APPROXIMATE SOLUTIONS AND CONSERVATION LAWS OF THE PERIODIC BASE TEMPERATURE OF CONVECTIVE LONGITUDINAL FINS IN THERMAL CONDUCTIVITY

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In this paper, the residual power series method (RPSM) is used to study the numerical approximations of a model of oscillating base temperature processes occurring in a convective rectangular fin with variable thermal conductivity. It is shown that the RPSM is efficient for examining numerical behavior of nonlinear models. Further, the conservation of heat is studied using the multiplier method.

Key Words: *Residual power series method, numerical approximations, conservation laws.*

1. Introduction

It is well-known that the majority of the real-world physical phenomena are modeled by mathematical equations, especially partial differential equations (PDEs) [1]. The investigations of the exact and numerical solutions of various PDEs have become a very important practice by different scholars. The best test of a numerical method is whether it gives the exact solution at lower cost than its competitors. It is also worthwhile to remember that a single numerical method may not be the best for all problems [3]. So, for assessing the accuracy of a numerical method, comparison with the exact solution of the problem (which includes any errors due to model inaccuracy) is a better test than comparison with experiments. Errors are useful in statistics, computer programming, advanced mathematics and much more [3]. We observe many new progresses in this field [4-19]. The RPSM is constituted with a repeated series algorithm to derive the residual power series (RPS) solutions of PDEs. It has been successfully used to handle the approximate solutions of many nonlinear models [4,5]. The model that will be studied in this paper is given by [6,7]

$$u_t = -K^2u^2 + \varepsilon u_x^2 + (1+\varepsilon u)u_{xx}. \tag{1}$$

Where K depends on the physical properties and design parameters, and where $u(x,t)$ has the domain of definition $x \in [0,1]$, $t \in [0,1]$, and subject to a mixed set of homogeneous Neumann and inhomogeneous Dirichlet boundary conditions, which includes a sinusoidally varying boundary value,

$$u(1,t) = 1 + s\cos(Bt), \quad u_x(0,t) = 0. \tag{2}$$

The parameter u represent the dimensionless temperature, x is the distance, t is the time, ε represent the thermal conductivity, K is the fin parameter, S the amplitude of oscillation and B is the frequency of oscillation [8]. Further details in regard to the derivation and design limitations of the model can be found in [6].

This study is aimed at investigating the numerical approximations to the periodic base temperature of convective longitudinal fins in thermal conductivity using the RPSM [12].

2. Numerical approximation using the RPSM

The RPSM is effective and easy to derive power series solutions of nonlinear equations. The method does not require perturbation, discretization or linearization from which the numerical results can be investigated. The RPSM converge to the exact solution with only few iterations taken into consideration.

To apply the RPSM [4], we consider

$$u = \sum_{n=0}^{\infty} f_n t^n, 0 \le t \le R, x \in I, \tag{3}$$

Let u_k to represent the k^{th} series of u

$$u_k = \sum_{n=0}^{k} f_n t^n, 0 \le t \le R, x \in I, \tag{4}$$

$u_0 = f(x)$. To derive the value of $f_n(x)$ $n = 1,2,...,k$ in series expansion of Eq.(1), we use the residual function Res, for Eq.(1) as

$$Res = u_t - \frac{1}{2}uu_{xx} - 2u^2u_x - (u_x)^2 = 0, \tag{5}$$

and the k^{th} residual series Res_k is given by

$$Res_k = (u_k)_t - \frac{1}{2}u_k(u_k)_{xx} - 2(u^2)_k(u_k)_x - (u_k^2)_x = 0. \tag{6}$$

With initial condition

$$u_0 = 1 + s\cos(Bt), \quad u_x(1,t), \tag{7}$$

where $u(1,t)$ is to be obtained from a known exact solution Eq.(1).

• To find first approximation solution u_1, we set $k = 1$ in Eq.(6)

$$Res_1 = -K^2u_1^2 - (u_1)_t + \varepsilon(u_1^2)_x + (1 + \varepsilon u_1)(u_1)_{xx} = 0, \tag{8}$$

where

$$u_1(x,t) = 1 + s\cos(Bt) + tf_1. \tag{9}$$

From Eq.(8), we conclude that $\{\frac{\partial Res_1}{\partial t}\}_{t=0}$ and we get

$$f_1 = -K^2(1+s)^2t. \tag{10}$$

The 1^{st} approximate residual power series solution is given by

$$u_1(x,t) = 1 + s\cos(\mathrm{Bt}) - K^2(1+s)^2t. \tag{11}$$

• To find first approximation solution u_2, we set $k = 2$ in Eq.(6)

$$Res_2 = -K^2u_2^2 - (u_2)_t + \varepsilon(u_2^2)_x + (1 + \varepsilon u_2)(u_2)_{xx} = 0, \tag{12}$$

where

$$u_2(x,t) = 1 + s\cos(\mathrm{Bt}) - K^2(1+s)^2t + t^2f_2. \tag{13}$$

From Eq.(12), and using the fact that $\{\frac{\partial Res_2}{\partial t}\}_{t=0}$, we get

$$f_2 = \frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2}. \tag{14}$$

The 2^{nd} approximate residual power series solution solution is given by

$$u_2(x,t) = 1 + s\cos(Bt) - K^2(1+s)^2t + \left(\frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2}\right)t^2. \tag{15}$$

• To find first approximation solution u_3, we set $k = 3$ in Eq.(6)

$$Res_3 = -K^2u_3^2 - (u_3)_t + \varepsilon(u_3^2)_x + (1 + \varepsilon u_3)(u_3)_{xx} = 0, \tag{16}$$

where

$$u_3 = 1 + s\cos(Bt) - K^2\left((1+s)^2t + \left(\frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2}\right)t^2\right) + t^3f_3. \tag{17}$$

From Eq.(16), and using the fact that $\{\frac{\partial Res_3}{\partial t}\}_{t=0}$, we get

$$f_3 = \frac{-6K^6-24K^6s-36K^6s^2-24K^6s^3-6K^6s^4}{6}. \tag{18}$$

The 3^{rd} approximate residual power series solution solution is given by

$$u_3 = 1 + s\cos(Bt) - K^2(1+s)^2 t + \left(\frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2}\right)t^2 + \left(\frac{-6K^6-24K^6s-36K^6s^2-24K^6s^3-6K^6s^4}{6}\right)t^3. \quad (19)$$

• To find first approximation solution u_4, we set $k = 4$ in Eq.(6)

$$Res_4 = -K^2u_4^2 - (u_4)_t + \varepsilon(u_4^2)_x + (1+\varepsilon u_4)(u_4)_{xx} = 0, \quad (20)$$

where

$$u_4 = 1 + s\cos(Bt) - K^2(1+s)^2 t + \frac{2K^4(1+s)^2+s\left(B^2+2K^4(1+s)^2\right)t^2}{2} + \left(\frac{-6K^6-24K^6s-36K^6s^2-24K^6s^3-6K^6s^4}{6}\right)t^3 + f_4t^4. \quad (21)$$

From Eq.(20), and using the fact that $\{\frac{\partial Res_4}{\partial t}\}_{t=0}$, we get

$$f_4 = \left(\frac{24K^8-B^4s+120K^8s+240K^8s^2+240K^8s^3+120K^8s^4+24K^8s^5}{24}\right). \quad (22)$$

The 4^{th} approximate residual power series solution solution is given by

$$u_4(x,t) = 1 + s\cos(Bt) - K^2(1+s)^2 t + \frac{2K^4(1+s)^2+s\left(B^2+2K^4(1+s)^2\right)t^2}{2} + \left(\frac{2K^6(1+s)^4+2K^2(1+s)\left(-B^2s+2K^4(1+s)^2+s\left(B^2+2K^4(1+s)^2\right)\right)}{6}\right)t^3 + \left(\frac{24K^8-B^4s+120K^8s+240K^8s^2+240K^8s^3+120K^8s^4+24K^8s^5}{24}\right)t^4. \quad (23)$$

3. Numerical results and discussions

This section provide the solutions by numerical simulations. Table 1 showed the error observed in the numerical computations making comparison with the exact solution Eq.(2) and the 4^{th} approximate residual power series solution Eq.(23) at different times. It is clear that, the RPSM is accurate and provides efficient results and a rapidly convergent series. It is observed that the numerical solutions are in close agreement with the exact solutions. Figures 1-4 showed 3D and contour surfaces of the fourth iteration $u_4(x,t)$ for the exact solutions and RPS at small time. In the numerical computation, we set the constants $s = 0.1$, $K = 0.006$ and $B = 0.1$. And we considered the test points for t (0.01,0.02,0.03,…,0.09) to illustrate the convergence of the residual power series solutions.

Table 1: The absolute error.

t	$\lvert u_{exact} - u_{RPSM}\rvert$
0.01	3.856×10^{-7}
0.02	6.71199×10^{-7}
0.03	8.56799×10^{-7}
0.04	9.4298×10^{-7}
0.05	9.27998×10^{-7}
0.06	8.13599×10^{-7}
0.07	5.99202×10^{-7}
0.08	2.84806×10^{-7}
0.09	1.29587×10^{-7}

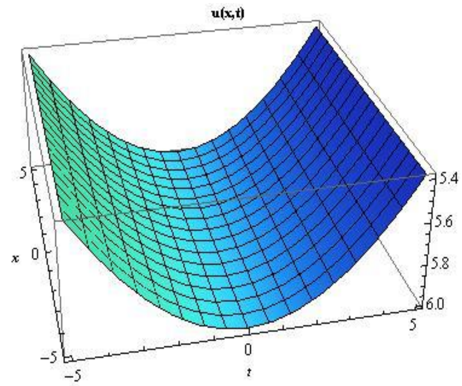

Figure 1: 3D surface of the exact solution Eq.(2)

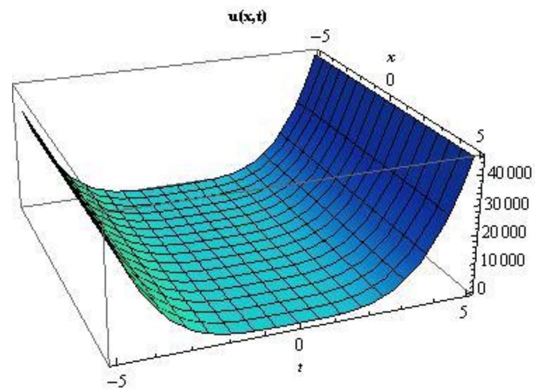

Figure 2: 3D surface of the RPS Eq.(23).

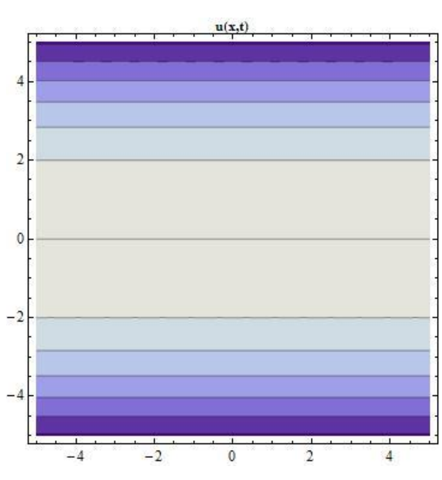

Figure 3: Contour surface of the exact solution Eq.(2).

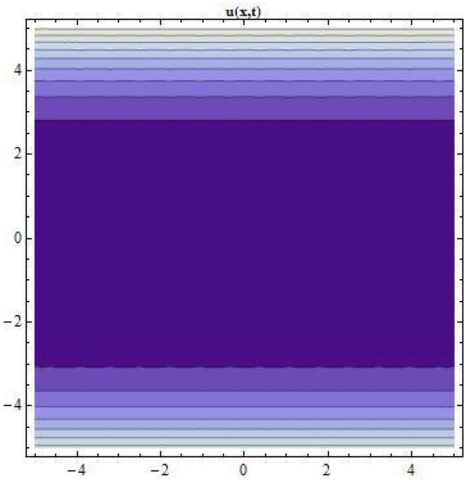

Figure 4: Contour surface of the RPS Eq.(23).

4 . Conservation Laws

In this section, the Cls of Eq.(1) will be studied using the multiplier technique [10,11]. Let $x = (x_1, x_2, \ldots, x_n)$ and $u = (\overline{u}^1, \overline{u}^2, \ldots, \overline{u}^m)$ be a set of n independent variables and m dependent variables. Consider the following r PDEs of k^{th} −order [10]

$$P_\alpha[\overline{u}] = P_\alpha(x, \overline{u}, \overline{u}_{(1)}, \overline{u}_{(2)}, \ldots, \overline{u}_{(k)}), \quad \alpha = 1,2,\ldots,r. \tag{24}$$

with $\overline{u}_{(1)} = \{\overline{u}^\alpha_{(i)}\}, \overline{u}_{(2)} = \{\overline{u}^\alpha_{(ij)}\}, \{\overline{u}^\alpha_{(i)}\} = \frac{\partial \overline{u}^\alpha_i}{\partial x_i}, \{\overline{u}^\alpha_{(ij)}\} = \frac{\partial^2 \overline{u}^\alpha}{\partial x_i \partial x_j}, \ldots$ Let $\overline{u} = (\overline{u}^2, \overline{u}^2, \ldots, \overline{u}^N)$ represents functions of the independent variables x and denoting partial derivatives $\frac{\partial}{\partial x_i}$ by subscripts i. [10], i.e, $\overline{u}^\sigma_i = \frac{\partial \overline{u}^\sigma}{\partial x_i}, \overline{u}^\sigma_{ij} = \frac{\partial^2 \overline{u}^\sigma}{\partial x_i x_j}$, e.t.c

1. $$D_i = \frac{\partial}{\partial x_i} + \overline{u}^\alpha_i \frac{\partial}{\partial \overline{u}^\alpha_i} + \overline{u}^\alpha_{ij} \frac{\partial}{\partial \overline{u}^\alpha_i} + \overline{u}^\alpha_{ijk} \frac{\partial}{\partial \overline{u}^\alpha_{jk}} + \ldots \tag{25}$$

where $i, j, k, \ldots = 1,2,\ldots,m$

2. Multipliers of Eq.(24) are the functions $\{\Lambda^\alpha[\overline{u}]\}$ which satisfy

$$\Lambda^\alpha[\overline{u}] P_\alpha[\overline{u}] = D_i T^i[\overline{u}], \tag{26}$$

for some certain functions $T^i[\overline{u}]$. If $\overline{u}^\sigma = \overline{u}^\sigma(x)$ is solution of Eq.(24), from Eq.(26), we acquire the Cls [10]

$$D_i T^i[\overline{u}] = 0 \tag{27}$$

of Eq.(42) and for each i, $T^i[\overline{u}]$ is a flux.

3. The Euler operators w.r.t the differential function U^j and the derivatives $\overline{u}^i_j, \overline{u}^j_{i_1 i_2}\ldots$ are defined by

$$E^j_{\overline{u}} = \frac{\partial}{\partial \overline{u}^j} - D_i \frac{\partial}{\partial \overline{u}^j_i} + \ldots + (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial \overline{u}^j_{i_1 \ldots i_s}} \tag{28}$$

for each $j = 1,2,\ldots,m$, $\{\Lambda^\alpha[\overline{u}]\}$ gives a the multipliers of Cls of Eq.(24) iff each operator in Eq.(29) annihilates the left-hand side of Eq.(26)

$$E^j_{\overline{u}}(\Lambda^\alpha[\overline{u}] P_\alpha[\overline{u}]) \equiv 0, \quad j = 1,\ldots,n \tag{29}$$

for arbitrary $\overline{u}, \overline{u}_i, \overline{u}_{ij}$. ..e.t.c.

To construct the Cls of Eq.(1) using the above described technique, we apply Eq.(26) to get the following determining equations

$$\Lambda_{xx} = \frac{2\Lambda K^2}{\varepsilon}, \Lambda_t = \frac{-2\Lambda K^2}{\varepsilon}, \Lambda = 0. \tag{30}$$

Solving Eq.(30), we acquire the following multiplier $\Lambda(x,t,u)$ given by

$$\Lambda = \{c_1 e^{\sqrt{\frac{2}{\varepsilon}}Kx} + c_2 e^{-\sqrt{\frac{2}{\varepsilon}}Kx}\} \times e^{\frac{-2K^2 t}{\varepsilon}}. \quad (31)$$

c_1 and c_2 are arbitrary constants. We derive the following multipliers for four fluxes based on the constants c_1 and c_2 follows:

1. If $c_1 = 1, c_2 = 0$, then we have the following multipliers:

$\Lambda = e^{-\frac{2K\sqrt{\varepsilon}t - \sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}$. Subsequently, we obtain the following fluxes

$$T_1^x = -(u\varepsilon + 1)u_x \times e^{-\frac{2K\sqrt{\varepsilon}t - \sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}},$$
$$T_1^t = -\frac{1}{2K}\{\sqrt{2}\varepsilon^{\frac{3}{2}}uu_x + Ku^2\varepsilon + \sqrt{2}\varepsilon u_x\} \times e^{-\frac{2K\sqrt{\varepsilon}t - \sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}. \quad (32)$$

2. If $c_2 = 1, c_1 = 0$, then we have the following multipliers:

$\Lambda = e^{-\frac{2K\sqrt{\varepsilon}t + \sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}$. Subsequently, we obtain the following fluxes

$$T_2^x = -(u\varepsilon + 1)u_x \times e^{-\frac{2K\sqrt{\varepsilon}t + \sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}},$$
$$T_2^t = -\frac{1}{2K}\{-\sqrt{2}\varepsilon^{\frac{3}{2}}uu_x + Ku^2\varepsilon - \sqrt{2}\varepsilon u_x\} \times e^{-\frac{2K\sqrt{\varepsilon}t + \sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}. \quad (33)$$

5. Concluding remarks

In this paper, we have successfully applied the RPSM to study the numerical approximations to a model of oscillating base temperature processes occurring in a convective rectangular fin with variable thermal conductivity. We showed that the RPSM is efficient for examining numerical behavior of nonlinear models. Some interesting figures are shown to show the reliability of the method. We have confirmed the conservation of heat and temperature using the multiplier method of conservation laws.

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