ADOMIAN-PADÉ APPROXIMATE SOLUTIONS TO THE CONFORMABLE NONLINEAR HEAT TRANSFER EQUATION

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This paper adopts the Adomian decomposition method and the Padé approximation technique to derive the approximate solutions of a conformable heat transfer equation by considering the new definition of the Adomian polynomials (APs). The Padé approximate solutions are derived along with interesting figures showing the approximate solutions.

Key Words: *Adomian decomposition method, conformable heat transfer equation, Padé approximation.*

1. Introduction

It is well-known that the majority of the real-world physical phenomena are modeled by mathematical equations, especially partial differential equations (PDEs) [1]. The investigations of the exact and numerical solutions of various PDEs have become a very important practice by different scholars [2]. In recent time, fractional derivatives have been applied in various fields of physical sciences such as reaction diffusion, heat transform, control, and many more. Fractional derivatives have many kinds of definitions [3-5]. We consider the nonlinear conformable heat transfer equation in the following form [6]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - 2u^3 \tag{1}$$

with the initial condition:

$$u(x,0) = \frac{1+2x}{x^2+x+1}. \tag{2}$$

Eq.(1) has been solved using different techniques, including the fractional complex transform along with He's variational iteration method [6], homotopy perturbation method [7], sub-equation method [8]. This paper applies the Adomian decomposition method and the Padé approximation technique [9] to search for an approximate solutions of the model problem. Several research results have been report in the last few decades in this avenue [10-20].

2. Conformable fractional derivative

In what follows, we give a brief description of the conformable derivative given by Khalil et al. [5] as follows:

Definition 1.

$$Let f: [a,b] \times (0,\infty) \to \mathbb{R}, \tag{3}$$

then the conformable fractional derivative of f is defined as

$$ {}_t\mathbb{T}_\alpha^\alpha(f)(x,t) = \lim_{\varepsilon\to 0} \frac{f(x,t+\varepsilon(t-\alpha)^{1-\alpha}) - f(x,t)}{\varepsilon}, \quad \alpha \in (0,1], \tag{4}$$

for all $t > 0$.

When $a = 0$, we have ${}_t\mathbb{T}_\alpha^a(f)(x,t) = {}_t\mathbb{T}_\alpha(f)(x,t)$. This definition of fractional derivative satisfies the following properties.

Theorem 1. Let $\alpha \in (0,1]$ *and* $a, b \in \mathbb{R}$, *then*

1. ${}_t\mathbb{T}_\alpha(au + bv) = a_t\mathbb{T}_\alpha(u) + b_t\mathbb{T}_\alpha(v)$.
2. ${}_t\mathbb{T}_\alpha(t^\lambda) = \lambda t^{\lambda-\alpha}, \lambda \in \mathbb{R}$.
3. ${}_t\mathbb{T}_\alpha(uv) = u_t\mathbb{T}_\alpha(v) + v_t\mathbb{T}_\alpha(u)$.
4. ${}_t\mathbb{T}_\alpha(\frac{u}{v}) = \frac{u_t\mathbb{T}_\alpha(v) - v_t\mathbb{T}_\alpha(u)}{v^2}$.
5. ${}_t\mathbb{T}_\alpha(u)(t) = t^{1-\alpha}u'(t), u \in C^1$.
6. ${}_t\mathbb{T}_\alpha(u)(t) = (t-\alpha)^{1-\alpha}u'(t), u \in C^1$.

3. Padé Approximation

Consider a system of fractional differential equations [9]

$$\begin{cases} L_1u_1 + R_1(u_1, u_2, \dots, u_s) + N_1(u_1, u_2, \dots, u_s) = g_1, \\ L_2u_1 + R_2(u_1, u_2, \dots, u_s) + N_2(u_1, u_2, \dots, u_s) = g_2, \\ \dots, \\ L_su_1 + R_s(u_1, u_2, \dots, u_s) + N_s(u_1, u_2, \dots, u_s) = g_s. \end{cases} \tag{5}$$

Subject to the initial or boundary conditions

$$\begin{aligned} B_1(u_1, \dots, u_s) &= \sum\sum \rho_{i,n}^1 \frac{\partial_{(n\alpha)} u_i(t,x,t,y,\dots)}{\partial t^{n\alpha}}\Big|_{t=b_{i,n}} = p_1(x,y,\dots), \\ B_2(u_1, \dots, u_s) &= \sum\sum \rho_{i,n}^2 \frac{\partial_{(n\alpha)} u_i(t,x,t,y,\dots)}{\partial t^{n\alpha}}\Big|_{t=b_{i,n}} = p_2(x,y,\dots). \end{aligned} \tag{6}$$

Applying the inverse L_i^{-1} on both sides of Eq.(5) gives

$$\begin{cases} u_1 = \phi_1 + L_1^{-1}g_1 - L_1^{-1}R_1(u_1, u_2, \dots, u_s) - L_1^{-1}N_1(u_1, u_2, \dots, u_s), \\ u_2 = \phi_2 + L_2^{-1}g_2 - L_2^{-1}R_2(u_1, u_2, \dots, u_s) - L_2^{-1}N_2(u_1, u_2, \dots, u_s), \\ \dots \\ u_s = \phi_s + L_s^{-1}g_s - L_s^{-1}R_s(u_1, u_2, \dots, u_s) - L_s^{-1}N_s(u_1, u_2, \dots, u_s), \end{cases} \tag{7}$$

in which the ϕ_i engulf the constants of integration w.r.t the variable t. The solution u_i can be dissolve into a series as [9]

$$u_i = \sum_{m=0}^{\infty} u_{i,m}, \quad i = 1,2,\dots,s, \tag{8}$$

$$[\![\textstyle\sum_{n=0}^{\infty} A_n]\!]_{n\geq m} = \left[\!\!\left[\left[\!\!\left[\sum_{n=0}^{\infty} \left[\!\!\left[\frac{1}{n!}\sum_{v=0}^{\infty} (u_v - u_0)^n \frac{\partial^n N(u_0)}{\partial u_0^n}\right]\!\!\right]_{v\geq p_1(m)}\right]\!\!\right]_{n\geq p_1(m)}\right]\!\!\right]_{\sum jv_j \geq p_3(m)}, \tag{9}$$

and the nonlinear terms $N_i(u_1, u_2, \dots, u_s)$ are assumed to be decomposed into a multivariable APs given

$$N_i(u_1, u_2, \dots, u_s) = \sum_{m=0}^{\infty} A_{i,m}(u_{1,0}, \dots, u_{2,0}, \dots, u_{2,m}, u_{s,0}, \dots, u_{s,m})\lambda^m, \tag{10}$$

which are defined by $A_{i,m} = \frac{1}{m!}\frac{d^m}{d\lambda^m}N_i^{(m)}(u_1(\lambda), \dots, u_s(\lambda))\Big|_{\lambda=0}$ and where $A_{i,m}$ can be generated for the four classes of the APs as shown by the cases below:

• *Class I*. APs $A_{i,m}^{(I)}$, is represented by

$$\theta_{i,m}^{(I)} = \sum_{n=0}^{\infty} \frac{(\phi_{i,m}^{(I)} - u_{i,0})}{n!} N_i^n(u_i, 0), \quad m \geq 1, \tag{11}$$

and

$$\begin{aligned} \phi_{i,m}^{(I)} &= \sum_{v=0}^{m-1} u_{i,v}^{(I)}, \\ A_{i,0}^{(I)} &\equiv \theta_{i,1}^{(I)}, \end{aligned} \tag{12}$$

for $m \geq 1$: $A_{i,m}^{(I)} = \theta_{i,m+1}^{(I)} - \theta_{i,m}^{(I)}$

• *Class II*. APs $A_{i,m}^{(II)}$ is represented by

$$\theta_{i,m}^{(II)} = \sum_{n=0}^{\infty} \frac{(\phi_{i,m}^{(II)} - u_{i,0})}{n!} N_i^n(u_i, 0), \quad m \geq 1, \tag{13}$$

and

$$\begin{aligned} \phi_{i,m}^{(II)} &= \sum_{v=0}^{m-1} u_{i,v}^{(II)}, \\ A_{i,0}^{(II)} &\equiv \theta_{i,1}^{(II)}, \end{aligned} \tag{14}$$

for $m \geq 1$: $A_{i,m}^{(II)} = \theta_{i,m+1}^{(II)} - \theta_{i,m}^{(II)}$.

• *Class III*. APs $A_{i,m}^{(III)}$ is represented by

$$\theta_{i,m}^{(III)} = \sum_{n=0}^{\infty} \frac{(\phi_{i,m-n+1}^{(III)} - u_{i,0})}{n!} N_i^n(u_i, 0), \quad m \geq 1, \tag{15}$$

and

$$\begin{aligned} \phi_{i,m}^{(III)} &= \sum_{v=0}^{m-1} u_{i,v}^{(III)}, \\ A_{i,0}^{(III)} &\equiv \theta_{i,1}^{(III)} \end{aligned} \tag{16}$$

for $m \geq 1$: $A_{i,m}^{(III)} = \theta_{i,m+1}^{(III)} - \theta_{i,m}^{(III)}$.

• *Class IV*. APs $A_{i,m}^{(IV)}$ is represented by

$$\theta_{i,m}^{(III)} = \left[\!\left[\sum_{n=0}^{\infty} \frac{(\phi_{i,m-n+1}^{(III)} - u_{i,0})}{n!} N_i^n(u_i, 0) \right]\!\right]_{\sum j v_j \geq m}, \ m \geq 1, \tag{17}$$

and

$$\begin{aligned} \phi_{i,m-n+1}^{(IV)} &= \sum_{v=0}^{m-1} u_{i,v}^{(IV)}, \\ A_{i,0}^{(IV)} &\equiv \theta_{i,1}^{(IV)} \end{aligned} \tag{18}$$

for $m \geq 1$: $A_{i,m}^{(IV)} = \theta_{i,m+1}^{(IV)} - \theta_{i,m}^{(IV)}$.

Substitution Eq.(8) and Eq.(10), and utilizing the Adomian-Rach double decomposition approach, one can get

$$\begin{cases} \sum_{m=0}^{\infty} u_{1,m} = \sum_{m=0}^{\infty} (\phi_{1,m} + L_1^{-1} g_1 - L_1^{-1} R_1(\sum_{m=0}^{\infty} u_{1,m}, \ldots, \sum_{m=0}^{\infty} u_{s,m}) - L_1^{-1} \sum_{m=0}^{\infty} A_{1,m}, \\ \sum_{m=0}^{\infty} u_{2,m} = \sum_{m=0}^{\infty} (\phi_{2,m} + L_2^{-1} g_1 - L_1^{-1} R_2(\sum_{m=0}^{\infty} u_{1,m}, \ldots, \sum_{m=0}^{\infty} u_{s,m}) - L_2^{-1} \sum_{m=0}^{\infty} A_{2,m}, \\ \ldots, \\ \sum_{m=0}^{\infty} u_{s,m} = \sum_{m=0}^{\infty} (\phi_{s,m} + L_s^{-1} g_1 - L_s^{-1} R_s(\sum_{m=0}^{\infty} u_{1,m}, \ldots, \sum_{m=0}^{\infty} u_{s,m}) - L_s^{-1} \sum_{m=0}^{\infty} A_{s,m}. \\ . \end{cases} \tag{19}$$

For the specified initial value problems [12], with $L = \frac{d^{\alpha}}{dt^{\alpha}}$ being the operator, $k_i - 1 < \alpha \leq k_i$, we have

$$\phi_{i,m} = C_{i,1}^m + C_{i,2}^m t + \ldots, C_{i,k_i}^m t^{k_i} \tag{20}$$

with $C_{i,j}^m$, $j = 1, \ldots, k_i$. For fractional IVPs, we have

$$\phi_{i,m} = \sum_{l=0}^{k_i - 1} t^{l\alpha} C_{i,l}^m \tag{21}$$

in which $C_{i,j}^m$, $j = 1, \ldots, k_i$ are computed using the conditions.

Thus, $u_{i,m} (m \geq 0)$ can be derived from

$$\begin{cases} u_{i,0} = \phi_{i,0} + L_i^{-1} g_i, \\ u_{i,m+1} = \phi_{i,m+1} - R_i(u_{1,m}, \ldots, u_{s,m}) - L_i^{-1} A_{i,m}, \\ m \geq 0. \end{cases} \tag{22}$$

The solutions of the approximants are

$$\phi_{i,r} = \sum_{m=0}^{r-1} u_{i,m}. \tag{23}$$

From Eq.(23), using $u_{i,0}$ all other solution terms $u_{i,j},\ j = 1,2,\ldots,$ can be obtained. Thus, the approximation $\phi_{i,r}$ for the solutions $u_i,\ i = 1,2,\ldots,s$ are derived.

3.1 The Padé Approximation

To fasten the convergence of the series solutions, the Padé approximants method is also assimilated into the above described algorithm given as: The main concept of Padé approximants is to replace a series function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ [11] by

$$\frac{A^{[L/M]}(z)}{B^{[L/M]}(z)} = \frac{a_0 + a_1 z + \ldots, a_L z^L}{1 + b_1 + \ldots, b_M z^M}, \tag{24}$$

given by $[L/M]$ called the Padé approximants.

For a function $f(z) = c_0 + c_1 z + \ldots, c_2 z^2 + \ldots,$ the coefficients $a_0, a_1, \ldots, a_L, b_0, b_1, \ldots, b_M$ can be calculated by matching the series coefficients. Therefore, one can acquire the following sequence

- $[1/1] = \frac{a_0 + a_1 z}{1 + b_1 z}, \quad \lim_{z\to\infty}[1/1] = \frac{a_1}{b_1},$
- $[2/2] = \frac{a_0 + a_1 z + a_2 z^2}{1 + b_1 z + b_2 z^2}, \quad \lim_{z\to\infty}[2/2] = \frac{a_2}{b_2},$
- $[3/3] = \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3}{1 + b_1 z + b_2 z^2 + b_3 z^3}, \quad \lim_{z\to\infty}[3/3] = \frac{a_3}{b_3},$
- ...
- $\lim_{z\to\infty}[m/m] = \frac{a_m}{b_m}.$

3.2 Application to the Conformable Heat Transfer Equation

We will consider the class (IV) APs Eq.(17) to compute the truncated series solutions and then compare the exact Eq.(2) and approximate solutions. For each of the cases to be considered, we will use Padé approximant sizes of $[10/10]$ and $[20/20]$.

• If we select $\alpha = 0.5,$ using 7 terms of an Adomian power series, Padé approximant size of $[10/10]$ and using the class (IV) APs, we obtain the following expression of the Padé approximant for this case as

$$\begin{aligned} \mathrm{Pade}_4[10/10] = 1 - 2.25676\sqrt{t} + 44.34929t^2 - 22.98715t^{\frac{3}{2}} + 8t \\ -1472.2094t^3 + 42956.36098t^{\frac{7}{2}} - 428.22830t^{\frac{5}{2}}. \end{aligned} \tag{25}$$

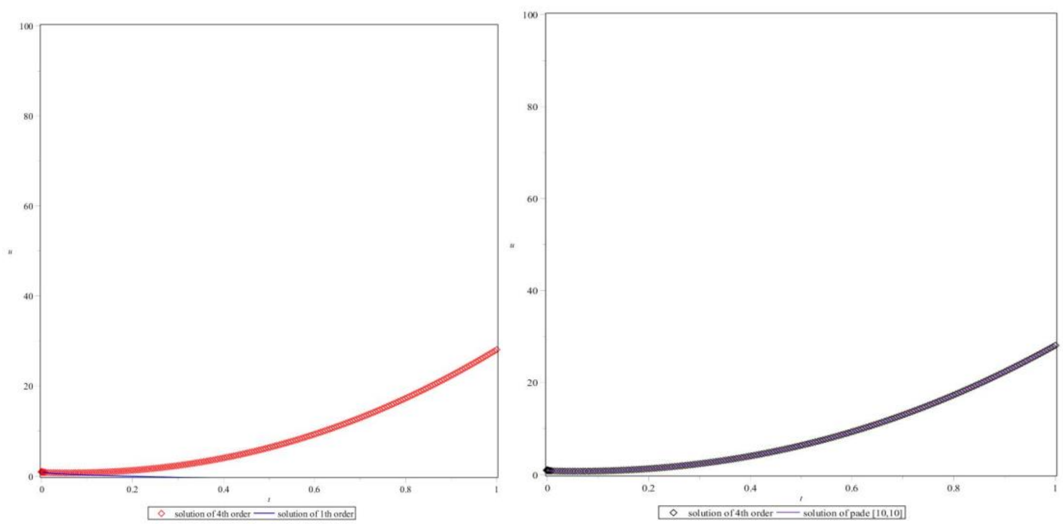

Figure 1: Comparison of the Adomian-Padé approximate solutions Eq.(25) for the different steps and $0 \leq t \leq 1$ and $0 \leq u \leq 100$.

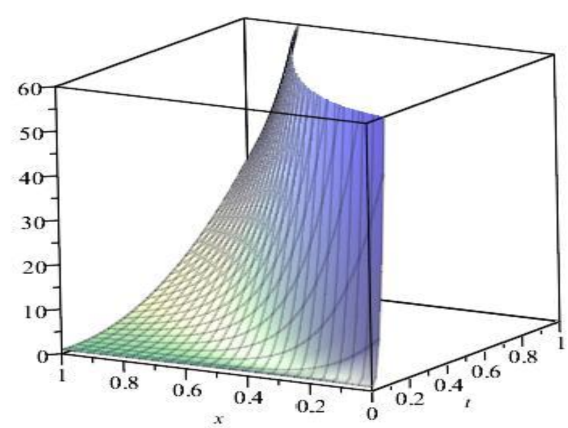

Figure 2: The decomposition solution for the 3^{rd} −order series for $0 \leq x \leq 1$, $0 \leq t \leq 1$ and $0 \leq u \leq 60$ with Eq.(2).

• If we select $\alpha = 1$, using 10 terms of an Adomian power series, Padé approximant size of $[10/10]$ and using the class (IV) APs, we obtain the following expression of the Padé approximant for this case as

$$\text{Pade}_4\left[\frac{10}{10}\right] = 1 - 2t + 4t^2 - 8t^3 + 16t^4 - 32t^5 + 64t^6 \\ -128t^7 + 256t^8 - 512t^9 + 1024t^{10} \tag{26}$$

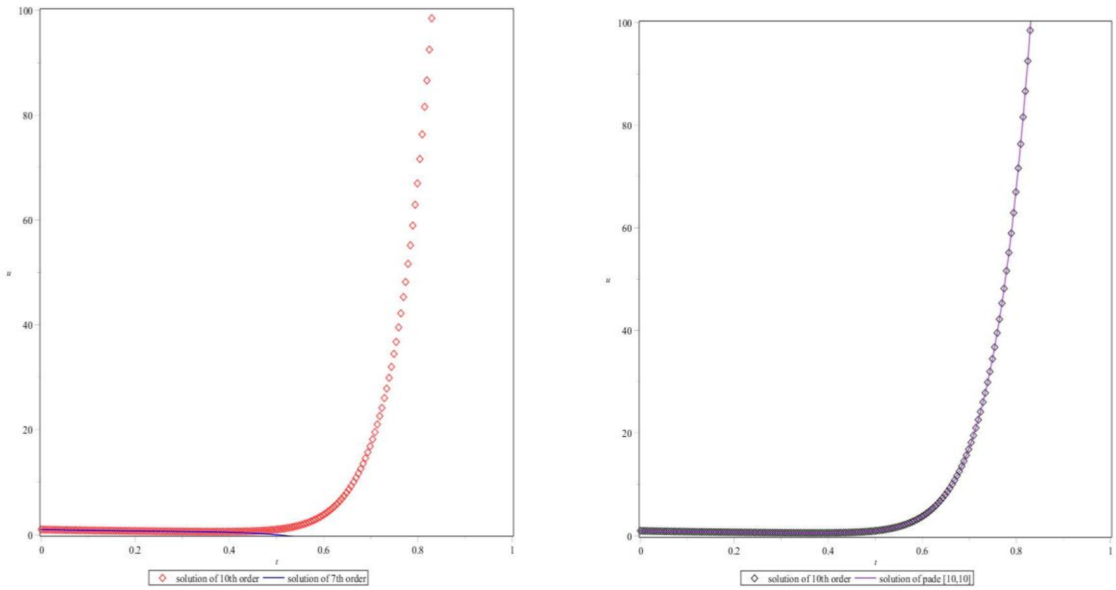

Figure 3: Comparison of the Adomian-Padé approximate solutions Eq.(26) for the different steps $0 \leq t \leq 1$ and $0 \leq u \leq 100$.

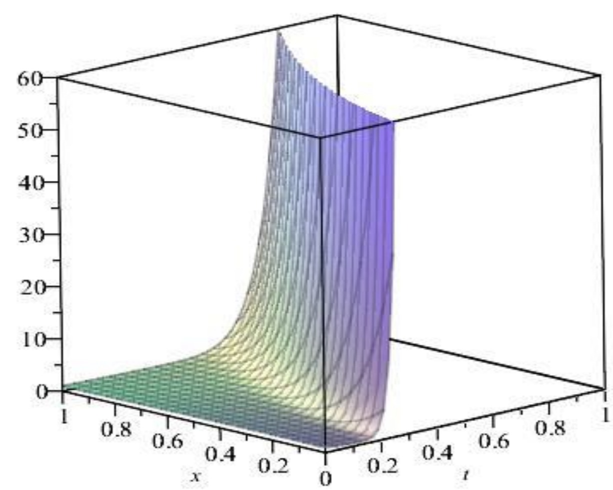

Figure 4: The decomposition solution for the 10 th- order series for $0 \leq x \leq 1,\ 0 \leq t \leq 1$ and $0 \leq u \leq 60$ with Eq.(2).

• If we select $\alpha = 0.8,$ using 10 terms of an Adomian power series, Padé approximant size of $[20/20]$ and using the class (IV) APs, we obtain the following expression of the Padé approximant for this case as

$$\mathrm{Pade}_4[20/20] = 1 - 2t + 4t^2 - 8t^3 + 16t^4 - 32t^5 + 64t^6 - 128t^7 + 256t^8 - 512t^9 + 1024t^{10}. \quad (27)$$

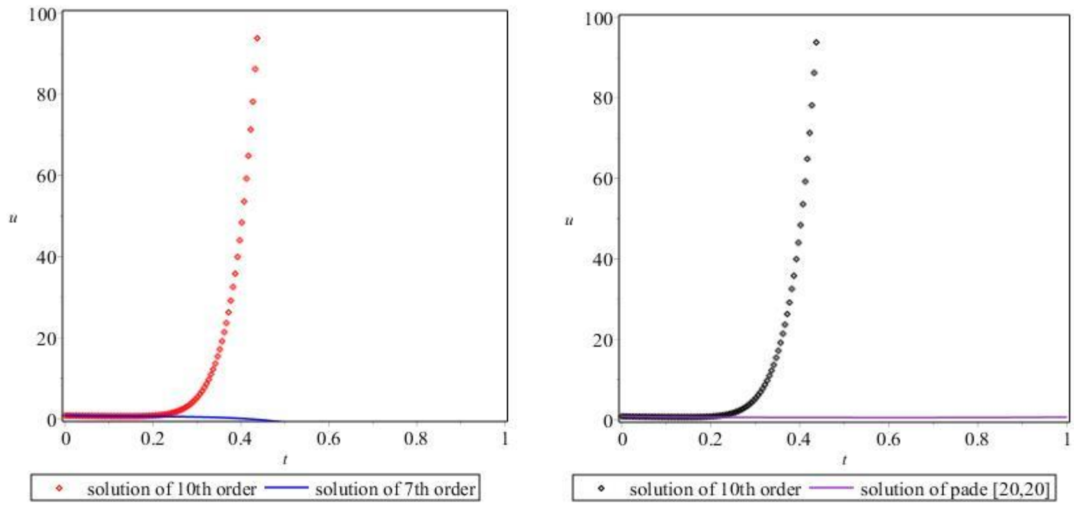

Figure 5: Comparison of the Adomian-Padé approximate solutions Eq.(27) for the different steps $0 \leq t \leq 1$ and $0 \leq u \leq 100$.

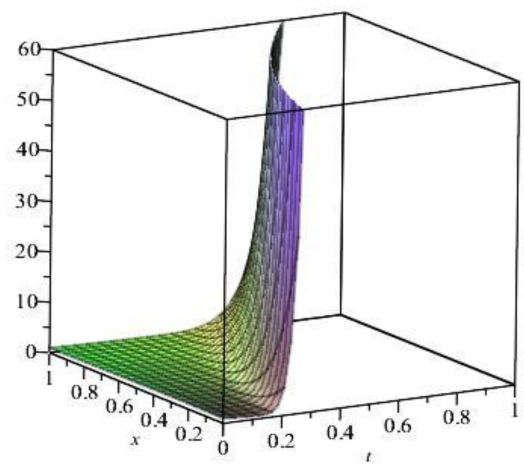

Figure 6: The decomposition solution for the 10 th- order series for $0 \leq x \leq 1,\ 0 \leq t \leq 1$ and $0 \leq u \leq 60$ with Eq.(2).

4 Concluding remarks

In this work, we have success fully used the Padé approximation technique to derive the approximate solutions of a conformable heat transfer equation. The result shows that the method is powerful and efficient for solving the nonlinear fractional differential equations arising in engineering and science. It is observed that the numerical solutions are in closed agreement with the exact solution. Some interesting figures have been given the comparison approximate solutions for the different steps in Figures 1-6.

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