AN EXPLICIT CHARACTERIZATION OF SPHERICAL CURVES ACCORDING TO BISHOP FRAME AND AN APPROXIMATELY SOLUTION

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In this paper, spherical curves are studied by using Bishop Frame. First, the differential equation characterizing the spherical curves is given. Then, we exhibit that the position vector of a curve which is lying on a sphere satisfies a third-order linear differential equation. Then we solve this linear differential equation by using Bernstein Series Solution Method.

Key Words: *Spherical Curve, Bishop Frame, Bernstein Series Solution Method.*

1. Introduction

As is well known fundamental structure of differential geometry is the curves. With in the process, the best part of classical differential geometry topics have been expanded to space curves. There are many studies which implies different characterizations of these curves. And also spherical curves are the special space curves which lies on the sphere. There are also many studies on spherical curves. Firstly, Wong [1] give a universal formulation of the condition for a curve to be on sphere in 1963. Then, Breuer and Gottlieb [2] shown that the differential equation characterizing a spherical curve can be solved explicitly to express the radius of curvature of the curve in terms of its torsion in 1971. Wong [3] proved that the "explicit characterization" of spherical curves recently obtained by Breuer and Gottlieb without any precondition on the curvature and torsion, a necessary and sufficient condition for a curve to be a spherical curve. The proof is based on an earlier result of the present author on spherical curves. Mehlum and Wimp [4] proof that the position vector of any 3-space curve lying on a sphere provides a third order linear differential equation in 1985. In 1988, Köse [5] give an explicit characterization of the dual spherical curve. Abdel Bakey [6] studied with dual spherical curves and obtained a differential equation which is characterizing the dual spherical curves. Then he give the explicit solution of this differential equation without the precondition on the dual torsion. And a necessary and sufficient condition for a dual curve to be spherical curve is given in 2002. İlarslan [7] present the spherical characterization of non-null regular curves in three dimensional Lorentzian space is given. Furthermore, the differential equation which expresses the mentioned characterization is solved in 2003. Kocayiğit [8] shown that The differential equation characterizing a spherical curve in n-dimensional Euclidean space $n \geq 3$ can be solved explicitly to express nth curvature function of the curve in terms of its curvatures and its other curvature functions in 2003. Ayyıldız [9] present the differential equation that is characterizing the dual Lorentzian spherical curves and then give an explicit solution of this differential equation are given in 2007. Camcı [10] studied with regular curves in 3-dimensional

Sasakian space and give the spherical characterizations of them. Furthermore, the differential equation which expresses the aforesaid characterization is solved in 2007.

In this paper, we give the differential equation characterizing the spherical curves by using Bishop Frame. Then, we present that the position vector of a curve lying on a sphere satisfies a third-order linear differential equation and The Solution is given with Bernstein Series Method.

2. Preliminaries

In Euclidean 3-space scalar product is given by

$$\langle \, , \rangle = x_1^2 + x_2^2 + x_3^2$$

here (x_1, x_2, x_3) is an arbitrary vector in E^3 and the norm of the vector $\vec{x} \in E^3$ is $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$. Let $\vec{\alpha} : I \subset IR \to E^3$ be arbitrary curve in the Euclidean space E^3. if $\langle \vec{\alpha}', \vec{\alpha}' \rangle = 1$, then the curve $\vec{\alpha}$ is said to be of unit speed (parametrized by arc length parameter s). $\{\vec{T}, \vec{N}, \vec{B}\}$ is named as Frenet frame of α and κ , τ are the curvature and the torsion of the curve $\vec{\alpha}$, respectively [10]. Here, curvature functions are defined by $\kappa = \kappa(s) = \|\vec{T}'(s)\|$, $\tau(s) = -\langle \vec{N}, \vec{B}' \rangle$. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors in E^3. Cross product of $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is defined by

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2, u_1 v_3 - u_3 v_2, u_1 v_2 - u_2 v_1)$$

The Bishop frame in other words parallel transport frame is a different point of view for describing a moving frame which is well-define even when the curve has vanishing second derivative. The parallel transport of an orthonormal frame along a curve can be expressed simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. Consequently, the Bishop (frame) formulas is expressed as.

$$\left.\begin{aligned} \frac{d\vec{T}}{ds} &= k_1 \vec{N}_1 + k_2 \vec{N}_2 \\ \frac{d\vec{N}_1}{ds} &= -k_1 \vec{T} \\ \frac{d\vec{N}_2}{ds} &= -k_2 \vec{T} \end{aligned}\right\} \qquad (1)$$

Here, we name the set $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$ as Bishop frame and k_1 and k_2 are the first and second Bishop curvature, respectively, [11]. There is a relation between Frenet and Bishop frame elements and this relation is given as follows:

$$\begin{aligned} \vec{T} &= \vec{T} \\ \vec{N} &= \cos\theta \vec{N}_1 + \sin\theta \vec{N}_2 \\ \vec{B} &= -\sin\theta \vec{N}_1 + \cos\theta \vec{N}_2 \end{aligned}$$

where $\theta(s)=\arctan\left(\frac{k_2}{k_1}\right)$, $\tau(s)=\theta'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$.

2.1. Bernstein Series Solution Method: Bernstein polynomials of nth-degree are defined by

$$B_{k,n}(x)=\binom{n}{k}\frac{x^k(R-x)^{n-k}}{R^n}, k=0,1,...,n \tag{2}$$

R is the maximum range of the interval $[0,R]$ over which the polynomials are defined to form a complete basis [12,13]. Let us explain briefly the Bernstein series solution method for the solution of the linear differential equations with variable coefficient defined as follows.

$$\sum_{k=0}^{m}P_k(s)y^{(k)}(s)=g(s) \qquad 0\le s\le R. \tag{3}$$

Suppose that the Eq. (2) has a solution, f . We wish to approximate f by

$$p_n(s)=\sum_{k=0}^{n}a_kB_{k,n}(s) \qquad n\ge 1 \tag{4}$$

such that $p_n(s)$ satisfies Eq. (3) on the nodes $0<s_i<s_{i+1}<...<s_{i+d}<R$. Putting $p_n(s)$ into Eq. (3), we obtain the system of linear equations depending on $a_0,a_1,...a_n$. Here a_k, $(k=0,1,...,n)$ are the elements of unknowns matrix A. This system of equations involving Bernstein polynomials is solved with the help of basic matrix equations. Thus the elements of unknowns matrix A are uniquely determined. By substituting these values in Eq. (4), an approximate solution of Eq. (3) is obtained.

3. Characterization of Spherical Curves according to Bishop Frame

Let $\langle\vec{x},\vec{x}\rangle-r^2=0$ be a sphere and $\vec{x}(s)$ be a spherical curve in E^3. Following results can be given according to Bishop Frame.

Theorem 1. Let $\vec{x}(s):I\to E^3$ be a spherical curve in E^3 with arc lenght parameter s. Then the expression of position vector $\vec{x}(s)$ in terms of Bishop frame elements is

$$\vec{x}(s)=\frac{k_2'}{k_1'k_2-k_1k_2'}\vec{N}_1+\frac{k_1'}{k_1'k_2-k_1k_2'}\vec{N}_2 .$$

Proof. The position vector of spherical curve can be written by linear combination of the frame elments

$$\vec{x}(s)=c_1\vec{T}+c_2\vec{N}_1+c_3\vec{N}_2,\ c_1,c_2,c_3=\text{constant}. \tag{5}$$

$\vec{x}$ be a spherical curve with the arc lengt parameter s. Then $\langle\vec{x},\vec{x}\rangle=r^2$. If we take the derivative according to s, we get

$$\langle\vec{T},\vec{x}\rangle=0. \tag{6}$$

By multiplying both sides of Eq (5) respectively with $\vec{T}$, $\vec{N}_1$ and $\vec{N}_2$, we respectively get

$c_1 = 0,\ c_2 = \langle \vec{N}_1, \vec{x} \rangle,\ c_3 = \langle \vec{N}_2, \vec{x} \rangle.$ If we take the derivative of Eq. (6) according to s, we obtain

$$\langle k_1 \vec{N}_1 + k_2 \vec{N}_2, \vec{x} \rangle = -1. \tag{7}$$

We take second derivative of Eq. (6) according to s, we have

$$k_1^{'} \langle \vec{N}_1, \vec{x} \rangle + k_2^{'} \langle \vec{N}_2, \vec{x} \rangle = 0. \tag{8}$$

From the equations below, we get

$$\langle \vec{N}_1, \vec{x} \rangle = \frac{k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}},\ \langle \vec{N}_2, \vec{x} \rangle = \frac{k_1^{'}}{k_1^{'} k_2 - k_1 k_2^{'}}. \tag{9}$$

As a result, the expression of position vector $\vec{x}(s)$ in terms of Bishop frame elements is

$$\vec{x}(s) = \frac{k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \vec{N}_1 + \frac{k_1^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \vec{N}_2. \tag{10}$$

Theorem 2. Let $\vec{x}(s): I \to E^3$ is a curve in E^3 with arc lenght parameter s. If $\vec{x} = \vec{x}(s)$ is spherical, then there is a correlation between the bishop curvatures given by

$$\frac{k_1^{'} k_2 + k_1 k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} = -1$$

Proof. Let $\vec{x}$ be a spherical curve with the arc lengt parameter s. If we take the derivative of Eq. (10) according to s, we get

$$\vec{T} = \left(\frac{k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \right)^{'} \vec{N}_1 + \left(\frac{k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \right) \left(-k_1 \vec{T} \right) + \left(\frac{k_1^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \right)^{'} \vec{N}_2 + \left(\frac{k_1^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \right) \left(-k_2 \vec{T} \right). \tag{11}$$

From equation (11), we obtain

$$\left(\frac{k_1^{'} k_2 + k_1 k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \right) = -1. \tag{12}$$

Theorem 3. Position vector of a curve lying on a sphere satisfies a third-order linear differential equation.

Proof. Let $\langle \vec{x} - \vec{m}, \vec{x} - \vec{m} \rangle - r^2 = 0$ be a sphere and $\vec{x} = \vec{x}(s)$ be a curve in E^3. Sphere and curve have a contact of order 3 at point P if $g(s) = \langle \vec{x} - \vec{m}, \vec{x} - \vec{m} \rangle - r^2 = 0$ and

$$\frac{dg}{ds} = \frac{d^2 g}{ds^2} = \frac{d^3 g}{ds^3} = 0 \tag{13}$$

With the aid of Eq. (1), the Eq. (13), respectively, give us

$$g' = \langle \vec{T}, \vec{x} - \vec{m} \rangle = 0 \tag{14}$$

$$g'' = k_1 \langle \vec{N}_1, \vec{x} - \vec{m} \rangle + k_2 \langle \vec{N}_2, \vec{x} - \vec{m} \rangle = -1 \tag{15}$$

$$g''' = k_1^{'} \langle \vec{N}_1, \vec{x} - \vec{m} \rangle + k_2^{'} \langle \vec{N}_2, \vec{x} - \vec{m} \rangle = 0\,. \tag{16}$$

From Eq. (15) and Eq. (16) we get

$$\langle \vec{N}_1, \vec{x} - \vec{m} \rangle = \frac{k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}}\,, \quad \langle \vec{N}_2, \vec{x} - \vec{m} \rangle = -\frac{k_1^{'}}{k_1 k_2^{'} + k_2 k_1^{'}}\,. \tag{17}$$

The vector $\vec{x} - \vec{m}$ can be written as the linear combination of the bishop frame elements

$$\vec{x} - \vec{m} = \alpha \vec{T} + \beta \vec{N}_1 + \gamma \vec{N}_2\,. \tag{18}$$

If we multiply both sides of the Eq. (18) respectively by $\vec{T}, \vec{N}_1, \vec{N}_2$, we get

$$\alpha = 0, \quad \beta = \frac{k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}}\,, \quad \gamma = \frac{k_1^{'}}{k_1 k_2^{'} - k_2 k_1^{'}} \tag{19}$$

If the center of the sphere is orijin then the expression of the position vector of a curve lying on sphere in terms of Bishop frame elements is

$$\vec{x} = \frac{k_2^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \vec{N}_1 - \frac{k_1^{'}}{k_1^{'} k_2 - k_1 k_2^{'}} \vec{N}_2\,. \tag{20}$$

If ve take the derivative of position vector three times, we respectively get

$$\vec{x}' = \vec{T} \tag{21}$$

$$\vec{x}'' = \vec{T}' = k_1 \vec{N}_1 + k_2 \vec{N}_2 \tag{22}$$

$$\vec{x}''' = -\left(k_1^2 + k_2^2\right) \vec{T} + k_1^{'} \vec{N}_1 + k_2^{'} \vec{N}_2 \tag{23}$$

From Eq. (22) and Eq. (23), we get

$$\vec{N}_1 = \frac{k_2 \vec{x}''' - k_2^{'} \vec{x}'' + k_2 \left(k_1^2 + k_2^2\right) \vec{x}'}{k_1^{'} k_2 - k_1 k_2^{'}} \tag{24}$$

$$\vec{N}_2 = \frac{k_1 \vec{x}''' - k_1^{'} \vec{x}'' + k_1 \left(k_1^2 + k_2^2\right) \vec{x}'}{k_1 k_2^{'} - k_1^{'} k_2} \tag{25}$$

If we put Eq. (24) and Eq. (25) in Eq. (20), we get

$$\vec{x} = \frac{k_1 k_1^{'} + k_2 k_2^{'}}{\left(k_1^{'} k_2 - k_1 k_2^{'}\right)^2} \vec{x}''' - \frac{k_1^{'2} + k_2^{'2}}{\left(k_1^{'} k_2 - k_1 k_2^{'}\right)^2} \vec{x}'' + \frac{\left(k_1^2 + k_2^2\right)\left(k_1 k_1^{'} + k_2 k_2^{'}\right)}{\left(k_1^{'} k_2 - k_1 k_2^{'}\right)^2} \vec{x}'. \tag{26}$$

If the Eq. (26) rearranged, we get

$$\left(k_1k_1^{'}+k_2k_2^{'}\right)\vec{x}'''-\left(k_1^{'2}+k_2^{'2}\right)\vec{x}''+\left(k_1^2+k_2^2\right)\left(k_1k_1^{'}+k_2k_2^{'}\right)\vec{x}'-\left(k_1^{'}k_2-k_1k_2^{'}\right)^2\vec{x}=0. \tag{27}$$

Equation (27) shows that the position vector of a curve lying on a sphere satisfies a third-order linear differential equation.

Corollary 1. Let $\vec{x}(s):I\to E^3$ is a spherical curve in E^3 with arc lenght parameter s. The centers of spheres with 3-contact points with $\vec{x}(s)$ in the point $\vec{x}(s)$ lie on straight line.

$$m=\vec{x}-\left(\frac{k_2^{'}}{k_1^{'}k_2-k_1k_2^{'}}\right)\vec{N}_1+\left(\frac{k_1^{'}}{k_1k_2^{'}-k_2k_1^{'}}\right)\vec{N}_2$$

3. Let $\vec{x}(s):I\to E^3$ is a spherical curve in E^3 with arc lenght parameter s. The radius of spheres with 3-contact points with $\vec{x}(s)$ is

$$r=\sqrt{\left(\frac{k_2^{'}}{k_1^{'}k_2-k_1k_2^{'}}\right)^2+\left(\frac{k_1^{'}}{k_1k_2^{'}-k_2k_1^{'}}\right)^2}\ .$$

4. The Solution with Bernstein Series Method

$$P_0(s)=-\left(k_1^{'}(s)k_2(s)-k_1(s)k_2^{'}(s)\right),\ P_1(s)=\left(k_1^2(s)+k_2^2(s)\right)\left(k_1(s)k_1^{'}(s)+k_2(s)k_2^{'}(s)\right)$$
$$P_2(s)=-\left[\left(k_1^{'}(s)\right)^2+\left(k_2^{'}(s)\right)^2\right],\ P_3(s)=k_1(s)k_1^{'}(s)+k_2(s)k_2^{'}(s)$$
$$F(s)=0 \text{ and } x^{(k)}(s)=y^{(k)}(s),\ k=0,1,2,3$$

By using the above equations the differential equation (27) characterizing the spherical curves according to Bishop frame can be rewritten as follows;

$$\sum_{k=0}^{m}P_k(s)y^{(k)}(s)=F(s)\quad m=3\quad 0\le s\le 2\pi. \tag{28}$$

Let f be a solution of Eq.(28). We wish to approximate f by

$$y=p_n(s)=\sum_{k=0}^{n}a_kB_{k,n}(s)\qquad n=4 \tag{29}$$

such that $p_n(s)$ satisfies Eq.(28) on the nodes $0\le s_0<s_1<...<s_4\le 2\pi$. Here $n=4$ is taken for simplicity. Putting $p_n(s)$ into Eq.(28), we get the system of linear equations depending on $a_0,a_1,,..,a_4$. Let us consider the Eq.(4.1) and find the matrix forms of each term in the equation. First we can convert the Bernstein series solution $y=p_n(s)$ defined by (4.2) and its derivatives $n=y^{(k)}(s)$ to matrix forms, for $n=4$ and $k=0,1,2,3$

$$y(s)=B_4(s)A \text{ and } y^{(k)}(s)=B_4^k(s)A \tag{30}$$

here $B_4(s)=\begin{bmatrix} B_{0,0}(s) & B_{1,0}(s) & \cdots & B_{4,0}(s) \\ B_{0,1}(s) & B_{1,1}(s) & \cdots & B_{4,1}(s) \\ \vdots & \vdots & \ddots & \vdots \\ B_{0,4}(s) & B_{1,4}(s) & \cdots & B_{4,4}(s) \end{bmatrix}$, $A=\begin{bmatrix} a_0 & a_1 & \cdots & a_2 \end{bmatrix}$

On the other hand, it can be written $\left[B_4(s)\right]^T$ as $\left[B_4(s)\right]^T=\left[D(S(s))\right]^T$ or

$$B_4(s)=S(s)D^T \tag{31}$$

for $d_{ij}=\begin{cases} \dfrac{(-1)^{j-i}}{R^j}\dbinom{n}{1}\dbinom{n-i}{j-i}, & \text{i} \le \text{j} \\ 0, & i>j \end{cases}$ the matrix D is calculated as follows

$$D=\begin{bmatrix} 1 & -2/\pi & 3/2\pi^2 & -1/2\pi^3 & 1/16\pi^4 \\ 0 & 2/\pi & -3/\pi^2 & 3/2\pi^3 & -1/4\pi^4 \\ 0 & 0 & 3/2\pi^2 & -3/2\pi^3 & 3/8\pi^4 \\ 0 & 0 & 0 & 1/2\pi^3 & -1/4\pi^4 \\ 0 & 0 & 0 & 0 & 1/16\pi^4 \end{bmatrix}$$

It is clearly seen that the relation between the matrix $S(s)$ and its derivative $S^{'}(s)$ is $S^{'}(s)=S(s)B$

$$B=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S(s)=\begin{bmatrix} 1 & s & s^2 & s^3 & s^4 \end{bmatrix}.$$

To obtain the matrix $S^{(k)}(s)$ in terms of the matrix $S(s)$, we can use the following procedure:

$$\begin{aligned} S^{''}(s)&=S^{'}(s)B=S(s)B^2 \\ S^{'''}(s)&=S^{''}(s)B=S(s)B^3 \end{aligned} \tag{32}$$

$$B^2=\begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B^3=\begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently, by substituting the matrix forms (31) and (32) into (30), we have the matrix relation.

$$
\begin{aligned}
y(s) &= S(s)D^T A \\
y'(s) &= S(s)BD^T A \\
y''(s) &= S(s)B^2 D^T A \\
y'''(s) &= S(s)B^3 D^T A.
\end{aligned}
\tag{33}
$$

Substituting the matrix relation (33) into (28) and then simplifying, we obtain the matrix equation

$$\sum_{k=0}^{m} P_k(s)S(s)B^k D^T A = f(s) \tag{34}$$

By using the nodes $\{s_i, i=0,1,...,4; 0 \le s_0 \le s_1 < ... < s_4 \le 2\pi\}$ in (34) we get the system of matrix equations $\sum_{k=0}^{m=3} P_k(s_i)S(s_i)B^k D^T A = f(s_i), \quad (i=0,1,...,4)$ and $s_0 = 0,\ s_1 = \frac{\pi}{2},\ s_2 = \pi,\ s_3 = \frac{3\pi}{2},\ s_4 = 2\pi$

$$P_k(s_i) = \begin{bmatrix} P_k(0) & 0 & 0 & 0 & 0 \\ 0 & P_k(\pi/2) & 0 & 0 & 0 \\ 0 & 0 & P_k(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_k(3\pi/2) & 0 \\ 0 & 0 & 0 & 0 & P_k(2\pi) \end{bmatrix}$$

$$S(s_i) = \begin{bmatrix} S(s_0) \\ S(s_1) \\ S(s_2) \\ S(s_3) \\ S(s_4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & (\pi/2) & (\pi/2)^2 & (\pi/2)^3 & (\pi/2)^4 \\ 1 & (\pi) & (\pi)^2 & (\pi)^3 & (\pi)^4 \\ 1 & (3\pi/2) & (3\pi/2)^2 & (3\pi/2)^3 & (3\pi/2)^4 \\ 1 & (2\pi) & (2\pi)^2 & (2\pi)^3 & (2\pi)^4 \end{bmatrix}, F(s_i) = \begin{bmatrix} f(0) \\ f(\pi/2) \\ f(\pi) \\ f(3\pi/2) \\ f(2\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The fundamental matrix equation can be written briefly as

$$\sum_{k=0}^{m} P_k S B^k D^T A = F \tag{35}$$

Hence, the fundamental matrix Eq.(35) corresponding to (29) can be written in the form

$$WA = F \text{ or } [W;F] = A,\ W = [w_{kh}],\ k,h = 0,1,...,4 \tag{36}$$

Also $W = \sum_{k=0}^{m} P_k S^k D^T$ and the Eq. (36) corresponds to a matrix of type (5x5). Now let us obtain the matrix equation of the conditions by means of the relation (36), as follows

$$S(0)B^k D^T A = [\alpha_k] \qquad k = 0,1,2$$

Firstly, the matrix forms for the conditions can be written as

$$U_k A = [\alpha_k] \text{ or } [U_k; \alpha_k] \quad k = 0,1,2 \tag{37}$$

for
$$U_0 = S(0)D^T = [u_{00}\ u_{01}\ \cdots\ u_{04}] = [1\ 0\ 0\ 0\ 0]$$
$$U_1 = S(0)B^1D^T = [u_{10}\ u_{11}\ \cdots\ u_{14}] = [-2/\pi\ \ 2/\pi\ \ 0\ 0\ 0]$$
$$U_2 = S(0)B^2D^T = [u_{20}\ u_{21}\ \cdots\ u_{24}] = [12/\pi^2\ \ -6/\pi^2\ \ 3/\pi^2\ \ 0\ 0]$$

Replacing the row matrices (37) by any $m=3$ rows of the matrix (36), we get the augmented matrix $[\tilde{W};\tilde{F}]$ as

$$[\tilde{W};\tilde{F}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & w_{04} & ; & f(0) \\ u_{00} & u_{01} & u_{02} & u_{03} & u_{04} & ; & \alpha_0 \\ u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & ; & \alpha_1 \\ u_{20} & u_{21} & u_{22} & u_{23} & u_{24} & ; & \alpha_1 \\ w_{50} & w_{51} & w_{52} & w_{53} & w_{54} & ; & f(2\pi) \end{bmatrix}$$

$w_{ij}\ (i=0,1\ j=0,1,...,4)$ obtained as follows;

$$w_{00} = P_0(0) - \frac{2}{\pi}P_1(0) + \frac{3}{\pi^2}P_2(0) - \frac{3}{\pi^3}P_3(0), w_{01} = \frac{2}{\pi}P_1(0) + \frac{6}{\pi^2}P_2(0) + \frac{9}{\pi^3}P_3(0)$$
$$w_{02} = \frac{3}{\pi^2}P_2(0) - \frac{9}{\pi^3}P_3(0), w_{03} = \frac{3}{\pi^3}P_3(0),\ \ w_{04} = 0, w_{50} = \frac{9}{\pi^2}P_2(2), w_{51} = -\frac{3}{\pi^3}P_3(2\pi)$$
$$w_{52} = \frac{3}{\pi^2}P_2(2\pi) + \frac{9}{\pi^3}P_3(2\pi), w_{53} = -\frac{2}{\pi}P_1(2\pi) - \frac{6}{\pi^2}P_2(2\pi) - \frac{9}{\pi}P_3(2\pi)$$
$$w_{54} = P_0(2\pi) + \frac{2}{\pi}P_1(2\pi) + \frac{3}{\pi^2}P_2(2\pi) + \frac{3}{\pi^3}P_3(2\pi)$$

As a result we can write the following matrix.

$$A = (\tilde{W})^{-1}\tilde{F} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \pi/2 & 0 & 0 \\ 0 & -2 & \pi & \pi^2/3 & 0 \\ R & M & T & K & V \\ Y & Z & Q & L & C \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}$$

$$R = w_{54} / (w_{03}w_{54} - w_{04}w_{53}),\ M = [(w_{50} + w_{51} + 2w_{52})w_{04} - (w_{00} + w_{01} + 2w_{02})w_{54}] / (w_{03}w_{54} - w_{04}w_{53})$$
$$T = \pi/2[(2w_{02} - w_{01})w_{54} - (2w_{52} - w_{51})w_{04}] / (w_{03}w_{54} - w_{04}w_{53}), L = 3[(w_{02}w_{53} - w_{03}w_{52}) / (w_{03}w_{54} - w_{04}w_{53})]$$

$$Z = [w_{53}(w_{00} + w_{01} - 2w_{02}) - w_{03}(w_{50} + w_{51} - 2w_{52})w_{54}] / (w_{03}w_{54} - w_{04}w_{53}), C = w_{03} / (w_{03}w_{54} - w_{04}w_{53})$$
$$Q = \pi/2\{[w_{53}(w_{01} + 2w_{02}) - w_{03}(w_{51} + 2w_{52})] / (w_{03}w_{54} - w_{04}w_{53})\}$$

and hence the elements $a_0, a_1, ..., a_4$ of A are uniquely determined as follow

$$a_o = \alpha_1,\ a_1 = \alpha_1 + \frac{\pi}{2}\alpha_2,\ a_2 = -2\alpha_1 + \pi\alpha_2 + \frac{\pi}{3}\alpha_3$$
$$a_3 = M\alpha_1 + T\alpha_2 + K\alpha_3,\ a_4 = Z\alpha_1 + Q\alpha_2 + T\alpha_3$$

If we put this a_k unknowns in equation (29), we obtain the Bernstein series solution $y = p_n(s) = m_1$ of the Eq.(28).

5. Conclusion

In the present paper we dealt with spherical curves. The expression of the position vector of a spherical curve in terms of Bishop frame elements and the relation between the first and second Bishop curvatures for the spherical curve is given. We obtain a third order linear differential equation which is characterizing the curve that lies on a sphere. And by using Bernstein series solution method, the solutions of the equation is approximately obtained.

References

[1] Wong , Y. C., On an Explicit Characterization of Spherical Curves, *Proc. Am. Math. Soc., 34* (1972), 1, pp. 239-242

[2] Breuer ,S., Gottlieb, D., Explicit Characterization of Spherical Curves, *Proc. Am. Math. Soc., 27* (1971), pp. 126-127

[3] Wong, Y. C., A global formulation of the condition for a curve to lie in a sphere, *Monatsh. Math., 67* (1963), 363-365

[4] Mehlum, E., Wimp ,J., Spherical Curves and Quadratic Relationships for Special Functions, *Austral. Mat. Soc., 27* (1985) pp. 111-124

[5] Kose ,O.,An Expilicit Characterization of Dual Spherical Curves, *Doğa Mat. 12* (1998), 3, pp. 105-113

[6] Abdel Bakey , R. A.,An Explicit Characterization of Dual Spherical Curve, *Commun. Fac. Sci. Univ. Ank. Series, 51* (2002), *2*, pp. 1-9

[7] Ilarslan , K., e*t all.*, On the explicit characterization of spherical curves in 3-dimensional Lorentzian space, *Journal of Inverse and Ill-posed Problems, 11* (2003), 4, pp. 389-397

[8] Kocayigit, H., *et all.*, On the explicit characterization of spherical curves in n-dimensional Euclidean space, *Journal of Inverse and Ill-posed Problems, 11* (2003), 3, pp. 245-254

[9] Ayyilidiz, N. , *et all.*, A Characterization of Dual Lorentzian Spherical Curves in the Dual Lorentzian Space, *Taiwanese Journal of Mathematics, 11* (2007), *4*, pp. 999-1018

[10] Camci, C., *et all.*, On the characterization of spherical curves in 3-dimensional Sasakian spaces, *J. Math. Anal. Appl., 342* (2008),pp. 1151–1159

[11] Bishop, L. R., There is More Than one Way to Frame a Curve, *Amer. Math. Monthly, 82* (1975), 3, pp. 246-251

[12] Bhatti, ,M.I., Brocken, B., Solutions of Differential Equations in a Bernstein Polynomial Basis, Bhatti *Journal of Computational And Applied Mathematics. 205* (2007), pp. 272-280

[13] Işik ,O.R., *et all.*, A rational approximation based on Bernstein polynomials for high order initial and boundary values problems, *Applied Mathematics and Computation, 217* (2011), pp. 9438-945