ANALYTICAL SOLUTIONS OF NONLINEAR KLEIN-GORDON EQUATIONS USING MULTISTEP MODIFIED REDUCED DIFFERENTIAL TRANSFORM METHOD

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This paper explores the approximate analytical solution of nonlinear Klein-Gordon equations (NKGE) by using Multistep Modified Reduced Differential Transform Method (MMRDTM). Through this proposed strategy, the nonlinear term is substituted by associating Adomian polynomials obtained by utilization of a multistep approach. The NKGE solutions can be obtained with a reduced number of computed terms. In addition, the approximate solutions converge rapidly in a wide time region. Three examples are provided to illustrate the effectiveness of the proposed method to obtain solutions for the NKGE. Graphical results are shown to represent the behavior of the solution so as to demonstrate the validity and accuracy of the MMRDTM.

Key Words*: Adomian polynomials, multistep approach, Reduced Differential Transform Method, nonlinear Klein-Gordon equations.*

1. Introduction

Klein-Gordon (KG) equation is an important equation which is related to the Schrödinger equation. It is applied widely in fields, such as quantum mechanics, solid state physics and nonlinear optics [1]. The KG equation is one of the important equations in solitons studies, particularly in the examination of solitons interactions for a collisionless plasma and the recurrence of initial states [2, 3].

Many techniques have been implemented to derive the approximate analytical solution of the Klein-Gordon equations. In 2011, Servi & Oturanç [4] executed Reduced Differential Transform Method (RDTM) to solve Klein-Gordon equation. On the other hand, to calculate the exact traveling wave solutions to the KG equation, Hafez, Alam, & Akbar [5] used the novel (G'/G)-expansion method. Meanwhile, Venkatesh, *et. al*, 2016 [6] used Lagendre wavelet-based approximations to solve KG equation that arise in quantum field theory using wavelets. Recently in 2018, Agom & Ogunfiditimi proposed the modified Adomian decomposition method (ADM) to find the exact solution to NKGE with quadratic nonlinearity [7].

Many partial differential equations, ordinary differential equations and delay differential equations have been solved by utilizing DTM and RDTM [8-14]. Ray [15] proposed an adjustment on the fractional RDTM and executed it to obtain solutions of fractional Korteweg de Vries (KdV) equations. Through this methodology, the modification included the substitution of the nonlinear term by relating Adomian polynomials. Therefore, the solutions of nonlinear initial value problem can be obtained in an easier way with reduced computed terms. Further, El-Zahar [16] introduced adaptive multistep DTM to obtain solution of singular perturbation initial-value problems. It yields the solution in a rapid convergent series which results in the solution converging in wide time region. Recently, C.Hussin *et. al.* [17] has proposed and implemented MMRDTM for solving Nonlinear Schrodinger Equations (NLSE). The results showed the approximate solutions of NLSE with high accuracy were obtained.

In this study, we combine modification in [15] and multistep approach in [16] to implement a new technique called Multistep Modified Reduced Differential Transform Method (MMRDTM). The key benefit of the proposed technique is that it produces an analytical approximation in a rapid convergent sequence with elegant computed terms.

2. Application of MMRDTM for the Solution of Nonlinear Klein-Gordon Equation

Let us use the general nonlinear Klein-Gordon equation of the form [18]

$$\boldsymbol{u_{tt} - \alpha u_{xx} + \beta u_t + \gamma u^k = f(x,t)} \qquad \boldsymbol{, x \in \Omega, \;\; 0 < t \leq T} \tag{1}$$

which is subjected to the following initial conditions

$$u(x,0) = u_0(x), \qquad a \leq x \leq b$$
$$u_t(x,0) = u_1(x), \qquad a \leq x \leq b$$

where $\Omega = [a,b] \subset R, u(x,t)$ denotes the wave displacement at position x and time t, $u_0(x)$ is a known function and α, β, and γ are real numbers $(\gamma \neq 0)$. $k = 2$ is the case of quadratic nonlinearity and $k = 3$ for a cubic nonlinearity.

Applying basic properties of Modified RDTM to Eq. (1), we obtain

$$U_{k+2,i}(x) = \left(\frac{1}{(k+2)(k+1)}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{k,i}(x)\right) - \gamma \sum_{k=0}^{n} A_{k,i} - \beta U_{k,i} + f(x,t)\right) \tag{2}$$

From the initial condition, we can write

$$U_0(x) = f(x). \tag{3}$$

The nonlinear term can be composed as follows

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(U_0(x), U_1(x), \dots, U_n(x))$$

We acquire the following $U_k(x)$ values by straightforward iterative estimation and substituting Eq. (3) into Eq. (2). Then, inversely, the transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives the n-terms estimation solution as follows,

$$u(x,t) = \sum_{k=0}^{K} U_k(x) t^k, \qquad t \in [0,T]$$

For $i = 1,2, \dots, M$, the interval $[0,T]$ is separated into N subintervals $[t_{i-1}, t_i]$ by using equal step size of $h = \frac{T}{M}$ and nodes $t_i = is$. Multistep RDTM is computed according to the following steps.

Firstly, the RDTM is applied to the initial value problem in interval $[0, t_1]$. From that point, the estimated result is obtained using the initial conditions $u(x,0) = f_0(x),\ u_1(x,0) = f_1(x)$.

$$u_1(x,t) = \sum_{k=0}^{K} U_{k,1}(x)t^k, \qquad\qquad t \in [0, t_1]$$

For $i \geq 2$, the initial conditions $u_i(x, t_{i-1}) = u_{i-1}(x, t_{i-1}), (\partial/\partial t)u_i(x, t_{i-1})$ are utilized at each subinterval $[t_{i-1}, t_i]$, and the Multistep RDTM is applied to the initial value problem in $[t_{i-1}, t_i]$, where t_0 is substituted by t_{i-1}. For $i = 1,2,\dots,M$, the procedure is continued and repeatedly performed to obtain estimated solutions $u_i(x,t)$ in sequence form that is

$$u_i(x,t) = \sum_{k=0}^{K} U_{k,i}(x)(t - t_{i-1})^k, \qquad t \in [t_{i-1}, t_i]$$

In fact, the MMRDTM expects the following solution:

$$u(x,t) = \begin{cases} u_1(x,t) & t \in [0, t_1] \\ u_2(x,t) & t \in [t_1, t_2] \\ \vdots & \vdots \\ u_M(x,t) & t \in [t_{M-1}, t_M] \end{cases}$$

We found that the calculation of MMRDTM is straightforward with better computational performance for all values of s. If the step size $h = T$, we can easily notice that the MMRDTM reduces to the MRDTM .

3. Numerical Results and Discussions

We give three test problems to evaluate if this method is efficient and accurate to solve nonlinear Klein-Gordon equations.

Example 1 Consider the second-order nonlinear Klein Gordon equation [19]

$$u_{tt} - u_{xx} + u^2 = 0 \tag{4}$$

with the following indicated initial condition

$$u(x,0) = 1 + \sin(x)$$
$$u_t(x,0) = 0$$

There is no exact solution for this equation. Using basic properties of MMRDTM then followed by utilizing MMRDTM for Eq. (4), we obtain the following equation

$$U_{k+2,i}(x) = \left(\frac{1}{(k+2)(k+1)}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{k,i}(x)\right) - \sum_{k=0}^{n} A_{k,i}\right) \tag{5}$$

From the initial condition, we compose

$$U_0(x) = 1 + \sin(x). \tag{6}$$

The following $U_k(x)$ values are obtained by substituting Eq. (6) into Eq. (5) by straightforward iterative calculation. Then, the inverse transformation of the set of values $\{U_6(x)\}_{k=0}^{6}$ gives the 6-terms estimated solution as follows,

$$\begin{aligned} u(x,t) = 1 + \sin(x) &+ \left(-\frac{3}{2}\sin(x) - \frac{3}{4} + \frac{1}{4}\cos(2x)\right)t^2 \\ &+ \left(\frac{25}{48}\sin(x) - \frac{1}{4}\cos(2x) + \frac{1}{4} - \frac{1}{48}\sin(3x\right)t^4 \\ &+ \left(-\frac{79}{480}\sin(x) + \frac{17}{144}\cos(2x) + \frac{41}{1440}\sin(3x) - \frac{263}{2880} - \frac{1}{576}\cos(4x)\right)t^6 \end{aligned}$$

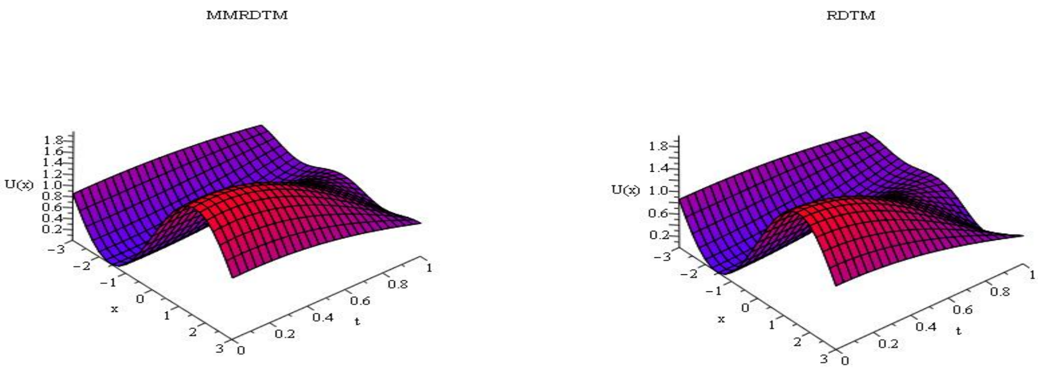
where $t \in [0.1, 0.3]$.

For $i = 1,2,3$, the interval $[0.1,0.3]$ is divided into 3 subintervals $[t_{i-1}, t_i]$ by equalizing step size by using the nodes $t_i = ih$. For $i \geq 2$, the initial conditions $u_i(x, t_{i-1}) = u_{i-1}(x, t_{i-1}), (\partial/\partial t)u_i(x, t_{i-1}) = (\partial/\partial t)u_{i-1}(x, t_{i-1})$ will be used at each subinterval $[t_{i-1}, t_i]$ Then, the Modified RDTM is utilized to the initial value problem in interval $[t_{i-1}, t_i]$, where t_0 is substituted by t_{i-1}. The process is continuously and repeatedly performed to obtain estimated solutions $u_i(x, t), i = 1,2,3$ for the solution $u(x, t)$, such as

$$u_i(x, t) = \sum_{k=0}^{K} U_{k,i}(x)(t - t_{i-1})^k, \qquad t \in [t_{i-1}, t_i]$$

Table 1 Results of MMRDTM and RDTM approximate solutions for Example 1

x	$t = 0.1$		
	MMRDTM	**RDTM**	**DTM** [17].
0.0	0.9950000000	0.9950000250	0.995000000
0.1	0.9950000250	1.0932911790	1.093336821
0.2	0.9950000011	1.1905030880	1.190602734
0.3	0.9950000026	1.2856688480	1.285829872
0.4	0.9950000000	1.3778447100	1.378073322
0.5	0.9949999983	1.4661192190	1.466420573
0.6	0.9949999998	1.5496219390	1.550000812
0.7	0.9950000005	1.6275316940	1.627994045
0.8	0.9950000026	1.6990842440	1.699640074
0.9	0.9950000002	1.7635793560	1.764245622
1.0	0.9949999983	1.8203872150	1.821201388
x	$t = 0.2$		
	MMRDTM	**RDTM**	**DTM**
0.0	0.9800000001	0.980001600	0.980000000
0.1	0.9799999992	1.073726340	1.073725261
0.2	0.9800000009	1.166138067	1.166138050
0.3	0.9800000001	1.256331039	1.256328927
0.4	0.9799999995	1.343432093	1.343427256
0.5	0.9800000009	1.426608233	1.425698958
0.6	0.9800000016	1.505073435	1.505058688
0.7	0.9799999973	1.578094717	1.578075355
0.8	0.9799999992	1.644997412	1.644678005
0.9	0.9800000004	1.705169747	1.705161053
1.0	0.9799999985	1.758066717	1.758088889
x	$t = 0.3$		
	MMRDTM	**RDTM**	**DTM**
0.0	0.9550000001	0.955018225	0.955000000
0.1	0.9550000006	1.041329837	1.041318399
0.2	0.9550000012	1.125975246	1.125970235
0.3	0.9550000008	1.208148067	1.208145667
0.4	0.9549999994	1.287088561	1.287081794
0.5	0.9550000000	1.362088667	1.362067708
0.6	0.9549999996	1.432495777	1.432448098
0.7	0.9550000020	1.497715374	1.497625423
0.8	0.9550000030	1.557212650	1.557060645
0.9	0.9550000010	1.610513250	1.610272513
1.0	0.9549999990	1.657203346	1.656835416



(a) (b)

Figure 1. Approximation result of MMRDTM and RDTM for Example 1

We compare the approximate results with DTM and RDTM. It is clear that the MMRDTM result converges for $x = 0$ until $x = 1$ compared with the DTM results obtained by Kanth and Aruna[16]. From the results, Fig. 1(a) and 1(b) show approximation of MMRDTM and RDTM for $t \in [0,1]$ and $x \in [-4,4]$ respectively. The performance error analyses obtained by MMRDTM are tabulated in Tab. 1.

Example 2 Consider the second-order nonlinear Klein Gordon [17]

$$u_{tt} - u_{xx} + u^2 = -x\cos(t) + x^2\cos^2(t) \tag{7}$$

which is subjected to the initial condition

$$u(x,0) = x$$
$$u_t(x,0) = 0$$

The exact solution of this equation is $x\cos(t)$.

Using basic properties of MMRDTM and then applying MMRDTM to Eq. (7), we obtain

$$U_{k+2,i}(x) = \left(\frac{1}{(k+2)(k+1)}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{k,i}(x)\right) - \sum_{k=0}^{n} A_{k,i} - \frac{x}{k!}\cos\left(\frac{\pi k}{2}\right) + \frac{x^2}{k!}\cos^2\left(\frac{\pi k}{2}\right)\right) \tag{8}$$

From the initial condition, we write

$$U_0(x) = x. \tag{9}$$

$U_k(x)$ values is obtained by substituting Eq. (9) into Eq. (8) by straightforward iterative calculation. Then, the inverse transformation of the set of values $\{U_6(x)\}_{k=0}^{6}$ gives the 6-terms approximation solution as follows,

$$u(x,t) = \mathrm{x} - \frac{1}{2}\mathrm{x}\,t^2 + \left(\frac{1}{8}x^2 + \frac{1}{24}x\right)t^4 + \left(\frac{1}{120} - \frac{1}{15}x\left(\frac{1}{8}x^2 + \frac{1}{24}x\right) - \frac{1}{144}x^2 - \frac{1}{720}\,x\right)t^6 \qquad , t \in [0,1]$$

Divide the interval [0,1] into 10 subintervals $[t_{i-1}, t_i], i = 1,2, \dots ,10,$ by using equal step size $h = 0.1$ and the nodes $t_i = ih$. The core ideas of the MMRDTM are as follows. Firstly, the RDTM is applied to the initial value problem over the interval $[0, t_1]$.

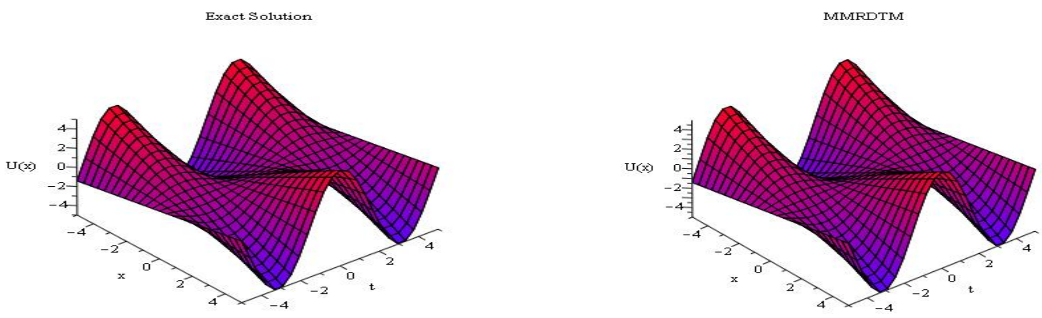
For $i \geq 2,$ we use the initial conditions $u_i(x, t_{i-1}) = u_{i-1}(x, t_{i-1}), (\partial/\partial t)u_i(x, t_{i-1}) = (\partial/\partial t)u_{i-1}(x, t_{i-1})$ at each subinterval $[t_{i-1}, t_i]$, and the MRDTM is applied to the initial value problem over the interval $[t_{i-1}, t_i]$, where t_0 is replaced by t_{i-1}. Next, the multistep scheme for repeating process are $u(x, 0) = f_0(x), u_1(x, 0) = 0$. The process is continued and repeated to generate a sequence of approximate solutions $u_i(x, t), i = 1,2, \dots ,10,$ for the solution $u(x, t)$ such as

$$u_i(x,t) = \sum_{k=0}^{K} U_{k,i}(x)(t - t_{i-1})^k, \qquad t \in [t_{i-1}, t_i]$$

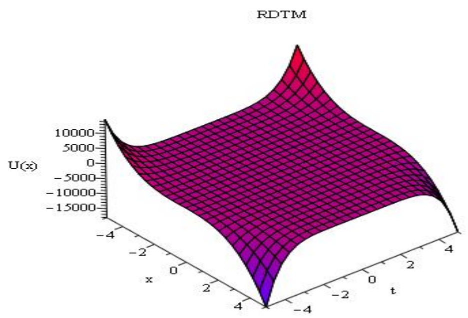
Table 2 Comparison error results of MMRDTM and RDTM approximate solutions for Example 2

T	Exact Solution	Absolute Error (MMRDTM)	Absolute Error (RDTM)
0.1	0.09950041653	$8.000000\ 10^{-11}$	$1.33230000\ 10^{-7}$
0.2	0.19601331550	$1.000000\ 10^{-10}$	$8.50420000\ 10^{-6}$
0.3	0.28660094670	$5.000000\ 10^{-10}$	$9.63977000\ 10^{-5}$
0.4	0.36842439760	$6.600000\ 10^{-9}$	$5.37570800\ 10^{-4}$
0.5	0.43879128100	$4.820000\ 10^{-8}$	$2.02903150\ 10^{-3}$
0.6	0.49520136890	$2.489000\ 10^{-7}$	$5.97327430\ 10^{-3}$
0.7	0.53538953110	$9.954000\ 10^{-7}$	$1.478879110\ 10^{-2}$
0.8	0.55736536740	$3.305200\ 10^{-6}$	$3.219962880\ 10^{-2}$
0.9	0.55944897150	$9.522700\ 10^{-6}$	$6.343567530\ 10^{-2}$
1.0	0.54030230590	$2.452810\ 10^{-5}$	$1.1152532497\ 10^{-1}$

Fig. 2(a) shows the exact solution, Fig. 2(b) shows the graph of approximate solution MMRDTM for $t \in$ [-5,5] and $x \in$ [-5,5] while Fig. 2(c) shows the graph of approximate solution RDTM for $t \in$ [-5,5] and $x \in$ [-5,5]. Obviously, the multistep approximate solutions for this type of nonlinear Klein Gordon equations are very close to the exact solutions. The performance error analyses obtained by MMRDTM are summarized in Tab. 2.



(a) (b)



(c)

Figure 2. Exact Solution,Approximation result of MMRDTM and RDTM for Example 2

Example 3 Consider the second-order nonlinear Klein Gordon equation [17]

$$u_{tt} - u_{xx} + \frac{\pi^2}{4}u + u^2 = x^2 \sin^2\left(\frac{\pi}{2}t\right) \tag{10}$$

subject to the initial condition

$$u(x,0) = 0$$
$$u_t(x,0) = \frac{\pi}{2}x$$

The exact solution of this equation is $x\, sin\left(\frac{\pi}{2}t\right)$.

Using fundamental properties of MMRDTM then utilizing MMRDTM to Eq. (10), we can get

$$U_{k+2,i}(x) = \left(\frac{1}{(k+2)(k+1)}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{k,i}(x)\right) - \left(\frac{\pi^2}{4}U_{k,i}(x)\right) - \sum_{k=0}^{n} A_{k,i} + \frac{x^2\left(\frac{\pi}{2}\right)^k}{k!}\sin^2\left(\frac{\pi k}{2}\right)\right) \tag{11}$$

From the initial condition, we write

$$U_0(x) = 0. \tag{12}$$

Separate the interval [0,1] into 10 subintervals $[t_{i-1}, t_i], i = 1,2, \dots ,10$, of equal step size $h = 0.1$ and use the nodes $t_i = ih$. The main ideas of the MMRDTM are as follows. Firstly, the RDTM is applied to the initial value problem over the interval $[0, t_1]$.

Table 3 Comparison error results of MMRDTM and RDTM approximate solutions for Example 3

T	Exact Solution	Absolute Error (MMRDTM)	Absolute Error (RDTM)
0.1	0.01564344651	$2.000000\ 10^{-11}$	$2.66113000\ 10^{-6}$
0.2	0.06180339888	$1.000000\ 10^{-11}$	$7.83019000\ 10^{-5}$
0.3	0.13619714990	$1.000000\ 10^{-9}$	$5.44701300\ 10^{-4}$
0.4	0.23511410100	$1.680000\ 10^{-8}$	$2.09835650\ 10^{-3}$
0.5	0.35355339060	$1.558000\ 10^{-7}$	$5.85571500\ 10^{-3}$
0.6	0.48541019660	$9.623000\ 10^{-7}$	$1.33725081\ 10^{-2}$
0.7	0.62370456700	$4.483000\ 10^{-6}$	$2.67454432\ 10^{-2}$
0.8	0.7608452132	$1.698320\ 10^{-5}$	$4.89435943\ 10^{-2}$
0.9	0.8889195065	$5.493920\ 10^{-5}$	$8.45755605\ 10^{-2}$
1.0	1.0000000000	$1.568987\ 10^{-4}$	$1.41378353\ 10^{-1}$

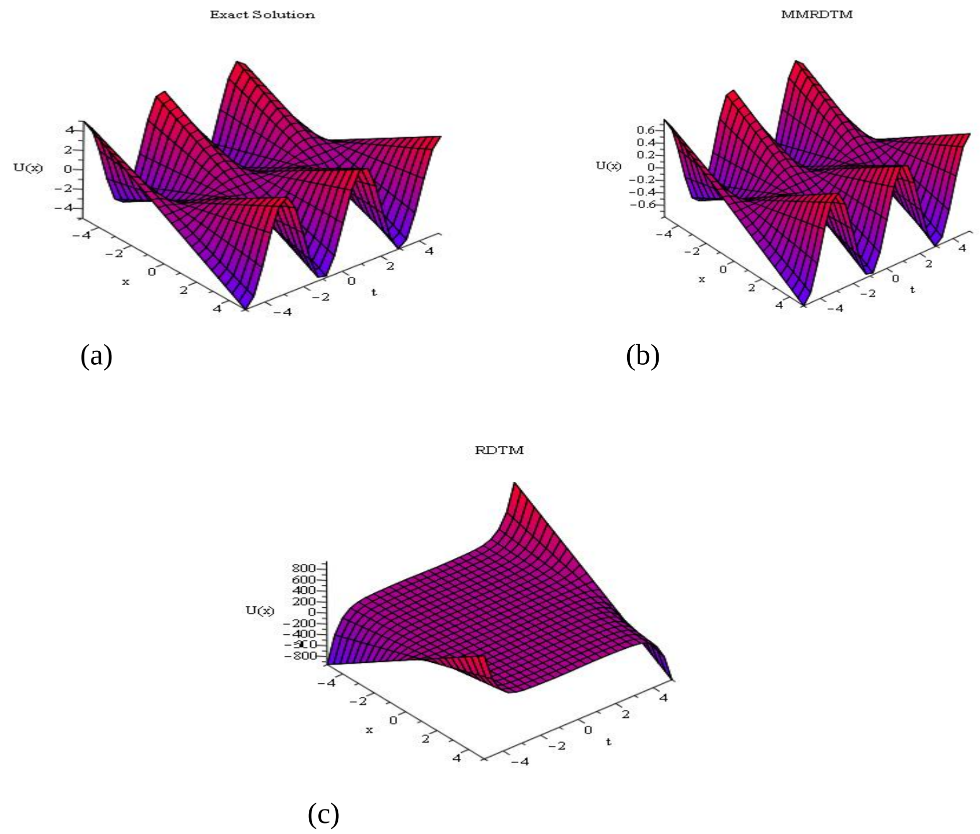


Figure 3. Exact Solution, Approximation result of MMRDTM and RDTM for Example 3

Fig. 3(a) shows the exact solution, Fig. 3(b) illustrates the graph of approximate solution MMRDTM for $t \in [-5,5]$ and $x \in [-5,5]$ while Fig. 3(c) illustrates the graph of approximate solution RDTM for $t \in [-5,5]$ and $x \in [-5,5]$. Therefore, as we can see the multistep approximate solutions for this type of nonlinear Klein Gordon equations are very close to the exact solutions. The performance error analyses obtained by MMRDTM are summarized in Tab. 3.

4. Conclusion

In this paper, we proposed and applied an approximate analytical method which is called the Multistep Modified Reduced Differential Transform Method (MMRDTM) to solve the one-dimensional nonlinear Klein-Gordon equations. In this new strategy, the modification involves the replacement of nonlinear term by its Adomian polynomials and a multistep approach. The results demonstrate that the approximate solutions of nonlinear Klein Gordon equation have high precision. In conclusion, we can state that the MMRDTM is a valid and efficient method for finding analytic approximate solution for these types of equations. The computations in this paper were obtained by utilizing Maple 13.

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