ALTERNATING DIRECTION IMPLICIT (ADI) METHOD FOR NUMERICAL SOLUTIONS OF TWO-DIMENSIONAL BURGERS EQUATIONS

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In this study, the system of two-dimensional Burgers equations is numerically solved by using Alternating Direction Implicit (ADI) method. Two model problems are studied to demonstrate the efficiency and accuracy of the ADI method. Numerical results obtained by present method are compared with the exact solutions and numerical solutions given by other researchers. It is displayed that the method is unconditionally stable by using the von-Neumann (Fourier) stability analysis method. It is shown that all results are in good agreement with the results given by existing numerical methods in the literature.

Key Words: *Alternating Direction Implicit (ADI) method, Two-dimensional Burgers equation, von-Neumann stability analysis.*

1. Introduction

Consider the system of two-dimensional Burgers' equations:

$$u_t + uu_x + vu_y = \frac{1}{\text{Re}}\left(u_{xx} + u_{yy}\right) \tag{1}$$

$$v_t + uv_x + vv_y = \frac{1}{\text{Re}}\left(v_{xx} + v_{yy}\right) \tag{2}$$

with initial conditions:

$$u(x,y,0) = f(x,y),\ v(x,y,0) = g(x,y),\ (x,y) \in D, \tag{3}$$

and boundary conditions:

$$u(x,y,t) = \varphi(x,y,t),\ v(x,y,t) = \psi(x,y,t),\ x,y \in \partial D,\ t > 0, \tag{4}$$

where; $D = \{(x,y) : a \le x \le b,\ a \le y \le b\}$ and ∂D is its boundary, f, g, φ and ψ are known functions, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined and Re is the Reynolds number.

Burgers equation appears as a mathematical model of the many physical events, such as gas dynamic, turbulence and shock wave theory [22], [23], [24]. Burgers model of turbulence is a very important fluid dynamic model and the study of this model and the theory of shock waves has been considered by many authors both for conceptual understanding of a class of physical flows and for testing various numerical methods [21]. In 1983, Flecther [1] has obtained analytical solution of the system eqs. (1)-(2) by using the Hopf-Cole transformation. Numerical solutions of the system of equations have been studied by many researchers. Jain and Holla [2] have proposed two algorithms based on cubic spline function technique for obtaining numerical solutions of one and two dimensional

Burgers equations. Fletcher [3] studied numerical solutions of one and two dimensional Burgers equations by finite element and finite difference methods and compared the results obtained by the two methods. Wubs and Goede [4] used an explicit and implicit method. Goyon [5] used variety multilevel approach to solve the equation. Bahadır [6] has obtained the numerical solutions of the system of equations by using the fully implicit finite difference method. El-Sayed and Kaya [7] have obtained the numerical solutions of the system of two dimensional Burgers equations by using the decomposition method. Abdou and Soliman [8] used the variational iteration method to solve the two dimensional Burgers equation. Mittal and Jiwari [9] have obtained the numerical solutions of the system for rather large Reynolds numbers (Re=1200) by using the differential quadrature method. Liu andWeiping [10] used the lattice Boltzmann method to obtain the numerical solutions of the system of two-dimensional equations. Zhao et al. [11] applied the Hopf-Cole transformation to the system of two-dimensional Burgers equations and then solved the resulting two-dimensional heat equation by local discontinuous Galerkin (LDG) finite element method. Srivastava et al. [12] used the Crank-Nicolsan finite difference method to solve the system of equations. Wani and Thakar [13] obtained the numerical solutions of the system by using a linear fnite difference method that approximate to nonlinear terms by using central difference scheme and second order terms by using Crank-Nicolsan scheme. Srivastava et al. [14, 15] obtained the numerical solutions of equations by using implicit exponential finite difference and implicit logarithmic finite difference methods. Shukla et al. [16] used the modified cubic B-spline differential quadrature method to obtain numerical solutions of equation and compared the obtained results with other studies in the literature. Aksan [20] used the cubic B-spline finite element method to solve the Burgers equation.

In this study, Alternating Direction Implicit (ADI) method is used to solve the system of eqs. (1)-(2). In order to show accuracy of the present method three model problems are investigated. Obtained results are compared with exact solutions and numerical solutions that were obtained in other studies in the literature. The results obtained by present method show that the results are acceptable and consistent with other studies in the literature.

2. Alternating Direction Implicit (ADI) Method

In general, ADI method approximate the solution of an initial-boundary value problem with a series of simpler problems [17]. The method is described in a comprehensive manner in Duffy [17], Chung [18] and Davis [19]' s books. In ADI method, problem is solved at two time legs: at the first leg approximation is implicit in *x*-direction and explicit in *y*-direction while at the second leg approximation is explicit in *x*-direction and implicit in *y*-direction [17]. The new approach moves from the time level n to $n+\frac{1}{2}$ and then to time level $n+1$ [17].

We indicate the discrete approximation of $u(x,y,t)$ and $v(x,y,t)$ at the grid point (ih_x, jh_y, nk) by $u_{i,j}^{n}$ and $v_{i,j}^{n}$, respectively ($i=0,1,2,\ldots,N_x;\ j=0,1,2,\ldots,N_y;\ n=0,1,2,\ldots$) where h_x and h_y the grid sizes in *x*-direction and *y*-direction, respectively and k represents the increment in time.

An Alternating Direction Implicit approximation to the eq. (1) is given by:

$$\frac{u_{i,j}^{n+\frac{1}{2}}-u_{i,j}^{n}}{k/2}+u_{i,j}^{n}\left(\frac{u_{i+1,j}^{n+\frac{1}{2}}-u_{i-1,j}^{n+\frac{1}{2}}}{2h_x}\right)+v_{i,j}^{n}\left(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n}}{2h_y}\right)=\frac{1}{\mathrm{Re}}\left(\frac{u_{i+1,j}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i-1,j}^{n+\frac{1}{2}}}{h_x^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{h_y^2}\right) \quad (5)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n+\frac{1}{2}}}{k/2}+u_{i,j}^{n+\frac{1}{2}}\left(\frac{u_{i+1,j}^{n+\frac{1}{2}}-u_{i-1,j}^{n+\frac{1}{2}}}{2h_x}\right)+v_{i,j}^{n+\frac{1}{2}}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2h_y}\right)=\frac{1}{\mathrm{Re}}\left(\frac{u_{i+1,j}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i-1,j}^{n+\frac{1}{2}}}{h_x^2}+\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h_y^2}\right). \quad (6)$$

For simplicity, we let $h_x = h_y = h$ hence Alternating Direction Implicit approximation to the eq. (1) is in the following form:

$$\left(-r_1\left(u_{i,j}^n\right)-r_2\right)u_{i-1,j}^{n+\frac{1}{2}}+\left(1+2r_2\right)u_{i,j}^{n+\frac{1}{2}}+\left(r_1\left(u_{i,j}^n\right)-r_2\right)u_{i+1,j}^{n+\frac{1}{2}}=\left(r_1\left(v_{i,j}^n\right)+r_2\right)u_{i,j-1}^n+\left(1-2r_2\right)u_{i,j}^n+\left(-r_1\left(v_{i,j}^n\right)+r_2\right)u_{i,j+1}^n \quad (7)$$

$$\left(-r_1\left(v_{i,j}^{n+\frac{1}{2}}\right)-r_2\right)u_{i,j-1}^{n+1}+\left(1+2r_2\right)u_{i,j}^{n+1}+\left(r_1\left(v_{i,j}^{n+\frac{1}{2}}\right)-r_2\right)u_{i,j+1}^{n+1}=\left(r_1\left(u_{i,j}^{n+\frac{1}{2}}\right)+r_2\right)u_{i-1,j}^{n+\frac{1}{2}}+\left(1-2r_2\right)u_{i,j}^{n+\frac{1}{2}}+\left(-r_1\left(u_{i,j}^{n+\frac{1}{2}}\right)+r_2\right)u_{i+1,j}^{n+\frac{1}{2}} \quad (8)$$

where $r_1 = \dfrac{k}{4h}$ and $r_2 = \dfrac{k}{2\,\mathrm{Re}\,h^2}$.

Similarly, an Alternating Direction Implicit approximation to the eq. (2) is given by:

$$\frac{v_{i,j}^{n+\frac{1}{2}}-v_{i,j}^n}{k/2}+u_{i,j}^n\left(\frac{v_{i+1,j}^{n+\frac{1}{2}}-v_{i-1,j}^{n+\frac{1}{2}}}{2h_x}\right)+v_{i,j}^n\left(\frac{v_{i,j+1}^n-v_{i,j-1}^n}{2h_y}\right)=\frac{1}{\mathrm{Re}}\left(\frac{v_{i+1,j}^{n+\frac{1}{2}}-2v_{i,j}^{n+\frac{1}{2}}+v_{i-1,j}^{n+\frac{1}{2}}}{h_x^2}+\frac{v_{i,j+1}^n-2v_{i,j}^n+v_{i,j-1}^n}{h_y^2}\right) \quad (9)$$

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n+\frac{1}{2}}}{k/2}+u_{i,j}^{n+\frac{1}{2}}\left(\frac{v_{i+1,j}^{n+\frac{1}{2}}-v_{i-1,j}^{n+\frac{1}{2}}}{2h_x}\right)+v_{i,j}^{n+\frac{1}{2}}\left(\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2h_y}\right)=\frac{1}{\mathrm{Re}}\left(\frac{v_{i+1,j}^{n+\frac{1}{2}}-2v_{i,j}^{n+\frac{1}{2}}+v_{i-1,j}^{n+\frac{1}{2}}}{h_x^2}+\frac{v_{i,j+1}^{n+1}-2v_{i,j}^{n+1}+v_{i,j-1}^{n+1}}{h_y^2}\right) \quad (10)$$

For simplicity, we let $h_x = h_y = h$ hence Alternating Direction Implicit approximation to the eq. (2) is in the following form:

$$\left(-r_1\left(u_{i,j}^n\right)-r_2\right)v_{i-1,j}^{n+\frac{1}{2}}+\left(1+2r_2\right)v_{i,j}^{n+\frac{1}{2}}+\left(r_1\left(u_{i,j}^n\right)-r_2\right)v_{i+1,j}^{n+\frac{1}{2}}=\left(r_1\left(v_{i,j}^n\right)+r_2\right)v_{i,j-1}^n+\left(1-2r_2\right)v_{i,j}^n+\left(-r_1\left(v_{i,j}^n\right)+r_2\right)v_{i,j+1}^n \quad (11)$$

$$\left(-r_1\left(v_{i,j}^{n+\frac{1}{2}}\right)-r_2\right)v_{i,j-1}^{n+1}+\left(1+2r_2\right)v_{i,j}^{n+1}+\left(r_1\left(v_{i,j}^{n+\frac{1}{2}}\right)-r_2\right)v_{i,j+1}^{n+1}=\left(r_1\left(u_{i,j}^{n+\frac{1}{2}}\right)+r_2\right)v_{i-1,j}^{n+\frac{1}{2}}+\left(1-2r_2\right)v_{i,j}^{n+\frac{1}{2}}+\left(-r_1\left(u_{i,j}^{n+\frac{1}{2}}\right)+r_2\right)v_{i+1,j}^{n+\frac{1}{2}} \quad (12)$$

where $r_1 = \dfrac{k}{4h}$ and $r_2 = \dfrac{k}{2\,\mathrm{Re}\,h^2}$.

3. Model Problems and Numerical Results

In order to illustrate the accuracy and efficiency of the present method two model problems are investigated.

Problem 1. We consider the system of two-dimensional Burgers eqs. (1)-(2) with exact solutions:

$$\begin{aligned} u(x,y,t) &= \frac{3}{4}-\frac{1}{4\left(1+\exp\left(\left(-4x+4y-t\right)\mathrm{Re}/32\right)\right)}, \\ v(x,y,t) &= \frac{3}{4}+\frac{1}{4\left(1+\exp\left(\left(-4x+4y-t\right)\mathrm{Re}/32\right)\right)}, \end{aligned} \quad (13)$$

that can be generated by using the Hopf-Cole transformation[1].

The initial and boundary conditions are taken from the exact solutions and the computational domain is $D=\{(x,y):0\le x\le 1,\ 0\le y\le 1\}$. In all numerical computations, we used the mesh width $h_x = h_y = h$ =0.05 and $\mathrm{Re}=100$.

Tab. 1 and Tab. 2 compare the numerical solutions obtained by present method with exact solutions and numerical solutions of existing methods [6, 12, 14, 15, 16] for $k=0.0001$ at $t=0.5$. Tab. 3 and Tab. 4 show the comparison of numerical solutions obtained by present method with exact solutions and numerical solutions obtained by existing methods [6, 12, 14, 15, 16] for $k=0.0001$ at $t=2$. It can be seen from the tables that the numerical solutions obtained by present method are better than the numerical solutions given by Bahadır [6] and also the results are compatible with the other studies [12, 14, 15, 16] in the literature.

Table 1. Comparison of numerical solutions and exact solutions of $u(x,y,t)$ for $\mathrm{Re}=100$, $k=0.0001$, $h=0.05$ and $t=0.5$.

(x,y)	ADI	Exact	Bahadır [6]	I-LFDM [15]	Expo-FDM[14]	CNS [12]	MCB-DQM[16]
(0.1,0.1)	0.54299	0.54332	0.54235	0.54300	0.54300	0.54300	0.54412
(0.5,0.1)	0.50034	0.50035	0.49964	0.50034	0.50034	0.50034	0.50037
(0.9,0.1)	0.50000	0.50000	0.49931	0.50000	0.50000	0.50000	0.50000
(0.3,0.3)	0.54268	0.54332	0.54207	0.54269	0.54270	0.54269	0.54388
(0.7,0.3)	0.50032	0.50035	0.49961	0.50032	0.50032	0.50032	0.50037
(0.1,0.5)	0.74215	0.74221	0.74130	0.74215	0.74215	0.74215	0.74196
(0.5,0.5)	0.54249	0.54332	0.54222	0.54251	0.54252	0.54250	0.54347
(0.9,0.5)	0.50030	0.50035	0.49997	0.50030	0.50030	0.50030	0.50035
(0.3,0.7)	0.74211	0.74221	0.74146	0.74211	0.74212	0.74212	0.74211
(0.7,0.7)	0.54245	0.54332	0.54243	0.54246	0.54247	0.54246	0.54327
(0.1,0.9)	0.74994	0.74995	0.74913	0.74994	0.74994	0.74995	0.74994
(0.5,0.9)	0.74210	0.74221	0.74201	0.74210	0.74210	0.74213	0.74219
(0.9,0.9)	0.54227	0.54332	0.54232	0.54228	0.54229	0.54640	0.54333

Table 2. Comparison of numerical solutions and exact solutions of $v(x,y,t)$ for $\mathrm{Re}=100$, $k=0.0001$, $h=0.05$ and $t=0.5$.

(x,y)	ADI	Exact	Bahadır [6]	I-LFDM [15]	Expo-FDM[14]	CNS [12]	MCB-DQM[16]
(0.1,0.1)	0.95701	0.95668	0.95577	0.95700	0.95700	0.95700	0.95589
(0.5,0.1)	0.99966	0.99965	0.99827	0.99966	0.99966	0.99966	0.99963
(0.9,0.1)	1.00000	1.00000	0.99861	1.00000	1.00000	1.00000	1.00000
(0.3,0.3)	0.95732	0.95668	0.95596	0.95731	0.95731	0.95731	0.95612
(0.7,0.3)	0.99968	0.99965	0.99827	0.99968	0.99968	0.99968	0.99964
(0.1,0.5)	0.75785	0.75779	0.75699	0.75785	0.75785	0.75785	0.75804
(0.5,0.5)	0.95751	0.95668	0.95685	0.95749	0.95749	0.95750	0.95654
(0.9,0.5)	0.99970	0.99965	0.99903	0.99970	0.99970	0.99970	0.99965
(0.3,0.7)	0.75789	0.75779	0.75723	0.75789	0.75789	0.75789	0.75789
(0.7,0.7)	0.95755	0.95668	0.95746	0.95754	0.95754	0.95754	0.95673
(0.1,0.9)	0.75006	0.75005	0.74924	0.75006	0.75006	0.75006	0.75006
(0.5,0.9)	0.75790	0.75779	0.75781	0.75790	0.75790	0.75787	0.75781
(0.9,0.9)	0.95773	0.95668	0.95777	0.95772	0.95772	0.95360	0.95667

Table 3. Comparison of numerical solutions and exact solutions of $u(x,y,t)$ for $\text{Re}=100$, $k=0.0001$, $h=0.05$ and $t=2$.

(x,y)	ADI	Exact	Bahadır [6]	I-LFDM [15]	Expo-FDM[14]	CNS [12]	MCB-DQM[16]
(0.1,0.1)	0.50047	0.50048	0.49983	0.50047	0.50047	0.50047	0.50050
(0.5,0.1)	0.50000	0.50000	0.49930	0.50000	0.50000	0.50000	0.50000
(0.9,0.1)	0.50000	0.50000	0.49930	0.50000	0.50000	0.50000	0.50000
(0.3,0.3)	0.50044	0.50048	0.49977	0.50044	0.50044	0.50044	0.50050
(0.7,0.3)	0.50000	0.50000	0.49930	0.50000	0.50000	0.50000	0.50000
(0.1,0.5)	0.55514	0.55568	0.55461	0.55515	0.55516	0.55515	0.55632
(0.5,0.5)	0.50041	0.50048	0.49973	0.50041	0.50041	0.50042	0.50050
(0.9,0.5)	0.50000	0.50000	0.49931	0.50000	0.50000	0.50001	0.50001
(0.3,0.7)	0.55480	0.55568	0.55429	0.55482	0.55482	0.55481	0.55597
(0.7,0.7)	0.50038	0.50048	0.49970	0.50038	0.50038	0.50068	0.50054
(0.1,0.9)	0.74419	0.74426	0.74340	0.74420	0.74420	0.74422	0.74406
(0.5,0.9)	0.55448	0.55568	0.55413	0.55450	0.55451	0.55980	0.55575
(0.9,0.9)	0.50052	0.50048	0.50001	0.50053	0.50053	0.51341	0.50052

Table 4. Comparison of numerical solutions and exact solutions of $v(x,y,t)$ for $\text{Re}=100$, $k=0.0001$, $h=0.05$ and $t=2$.

(x,y)	ADI	Exact	Bahadır [6]	I-LFDM [15]	Expo-FDM[14]	CNS [12]	MCB-DQM[16]
(0.1,0.1)	0.99953	0.99952	0.99826	0.99953	0.99953	0.99953	0.99950
(0.5,0.1)	1.00000	1.00000	0.99860	1.00000	1.00000	1.00000	1.00000
(0.9,0.1)	1.00000	1.00000	0.99861	1.00000	1.00000	1.00000	1.00000
(0.3,0.3)	0.99956	0.99952	0.99820	0.99956	0.99956	0.99956	0.99950
(0.7,0.3)	1.00000	1.00000	0.99860	1.00000	1.00000	1.00000	1.00000
(0.1,0.5)	0.94486	0.94432	0.94393	0.94485	0.94485	0.94485	0.94368
(0.5,0.5)	0.99959	0.99952	0.99821	0.99959	0.99959	0.99959	0.99950
(0.9,0.5)	1.00000	1.00000	0.99862	1.00000	1.00000	0.99999	0.99999
(0.3,0.7)	0.94520	0.94432	0.94409	0.94518	0.94518	0.94519	0.94403
(0.7,0.7)	0.99962	0.99952	0.99823	0.99962	0.99962	0.99932	0.99946
(0.1,0.9)	0.75581	0.75574	0.75500	0.75580	0.75580	0.75579	0.75595
(0.5,0.9)	0.94552	0.94432	0.94441	0.94550	0.94550	0.94020	0.94425
(0.9,0.9)	0.99948	0.99952	0.99846	0.99948	0.99948	0.98659	0.99948

Problem 2. We consider the system of eqs. (1) and (2) over the computational domain $D=\{(x,y): 0\le x\le 0.5,\ 0\le y\le 0.5\}$ with initial conditions:

$$u(x,y,0)=\sin(\pi x)+\cos(\pi y),\quad v(x,y,0)=x+y,\qquad (x,y)\in D, \tag{14}$$

and boundary conditions:

$$\left.\begin{aligned} &u(0,y,t)=\cos(\pi y), && v(0,y,t)=y,\\ &u(0.5,y,t)=1+\cos(\pi y), && v(0.5,y,t)=0.5+y,\\ &u(x,0,t)=1+\sin(\pi x), && v(x,0,t)=x,\\ &u(x,0.5,t)=\sin(\pi x), && v(x,0.5,t)=0.5+x, \end{aligned}\right\},\qquad t>0. \tag{15}$$

Tab. 5 and Tab. 6 present the comparison of computed values of $u(x,y,t)$ and $v(x,y,t)$ with those given by [2, 6, 12, 13, 14, 15, 16] for $\mathrm{Re}=50$, $h_x = h_y = h$ =0.025 and $k=0.0001$. As seen from the Tab. 5 and Tab. 6, the obtained results by using ADI method are in good agreement with the existing methods [2, 6, 12, 13, 14, 15, 16] in the literature. The numerical solutions obtained by present method are compared with the other numerical solutions obtained by existing methods [2, 6, 12, 13] for $\mathrm{Re}=500$, $h_x = h_y = h$ =0.025 and $k=0.0001$ in Tab. 7 and Tab. 8. It can be seen from the tables that computed values obtained by ADI method are in good agreement with those available in the literature. Also, we could get solutions for $k=0.001$, Reynolds numbers $\mathrm{Re}=1000$ and $\mathrm{Re}=1200$ (Tab. 9, Tab. 10). Fig. 1 and Fig. 2 show $u(x,y,t)$ and $v(x,y,t)$ numerical solutions for $\mathrm{Re}=50,500,$ $h_x = h_y = h$ =0.025 and $k=0.0001$ at $t=0.625$.

Table 5. Comparison of numerical solutions of $u(x,y,t)$ for $\mathrm{Re}=50$, $k=0.0001$, $h=0.025$ at $t=0.625$.

(x,y)	ADI	Bahadır [6]	Jain and Holla [2]	I-LFDM [15]	Expo-FDM [14]	CNS [12]	MCB-DQM [16]	LINEAR [13]
(0.1,0.1)	0.97146	0.96688	0.97258	0.97146	0.97146	0.97146	0.97056	0.971461
(0.3,0.1)	1.15282	1.14827	1.16214	1.15280	1.15280	1.15280	1.15152	1.152820
(0.2,0.2)	0.87310	0.85911	0.86281	0.86308	0.86308	0.86307	0.86244	0.863072
(0.4,0.2)	0.97980	0.97637	0.96483	0.97985	0.97985	0.97981	0.98078	0.979813
(0.1,0.3)	0.66316	0.66019	0.66318	0.66316	0.66316	0.66316	0.66336	0.663157
(0.3,0.3)	0.77229	0.76932	0.77030	0.77233	0.77233	0.77230	0.77226	0.772297
(0.2,0.4)	0.60488	0.57966	0.58070	0.58181	0.58181	0.58180	0.58273	0.581799
(0.4,0.4)	0.75853	0.75678	0.74435	0.75862	0.75862	0.75856	0.76179	0.758558

Table 6. Comparison of numerical solutions of $v(x,y,t)$ for $\mathrm{Re}=50$, $k=0.0001$, $h=0.025$ at $t=0.625$.

(x,y)	ADI	Bahadır [6]	Jain and Holla [2]	I-LFDM [15]	Expo-FDM [14]	CNS [12]	MCB-DQM [16]	LINEAR [13]
(0.1,0.1)	0.09869	0.09824	0.09773	0.09869	0.09869	0.09869	0.09842	0.098688
(0.3,0.1)	0.14158	0.14112	0.14039	0.14158	0.14158	0.14158	0.14107	0.141582
(0.2,0.2)	0.16754	0.16681	0.16660	0.16754	0.16754	0.16754	0.16732	0.167542
(0.4,0.2)	0.17109	0.17065	0.17397	0.17111	0.17111	0.17110	0.17223	0.171095
(0.1,0.3)	0.26378	0.26261	0.26294	0.26378	0.26378	0.26378	0.26380	0.263781
(0.3,0.3)	0.22654	0.22576	0.22463	0.22655	0.22655	0.22654	0.22653	0.226539
(0.2,0.4)	0.32851	0.32745	0.32402	0.32851	0.32851	0.32851	0.32935	0.328508
(0.4,0.4)	0.32499	0.32441	0.31822	0.32502	0.32502	0.32500	0.32884	0.324997

Table 7. Comparison of numerical solutions of $u(x,y,t)$ for $\mathrm{Re}=500$, $k=0.0001$, $h=0.025$ at $t=0.625$.

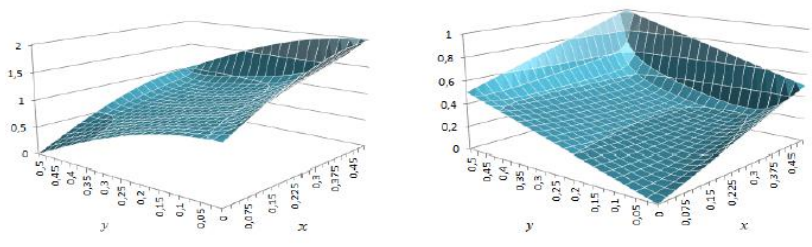
(x,y)	ADI	Bahadır [6]	Jain and Holla [2]	CNS [12]	LINEAR [13]
(0.15,0.1)	0.96870	0.96650	0.95691	0.96870	0.968969
(0.3,0.1)	1.03202	1.02970	0.95616	1.03200	1.032020
(0.1,0.2)	0.84619	0.84449	0.84257	0.86178	0.846187
(0.2,0.2)	0.87814	0.87631	0.86399	0.87814	0.878141
(0.1,0.3)	0.67920	0.67809	0.67667	0.67920	0.679202
(0.3,0.3)	0.79947	0.79792	0.76876	0.79947	0.799471
(0.15,0.4)	0.54674	0.54601	0.54408	0.66036	0.546743
(0.2,0.4)	0.58959	0.58874	0.58778	0.58959	0.589589

Table 8. Comparison of numerical solutions of $v(x,y,t)$ for $\mathrm{Re}=500$, $k=0.0001$, $h=0.025$ at $t=0.625$.

(x,y)	ADI	Bahadır [6]	Jain and Holla [2]	CNS [12]	LINEAR [13]
(0.15,0.1)	0.09043	0.09020	0.10177	0.09043	0.092303
(0.3,0.1)	0.10728	0.10690	0.13287	0.10728	0.107275
(0.1,0.2)	0.18010	0.17972	0.18503	0.17295	0.180103
(0.2,0.2)	0.16816	0.16777	0.18169	0.16816	0.168157
(0.1,0.3)	0.26268	0.26222	0.26560	0.26268	0.262677
(0.3,0.3)	0.23550	0.23497	0.25142	0.23550	0.235501
(0.15,0.4)	0.31799	0.31753	0.32084	0.29019	0.317991
(0.2,0.4)	0.30419	0.30371	0.30927	0.30419	0.304187

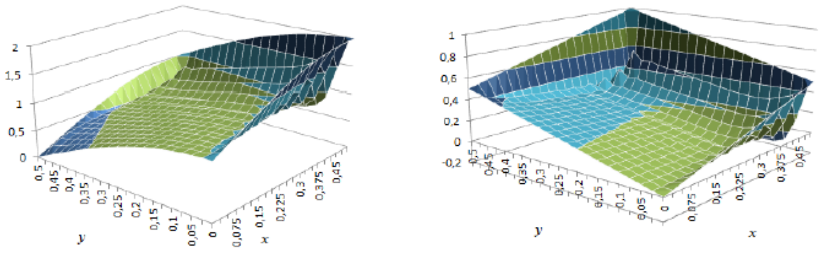
Table 10. Comparison of numerical solutions of $u(x,y,t)$ and $v(x,y,t)$ for $\mathrm{Re}=1200$, $k=0.001$ and different values of h at $t=0.625$.

(x,y)	$u(x,y,t)$			$v(x,y,t)$		
	$h=0.025$	$h=0.0125$	$h=0.00625$	$h=0.025$	$h=0.0125$	$h=0.00625$
(0.15,0.1)	1.05327	0.96257	0.96222	0.13234	0.08671	0.08652
(0.3,0.1)	1.24775	0.97834	0.96952	0.21535	0.08077	0.07637
(0.3,0.3)	0.93716	0.77936	0.77605	0.34114	0.21894	0.21637
(0.15,0.4)	0.57450	0.54886	0.54919	0.36607	0.31484	0.31455
(0.2,0.4)	0.64228	0.58927	0.58939	0.37999	0.29928	0.29890



(A) $u(x,y,t)$ (B) $v(x,y,t)$

Figure 1. Numerical solutions for $\mathrm{Re}=50$, $h_x=h_y=h=0.025$ and $k=0.0001$ at $t=0.625$.



(A) $u(x,y,t)$ (B) $v(x,y,t)$

Figure 2. Numerical solutions for $\mathrm{Re}=500$, $h_x = h_y = h = 0.025$ and $k = 0.0001$ at $t = 0.625$.

4. Stability Analysis

We use the von-Neumann stability analysis to prove unconditional stability of schemes (7)- (8) and (11)-(12). Let

$$u^n_{i,j} = u^n_{p,q} = \gamma^n e^{i\alpha ph} e^{i\beta qh} \tag{16}$$

where $i=\sqrt{-1}$.

In order to analyze the stability of the numerical schemes (7)-(8), the nonlinear terms uu_x and vu_y in the two-dimensional Burgers equation (1) have been linearized by replacing the quantities u and v by local contents. Thus the nonlinear terms uu_x and vu_y convert into $\hat{U}u_x$ and $\hat{V}u_y$ in equation (1) and in that case eq. (1) turns into

$$u_t + \hat{U}u_x + \hat{V}u_y = \frac{1}{\mathrm{Re}}\left(u_{xx} + u_{yy}\right). \tag{17}$$

If we write the ADI scheme for this linearized equation, newly approximations are taken as:

$$\left(-r_1\hat{U} - r_2\right)u^{n+\frac{1}{2}}_{i-1,j} + \left(1+2r_2\right)u^{n+\frac{1}{2}}_{i,j} + \left(r_1\hat{U} - r_2\right)u^{n+\frac{1}{2}}_{i+1,j} = \left(r_1\hat{V} + r_2\right)u^{n}_{i,j-1} + \left(1-2r_2\right)u^{n}_{i,j} + \left(-r_1\hat{V} + r_2\right)u^{n}_{i,j+1} \tag{18}$$

$$\left(-r_1\hat{V} - r_2\right)u^{n+1}_{i,j-1} + \left(1+2r_2\right)u^{n+1}_{i,j} + \left(r_1\hat{V} - r_2\right)u^{n+1}_{i,j+1} = \left(r_1\hat{U} + r_2\right)u^{n+\frac{1}{2}}_{i-1,j} + \left(1-2r_2\right)u^{n+\frac{1}{2}}_{i,j} + \left(-r_1\hat{U} + r_2\right)u^{n+\frac{1}{2}}_{i+1,j} \tag{19}$$

then substituting the (16) equality into the newly numerical schemes (18) and (19) we get the following growth factors:

$$\frac{\gamma^{n+\frac{1}{2}}}{\gamma^n} = \frac{1-\alpha_1}{1+\alpha_2}, \quad \frac{\gamma^{n+1}}{\gamma^{n+\frac{1}{2}}} = \frac{1-\alpha_2}{1+\alpha_1} \tag{20}$$

where

$$\alpha_1 = 4r_2\sin^2\frac{\alpha h}{2} + i\left(2r_1\hat{U}\sin\alpha h\right),\quad \alpha_2 = 4r_2\sin^2\frac{\beta h}{2} + i\left(2r_1\hat{V}\sin\beta h\right). \quad (21)$$

Hence

$$\frac{\gamma^{n+1}}{\gamma^n} = \frac{1-\alpha_2}{1+\alpha_1}\frac{1-\alpha_1}{1+\alpha_2}. \quad (22)$$

The growth factor from n to $n+1$ must be less than 1 in absolute value for unconditionally stability.

$$\left|\frac{\gamma^{n+1}}{\gamma^n}\right| = \sqrt{\frac{1+a_1^2-2a_1+b_1^2}{1+a_1^2+2a_1+b_1^2}}\sqrt{\frac{1+a_2^2-2a_2+b_2^2}{1+a_2^2-2a_2+b_2^2}} \quad (23)$$

where $a_1 = 4r_2\sin^2\frac{\alpha h}{2}$, $b_1 = 2r_1\hat{U}\sin\alpha h$, $a_2 = 4r_2\sin^2\frac{\beta h}{2}$ and $b_2 = 2r_1\hat{V}\sin\beta h$. Since $a_1, a_2 \geq 0$ it can be seen that $\left|\frac{\gamma^{n+1}}{\gamma^n}\right| \leq 1$. Hence schemes (7) and (8) are unconditionally stable.

Similarly we can prove the unconditional stability of the numerical schemes (11) and (12).

5. Conclusion

Alternating Direction Implicit (ADI) method is presented for numerical solutions of the system of two-dimensional Burgers equations. Two test problems are used to illustrate the accuracy of the present method. Comparisons are made with the existing methods in the literature. The method is analyzed by von-Neumann stability analysis method and it is displayed that the method is unconditionally stable. The obtained results show that the ADI method is a successful method to solve the system of two-dimensional Burgers equations.

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