, we obtain $p_{12}=-p_{21}$. Thus, the matrix of the derivation formulas of the adapted frame $\{N,N_1,N_2\}$ is

$$\begin{bmatrix} N' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & p_{01} & p_{02} \\ p_{01} & 0 & p_{12} \\ p_{02} & -p_{12} & 0 \end{bmatrix} \begin{bmatrix} N \\ N_1 \\ N_2 \end{bmatrix}.$$

Theorem 3.2. Let α be a spacelike curve with a spacelike binormal with unit speed. If $\{N,C,W\}$ is Serret-Frenet frame, and $\{N,N_1,N_2\}$ is Bishop frame, then there is a relation between two frames as follows

$$\begin{bmatrix} N \\ C \\ W \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N \end{bmatrix}.$$

Proof. The tangent vector of the adapted frame $\{N,N_1,N_2\}$ is identified by

$$W=\sin\theta N_1+\cos\theta N_2. \tag{1}$$

If we differentiate eq.(1) with respect to s, we obtain

$$W' = -g.C = (\theta' - p_{12})(\cos\theta N_1 - \sin\theta N_2) + (\sin\theta p_{01} + \cos\theta p_{02})N \tag{2}$$

eq.(2) shows that W is relatively parallel if and only if $\theta' - p_{12} = 0$. Since there is a solution for θ that satisfies all initial condition, there is a local relatively parallel normal field. In eq.(2), the results $\theta' = -g$ and $C = \cos\theta N_1 - \sin\theta N_2$ are obtained. In addition, we can take the coefficients as $p_{01} = k_1 = f\cos\theta$ and $p_{02} = k_2 = f\sin\theta$.

Theorem 3.3. Let α be a C^k regular spacelike curve with a spacelike binormal in Minkowski 3-space. There is a unique C^{k-1} which is a relatively parallel field X along such that α and the scalar product of the two fields is relatively constant.

Proof. If the scalar product $\langle X, Y\rangle_L$ is constant, then the derivative of the scalar product becomes zero. Therefore, we assume that and are normal with derivatives $\lambda_1 N$ and $\lambda_2 N$. Then, the derivative of $\langle X, Y\rangle_L$ is

$$\begin{aligned}\frac{d}{dt}\langle X,Y\rangle_L &= \langle X',Y\rangle_L + \langle X,Y'\rangle_L \\ &= \lambda_1\langle N,Y\rangle_L + \lambda_2\langle X,N\rangle_L = 0\end{aligned}$$

as desired. Thus, $\langle X, Y\rangle_L$ is constant, [1]. Consequently, the derivative matrix of the alternative frame $\{N, N_1, N_2\}$ of the spacelike curve with a spacelike binormal is obtained by

$$\begin{bmatrix} N' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} N \\ N_1 \\ N_2 \end{bmatrix}$$

where $f = \|N'\|_L = \sqrt{\tau^2 - \kappa^2} = \sqrt{k_1^2 + k_2^2}$ and $\theta = -\int_{s_0}^{s} g(t) = -\tan\left(\dfrac{k_2}{k_1}\right)$.

Theorem 3.4. Let $\alpha = \alpha(s)$ be a regular spacelike curve and k_1, k_2 be N-Bishop curvatures of this curve in the Minkowski 3-space. Then, the normal direction curve $\int N ds$ is a spherical curve if and only if $\lambda_1 k_1 + \lambda_2 k_2 - 1 = 0.$ Here, λ_1 and λ_2 are constant.

Proof. Let P be a point on sphere S^2. Taking the spherical curve as $\gamma(s) = \int N(s)ds$, we have $\langle \gamma - P, \gamma - P\rangle_L = r^2.$ The differentiation of both sides yields $\lambda_1 k_1 + \lambda_2 k_2 - 1 = 0$, where λ_1 and λ_2 are constant.

$$\begin{aligned}&\langle \gamma - P, \gamma - P\rangle = r^2 \\ &\left\langle (\gamma - P)', \gamma - P\right\rangle_L + \left\langle \gamma - P, (\gamma - P)'\right\rangle_L = 0 \\ &\langle N, \gamma - P\rangle_L = 0.\end{aligned}$$

From this equation, $\gamma - P = \lambda_1 N_1 + \lambda_2 N_2$. Because the vectors N_1, N_2 are spacelike vectors, the coefficients $\lambda_1 = \langle \gamma - P, N_1\rangle_L$ and $\lambda_2 = \langle \gamma - P, N_2\rangle_L$ are obtained. Differentiating the coefficients λ_1 and λ_2, we obtain

$$\begin{aligned}\lambda_1' &= \langle \gamma - P, N_1\rangle_L{}' \\ &= \langle N, N_1\rangle_L + k_1\langle \gamma - P, N\rangle_L \\ &= k_1\langle \gamma - P, N\rangle_L,\end{aligned} \quad \text{and} \quad \begin{aligned}\lambda_2' &= \langle \gamma - P, N_2\rangle_L{}' \\ &= \langle N, N_2\rangle_L + k_2\langle \gamma - P, N\rangle_L \\ &= k_2\langle \gamma - P, N\rangle_L.\end{aligned}$$

Furthermore, by differentiating $\langle \gamma - P, N\rangle_L$, we have

$$\langle \gamma - P, N\rangle_L' = -1 + \langle \lambda_1 N_1 + \lambda_2 N_2, k_1 N_1 + k_2 N_2\rangle_L$$
$$= -1 + \lambda_1 k_1 + \lambda_2 k_2.$$

As a result, we obtain the final equation $\lambda_1 k_1 + \lambda_2 k_2 - 1 = 0.$ Contrarily, we assume that (k_1, k_2) is on the line $\lambda_1 x + \lambda_2 y - 1 = 0$, where λ_1 and λ_2 are constant. From the derivation, we obtain

$$\overrightarrow{p\gamma} = \lambda_1 N_1 + \lambda_2 N_2$$
$$-p' = -\gamma' + (\lambda_1 N_1 + \lambda_2 N_2)'$$
$$= (-1 + \lambda_1 k_1 + \lambda_2 k_2) N.$$

We have $\lambda_1 k_1 + \lambda_2 k_2 - 1 = 0$. The derivative of the norm $\left\|\overrightarrow{p\gamma}\right\|_L^2 = \langle \gamma - p, \gamma - p\rangle$ is $\frac{d}{ds}\langle \gamma - p, \gamma - p\rangle_L = \langle N, \gamma - p\rangle_L = 0.$ Because of the inner product $\langle \gamma - p, \gamma - p\rangle = r^2$ is constant, we can take the constants r and p as radius and center of a spacelike curve on Lorentzian sphere. Thus the normal direction curve of a spacelike curve with spacelike binormal is a spherical curve in Minkowski 3-space.

Definition 3.1. Let $\chi = \chi(s)$ be a regular spacelike curve with a spacelike binormal and k_1, k_2 be the N-Bishop curvatures of this curve in the Minkowski 3-space. The vector $\omega = N \times_L N' = -k_2 N_1 + k_1 N_2$ is called the N-Bishop Darboux vector, and the unit vector of ω is

$$\Sigma = \frac{\omega}{\|\omega\|_L} = \frac{-k_2 N_1 + k_1 N_2}{\sqrt{k_1^2 + k_2^2}}.$$

Theorem 3.5. If $\chi = \chi(s)$ is a regular spacelike curve with spacelike binormal in Minkowski 3-space, the following formulas hold

a) $N' \times N'' = (k_1' k_2 - k_1 k_2') N - f^2 \omega$,

b) $\det(N, N', N'') = k_1 k_2' - k_1' k_2$,

c) $\dfrac{\det(N, N', N'')}{\|N \times N'\|_L^2} = \dfrac{k_1' k_2 - k_2' k_1}{k_1^2 + k_2^2}$,

where ω is the Darboux vector of the spacelike curve with a spacelike binormal; k_1, k_2 are N-Bishop curvatures, and $f = \sqrt{k_1^2 + k_2^2}$.

Proof. a) The derivative matrix of the N-Bishop frame shows that $N' = k_1 N_1 + k_2 N_2$. Thus, the second derivative of the principal normal vector is obtained by

$$N'' = (k_1 N_1 + k_2 N_2)'$$
$$= (k_1^2 + k_2^2) N + k_1' N_1 + k_2' N_2$$
$$= f^2 N + k_1' N_1 + k_2' N_2.$$

Furthermore, the inner products of N'' holds $\langle N'', N\rangle_L = -f^2, \langle N'', N_1\rangle_L = k_1', \langle N'', N_2\rangle_L = k_2'.$ From definition of Darboux vector $N' = \omega \times_L N$, we obtain

$$N' \times_L N'' = (\omega \times_L N) \times_L N''$$
$$= \langle N, N''\rangle_L \omega - \langle \omega, N''\rangle_L N$$
$$= (k_1' k_2 - k_1 k_2') N - f^2 \omega.$$

b) Using the property of the inner product in Minkowski space, we have

$$\langle N, N'\times N''\rangle_L = \langle N, (k_1'k_2 - k_1k_2')N - f^2\omega\rangle_L$$
$$= (k_1'k_2 - k_1k_2')\langle N, N\rangle_L - f^2\langle N,\omega\rangle_L$$
$$= k_1k_2' - k_1'k_2.$$

c) From the outer product $N' = \omega \times_L N$, we can also write $\omega = N \times_L N'$. The norm of is obtained by $\|N\times N'\|_L = \|\omega\|_L = \sqrt{k_1^2+k_2^2}$. The angle $\theta = -\tan^{-1}\left(\frac{k_2}{k_1}\right)$ can be also calculated by

$$\theta' = -\frac{\left(\frac{k_2}{k_1}\right)'}{1+\left(\frac{k_2}{k_1}\right)^2} = \frac{k_1'k_2 - k_2'k_1}{k_1^2+k_2^2} = -\frac{\langle N, N'\times N''\rangle_L}{\|N\times N'\|_L^2}$$
$$= \pm\frac{\det(N, N', N'')}{\|N\times N'\|_L^2} = \pm g(t).$$

The N-Bishop Darboux rotation of Bishop frame can be divided into two rotation motions. The vector N_1 rotates with a k_1 angular speed around N_2. Substituting Darboux vector to the following equations, we have

$$\begin{aligned} N_1' &= \omega \times_L N_1 \\ &= (-k_2N_1 + k_1N_2)_L \times_L N_1 \\ &= -k_2N_1\times_L N_1 + k_1N_2\times_L N_1 \\ &= k_1N, \end{aligned} \quad \text{and} \quad \begin{aligned} N_2' &= \omega \times_L N_2 \\ &= (-k_2N_1 + k_1N_2)\times_L N_2 \\ &= -k_2N_1\times_L N_2 + k_1N_2\times_L N_2 \\ &= k_2N. \end{aligned}$$

The vector $\Sigma = \frac{\omega}{\|\omega\|_L}$ rotates with $\theta' = \frac{k_1'k_2 - k_2'k_1}{k_1^2+k_2^2}$ around N and angular speed $\|\omega\|_L$ around $\frac{\omega}{\|\omega\|_L}$. From the definition of the N-Bishop Darboux axis, we have $N' = \omega\times_L N$. The unit vector of the N-Bishop Darboux vector E is calculated by $\Sigma = \frac{\omega}{\|\omega\|_L} = \frac{-k_2N_1 + k_1N_2}{\sqrt{k_1^2+k_2^2}}$. The derivative of the Bishop Darboux vector is $\omega' = (-k_2N_1 + k_1N_2)' = -k_2'N_1 + k_1'N_2$. The derivative of the norm of the N-Bishop Darboux vector is $\|\omega\|_L' = \|-k_2N_1 + k_1N_2\|_L' = \frac{k_1k_1' + k_1k_2'}{\sqrt{k_1^2+k_2^2}}$. Furthermore, the unit N-Bishop Darboux vector is differentiated as

$$\Sigma' = \left(\frac{\omega}{\|\omega\|_L}\right)' = \left(\frac{-k_2N_1 + k_1N_2}{\sqrt{k_1^2+k_2^2}}\right)'$$
$$= \frac{(k_1'k_2 - k_2'k_1)}{k_1^2+k_2^2}\frac{(k_1N_1 + N_2k_2)}{\sqrt{k_1^2+k_2^2}}$$
$$= \theta'(\Sigma\times N) + 0.N + 0.\Sigma.$$

The derivative of the principal normal vector is

$$N' = \|\omega\|_L \left(\frac{\omega}{\|\omega\|_L} \times_L N \right) = \left(\Sigma \times_L N \right) + 0.N + 0.\Sigma.$$

The derivative of the outer product of $\left(\Sigma \times_L N \right)$ is

$$\begin{aligned}
\left(\Sigma \times_L N \right)' &= \Sigma' \times_L N + \Sigma \times_L N' \\
&= \theta' \left(-\langle \Sigma, N \rangle_L N + \langle N, N \rangle_L \Sigma \right) - \|\omega\|_L \left(-\langle \Sigma, \Sigma \rangle_L N + \langle N, \Sigma \rangle_L \Sigma \right) \\
&= -\theta'.\Sigma + \|\omega\|_L \langle \Sigma, \Sigma \rangle_L N \\
&= 0.\left(\Sigma \times_L N \right) + \varepsilon_1 \|\omega\|_L N - \theta'.\Sigma.
\end{aligned}$$

where $\varepsilon_1 = \pm 1$. Consequently, the following derivative matrix can be written as

$$\begin{bmatrix} \left(\Sigma \times_L N \right)' \\ N' \\ \Sigma' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1 \|\omega\|_L & -\theta' \\ \|\omega\|_L & 0 & 0 \\ \theta' & 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma \times_L N \\ N \\ \Sigma \end{bmatrix},$$

where the first coefficient $\|\omega\|_L = \sqrt{k_1^2 + k_2^2}$ is greater than zero. The second coefficient is computed as $\theta' = \dfrac{k_1' k_2 - k_2' k_1}{k_1^2 + k_2^2}, k_2 \neq 0, |k_1| \neq |k_2|$. Therefore the vectors $\{ \left(\Sigma \times_L N \right), N, \Sigma \}$ describe a rotation motion with the rotation vector $\omega_1 = \varepsilon_1 \{ \theta' N + \|\omega\|_L \Sigma \} = \varepsilon_1 \{ \theta' N + \omega \} or \pm \left(\theta' N + \omega \right)$. Additionally, the momentum rotation vector of the N-Bishop Darboux vector is

$$N' = \omega_1 \times_L N$$

$$\left(\Sigma \times_L N \right)' = \omega_1 \times_L \left(\Sigma \times_L N \right)$$

$$\Sigma' = \omega_1 \times_L \Sigma.$$

Corollary 3.1 The rotational movement of the N-Bishop Darboux axis can be divided into two movements. The rotation vector ω_1 is the sum of rotation vectors of rotational motions. Similarly, the N-Bishop Darboux axis is consecutively rotated to obtain several N-Bishop Darboux vectors,

$$\omega_0 = \omega, \omega_1, \ldots$$

Conclusion

In this work, we studied the N-Bishop frame in Minkowski 3-space for the spacelike curves with spacelike binormal, which were first discovered in 2017 by Keskin et. al. Then, the N-Bishop Darboux axis was defined for the first time. Some features of N-Bishop Darboux rotation axis were investigated. We believe that this study will contribute to the investigations of many researchers, particularly those who work on Minkowski space.

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