ON EXACT SOLUTIONS FOR NEW COUPLED NONLINEAR MODELS GETTING EVOLUTION OF CURVES IN GALILEAN SPACE

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In this work, the new coupled nonlinear partial differential equations (CNLPDEs) getting the time evolution of the curvatures of the evolving curve are derived in the Galilean space. Exact solutions for these new CNLPDEs are obtained. Finally, Lie symmetry analysis is performed on these new CNLPDEs and the algebra of Lie point symmetries of these new equations is found.

Keywords: *Lie symmetry analysis, CNLPDEs, Soliton solutions, Bell-Shaped and Kink solitary wave, Curve evolution, Galilean space.*

1. Introduction

The issue of how to evolve spatial curves in time is of deep interest and has been examined in different planes and spaces by many researchers. The precursor in these types work is Hasimoto's paper which was first introduced the NLSE determining the motion of an isolated non-stretching thin vortex filament, [1]. Lamb extended Hasimoto's work to get connect other motions of curves to the mKdV and sine-Gordon equations [2]. Recently, Abdel-All et al. [3, 4] examined the evolution of curves using the velocities of the moving frame and investigated the evolution of plane curves. Also, there are a lot of studies about the motion of curves in [5-7].

In this study, we get (CNLPDEs) getting the time evolution of the curvatures of the curve in Galilean space. We derive the new geometric types of the evolution equations for curvatures from the main (CNLPDEs) in G_3. We get the exact solutions for these new equations and derive two types of nonlinear traveling solitary wave, which are well known, bell-shaped and kink solitary waves from these solutions [8-11]. The bell-shaped solitary wave appears in consequence of the balance between nonlinearity and dispersion. The balance between nonlinearity and dissipation supports known the nonlinear wave of stable shape as kink shaped wave.

Moreover, we study this new equation with the aid of Lie symmetry analysis method. This method is one of the most important and efficient techniques to find the exact solutions [12-19].

2. Time-evolution equations in G_3

A space curve in three-dimensional Galilean space G_3 is defined in parametric form by $r = r(s)$. Here r be an admissible curve of the class C^∞in G_3. Then Frenet formulas are given by

$$\begin{cases} T_s = \kappa N \\ N_s = \tau B \\ B_s = -\tau N \end{cases}, \tag{1}$$

where $T = r_s$, N and B are called the vectors of the unit tangent, principal normal and binormal of $r(s)$, respectively. Also κ and τ are geometric parameters that represent, respectively, the curvature and torsion of $r(s)$, [20]. Through this paper, the subscripts describe partial derivatives.

We know that an admissible curve is uniquely detected by two scalar quantities, is said to be the curvature and torsion, as functions. Then we can write the following general theorem:

*Theorem 1***.** (Fundamental existence and uniqueness theorem for space curves). Let $\kappa(s)$ and $\tau(s)$ be smooth functions on $c_1 \leq s \leq c_2$. Then there exists one and only one smooth curve r which is parametrized by arclength has the curvature $\kappa(s)$ and the torsion $\tau(s)$, [21].

If $r(s)$ moves with time t, then (1) are becoming functions of both s and t. We can write the evolution equations of $\{T, N, B\}$ quite generally, in a form similar to (1) as following form [22]

$$\begin{cases} T_t = \alpha N + \beta B \\ N_t = \gamma B \\ B_t = -\gamma N \end{cases} . \tag{2}$$

Clearly α, β and γ (which are the velocities of the moving frame) detect the motion of the curve. For non-stretching(inextensible) curves, the moving frame must be satisfied the compatibility conditions

$$T_{st} = T_{ts}, N_{st} = N_{ts} \text{ and } B_{st} = B_{ts}. \tag{3}$$

Here non-stretching(inextensible) curves imply that the flow described by (2) preserves the curves in arc-length parametrization [23].

If we substitute (1) and (2) into (3), then we get

$$\kappa_t N + \kappa\gamma B = \alpha_s N + \alpha\tau B + \beta_s B - \tau\beta N,$$
$$\tau_t B - \tau\gamma N = \gamma_s B - \tau\gamma N,$$
$$-\tau_t N - \tau\gamma B = -\gamma_s N - \tau\gamma B.$$

From the above equations, we obtain

$$\kappa_t = \alpha_s - \tau\beta, \tag{4}$$
$$\tau_t = \gamma_s,$$
$$\gamma = \frac{\tau\alpha + \beta_s}{\kappa}.$$

The evolution equations for κ and τ of r can be obtained with respect to $\{\alpha, \beta, \gamma\}$, which can be given as (CNLPDEs) such that,

$$\kappa_t = \alpha_s - \tau\beta, \tag{5}$$
$$\tau_t = \left(\frac{\tau\alpha + \beta_s}{\kappa}\right)_s.$$

The equation (5) is the main result of this paper. We determine the equations of motion of the curve for a given $\{\alpha, \beta, \gamma\}$. Then, we choose $\{\alpha, \beta, \gamma\}$ in terms of the $\{\kappa, \tau\}$. Moreover, from the above equations, we take into consideration that γ does not impact the final form of the evolving curve.

3. Applications for CNLPDEs

In this section, our starting point is to give some applications of the new geometric models of the evolution
equations for curvatures from (1) in G_3. The set of $\{\kappa, \tau, \alpha, \beta, \gamma\}$ becoming clear in (1) and (2) fundamentally determines a moving curve.

Type 1. The evolution equations for the curvatures of r in terms of the velocities $\{\alpha, \beta, \gamma\} = \{0, \kappa, \frac{\kappa_s}{\kappa}\}$ obtained using (5) as

$$\kappa_t = -\tau\kappa, \tag{6}$$
$$\tau_t = \frac{\kappa\kappa_{ss} - \kappa_s^2}{\kappa^2}.$$

So, we can have the general solutions of this system as follows

$$\kappa = A_1 \text{sech}[\lambda(s + vt)], \tag{7}$$
$$\tau = A_2 \tanh[\lambda(s + vt)],$$

where A_1, A_2 and λ are arbitrary real constants and v is the velocity of the solitary wave. Also, we see that v is equal to A_2/λ. In this equation, the bell shaped and kink solitary wave obtained, respectively.

Under these conditions, the general solutions are

$$\{\kappa,\tau,\alpha,\beta,\gamma\}=\{A_1\text{sech}(\lambda s+A_2t),A_2\tanh(\lambda s+A_2t),0,\\ A_1\text{sech}(\lambda s+A_2t),-A_1\lambda\tanh(\lambda s+A_2t)\}. \quad (8)$$

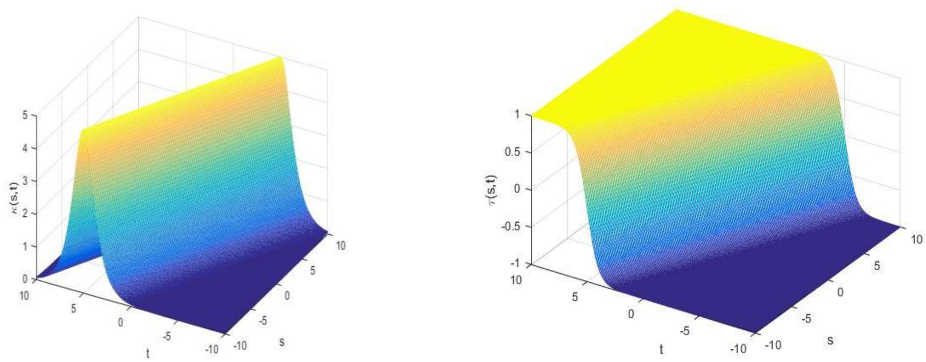

Figure 1: The bell-shaped and kink solitary wave solutions for Eq. (7)

If we put $A_1=5$, $A_2=1$ and $\lambda=0.5$ in Eq. (7), $\kappa(s,t)=5\text{sec}h(0.5s+t)$ and $\tau(s,t)=\tanh(0.2s+t)$. Under these conditions, we see that $\kappa\to 0$ and $\tau\to 0$ as $s\to\pm\infty$, in Figure1.

Type 2. The evolution equations for the curvatures of the curve in terms of $\{\alpha,\beta,\gamma\}=\{\kappa_{ss},0,\frac{\kappa_{ss}\tau}{\kappa}\}$ obtained using (5) as

$$\kappa_t=\kappa_{sss},\tau_t=\frac{(\tau\kappa_{sss}+\tau_s\kappa_{ss})\kappa-\tau\kappa_s\kappa_{ss}}{\kappa^2}. \quad (9)$$

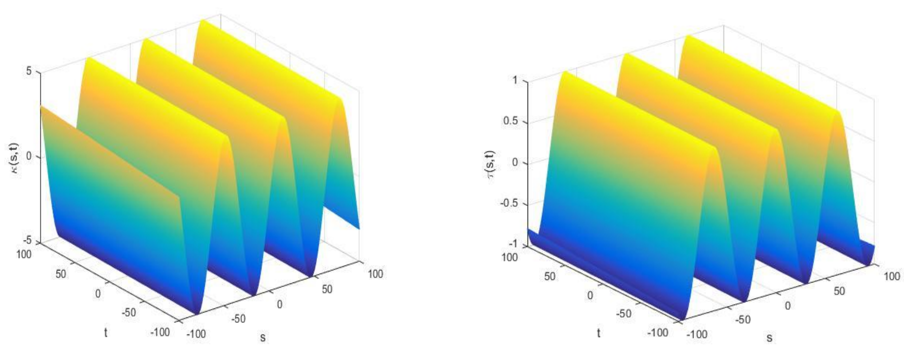

Figure 2: The trigonometric solutions for Eq. (10)

We can get the general solutions of this system as follows

$$\kappa=B_1\sin[\delta(s+\varpi t)],\\ \tau=B_2\cos[\delta(s+\varpi t)], \quad (10)$$

where B_1,B_2 and δ are the arbitrary real constants and ϖ is equal to $-\delta^2$. Under these conditions, the general solutions are

$$\{\kappa,\tau,\alpha,\beta,\gamma\}=\{B_1\sin[\delta(s-\delta^2t)],B_2\cos[\delta(s-\delta^2t)],\\ -B_1\delta^2\sin[\delta(s-\delta^2t)],0,-B_2\delta^2\cos[\delta(s-\delta^2t)]\}. \quad (11)$$

$\kappa(s,t)=5\sin[0.1(s-0.01t)]$ and $\tau(s,t)=10\cos[0.1(s-0.01t)]$ trigonometric function solutions are obtained for $B_1=5,B_2=10$ and $\delta=0.1$. In that cases, we see that $\kappa\to 0$ and $\tau\to 0$ as $s\to\pm\infty$, in

Figure 2.

Type 3. The evolution equations for the curvatures of the curve in terms of $\{\alpha, \beta, \gamma\} = \{\tau, \kappa, \frac{\kappa_s+\tau^2}{\kappa}\}$ obtained using (5) as

$$\kappa_t = \tau_s - \kappa\tau, \quad (12)$$
$$\tau_t = \frac{(\kappa_{ss} + 2\tau\tau_s)\kappa - (\kappa_s + \tau^2)\kappa_s}{\kappa^2}.$$

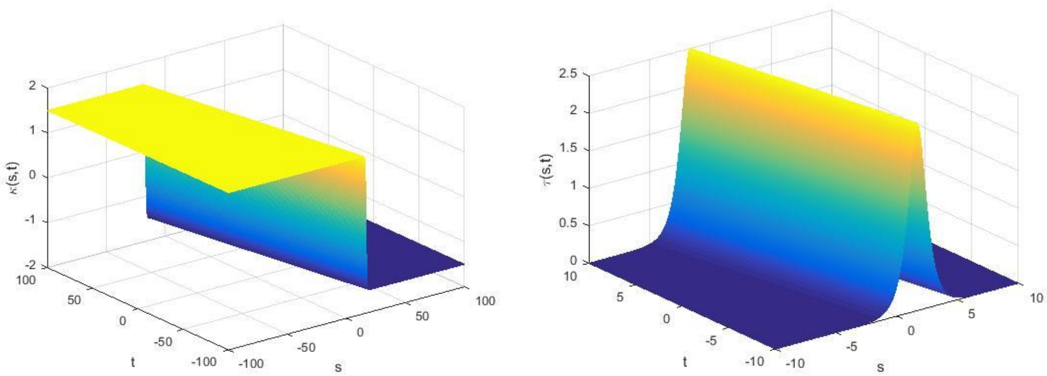

Figure 3: The solutions to Eq. (13)

We can write the general solutions of this system as follows

$$\kappa = C_1\text{sech}[\mu(s+vt)] + C_2\tanh[\mu(s+vt)], \quad (13)$$
$$\tau = C_3\text{sech}[\mu(s+vt)],$$

where C_1, C_2, C_3 and μ are arbitrary real constants. In this, μ and v are $C_1^2 - C_2$ and $\frac{(C_2+\mu)C_3}{C_1\mu}$, respectively. So, the general solutions are

$$\{\kappa, \tau, \alpha, \beta, \gamma\} = \{C_1\text{sech}[\mu(s+vt)] + C_2\tanh[\mu(s+vt)], C_3\text{sech}[\mu(s+vt)],$$
$$C_3\text{sech}[\mu(s+vt)], C_1\text{sech}[\mu(s+vt)] + C_2\tanh[\mu(s+vt)],$$
$$\frac{\text{sech}[\mu(s+vt)](C_3^2 + C_2\mu - C_1\mu\sinh[\mu(s+vt)])}{C_1 + C_2\sinh[\mu(s+vt)]}\}. \quad (14)$$

If we choose $C_1 = 0.1, C_2 = -1.5$ and $C_3 = 2.5$ in Eq. (13), $\kappa(s,t) = 0.1\text{sech}(1.51(s + 0.165t)) - 1.5\tanh(1.51(s + 0.165t))$ and $\tau(s,t) = 2.5\text{sech}(1.51(s + 0.165t))$. We see that $\kappa \to \pm 0.1$ and $\tau \to 0$ as $s \to \pm\infty$ in Figure 3.

3.1. Lie symmetry Analysis of CNLPDEs

In this section, our purpose is to give Lie symmetry analysis for CNLPDEs. Many researchers explain how to derive Lie symmetry analysis in many books, [24, 25].

We will now consider the one parameter group of point transformations of the form

$$\tilde{s} \to s + \varepsilon\xi(s,t,\kappa,\tau),$$
$$\tilde{t} \to t + \varepsilon\eta(s,t,\kappa,\tau),$$
$$\tilde{\kappa} \to \kappa + \varepsilon\zeta_1(s,t,\kappa,\tau),$$
$$\tilde{\tau} \to \tau + \varepsilon\zeta_2(s,t,\kappa,\tau),$$

where ε is a group parameter. The vector field related to the above transformations can be given by

$$V = \xi(s,t,\kappa,\tau)\frac{\partial}{\partial s} + \eta(s,t,\kappa,\tau)\frac{\partial}{\partial t} + \zeta_1(s,t,\kappa,\tau)\frac{\partial}{\partial \kappa} + \zeta_2(s,t,\kappa,\tau)\frac{\partial}{\partial \tau}. \quad (15)$$

The Lie point symmetries of the equations are generated by a vector field as the above form. Through this section C_1, C_2, C_3 and C_4 are arbitrary constants.

Type 1. We can show it considering a well established procedure that (6) admits the following infinitesimals

$$\begin{aligned} \xi(s,t,\kappa,\tau) &= 2C_2 s + C_4, \\ \eta(s,t,\kappa,\tau) &= 2C_2 t + C_3, \\ \zeta_1(s,t,\kappa,\tau) &= \kappa(C_1 \ln\kappa + C_2 \ln\kappa + F(s,t)), \\ \zeta_2(s,t,\kappa,\tau) &= C_1\tau - C_2\tau - F_t(s,t), \end{aligned}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants and $F(s,t)$ is an arbitrary function such that $F_{ss} = -F_{tt}$. The algebra of Lie point symmetries of (6) is generated by five vector fields

$$\begin{aligned} V_1 &= \kappa \ln\kappa \frac{\partial}{\partial \kappa} + \tau \frac{\partial}{\partial \tau}, \\ V_2 &= \kappa \ln\kappa \frac{\partial}{\partial \kappa} + 2t \frac{\partial}{\partial t} - \tau \frac{\partial}{\partial \tau} + 2s \frac{\partial}{\partial \mathrm{s}}, \\ V_3 &= \frac{\partial}{\partial t}, \\ V_4 &= \frac{\partial}{\partial \mathrm{s}}, \\ V_5 &= \kappa F(s,t) \frac{\partial}{\partial \kappa} - F_t(s,t) \frac{\partial}{\partial \tau}. \end{aligned}$$

Type 2. We can present it considering a well established procedure that (9) admits the following infinitesimals

$$\begin{aligned} \xi(s,t,\kappa,\tau) &= C_2 s + C_4, \\ \eta(s,t,\kappa,\tau) &= 3C_2 t + C_3, \\ \zeta_1(s,t,\kappa,\tau) &= \kappa C_1, \\ \zeta_2(s,t,\kappa,\tau) &= G(\kappa,\tau), \end{aligned}$$

where $C_i (i = 1,2,3,4)$ are arbitrary constants and $G(\kappa,\tau)$ satisfies the following equation

$$G = \kappa G_\kappa + \tau G_\tau.$$

The algebra of Lie point symmetries of (9) is constituted by five vector fields

$$\begin{aligned} V_1 &= \kappa \frac{\partial}{\partial \kappa}, \\ V_2 &= 3t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s}, \\ V_3 &= \frac{\partial}{\partial t}, \\ V_4 &= \frac{\partial}{\partial \mathrm{s}}, \\ V_5 &= G(\kappa,\tau) \frac{\partial}{\partial \tau}. \end{aligned}$$

Type 3. We can show it considering a well established procedure that (12) admits the following infinitesimals

$$\begin{aligned} \xi(s,t,\kappa,\tau) &= -C_1 s + C_3, \\ \eta(s,t,\kappa,\tau) &= -C_1 t + C_2, \\ \zeta_1(s,t,\kappa,\tau) &= \kappa C_1, \\ \zeta_2(s,t,\kappa,\tau) &= C_1\tau, \end{aligned}$$

where $C_i (i = 1,2,3)$ are arbitrary constants. The algebra of Lie point symmetries of (12) is generated by three vector fields

$$V_1 = \kappa \frac{\partial}{\partial \kappa} - t\frac{\partial}{\partial t} - s\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial \tau},$$
$$V_2 = \frac{\partial}{\partial t},$$
$$V_3 = \frac{\partial}{\partial s}.$$

4. Conclusion

In this work, we have derived the geometrical types for the motion of spatial curves in Galilean space. For each type, we get the evolution equations in terms of the curvatures. Solutions of these equations lead to the exact solutions as two types of nonlinear traveling solitary wave, which are well known, bell-shaped and kink solitary waves traveling wave, the trigonometric function solutions, and other similarity solutions. Moreover, we illustrated all the solutions by using Matlab in Figures 1-3. Finally, we applied Lie symmetry analysis to every geometric model.

References

[1] Hasimoto, H., A soliton on a vortex filament, *J. Fluid Mech., 51* (1972), 3, pp. 477-485

[2] Lamb, G. L., Solitons on moving space curves, *J. Math. Phys., 18* (1977), pp. 1654-1661

[3] Abdel-All, N. H., *et al*., Evolution of curves via the velocities of the moving frame, *J. Math. Comput. Sci., 2* (2012), 5, pp. 1170-1185

[4] Abdel-All, N. H., *et al*., Geometry of evolving plane curves problem via lie group analysis, *Stud.Math. Sci., 2* (2011), 1, pp. 51-62

[5] Ogrenmis, A.O. *et al.,* Inextensible curves in the Galilean space, *Int. J. Phys. Sci., 5* (2010), pp. 1424-1427

[6] Yuzbasi, Z.K., *et al.*, Inelastic flows of curves on lightlike surfaces, *Mathematics*, *6* (2018), 11, pp. 224

[7] Yuzbasi, Z.K., *et al.,* A Note on Inextensible Flows of Partially & Pseudo Null Curves in E_1^4, *Prespacetime J., 7* (2016), 5, pp. 818-827

[8] Khater, A. H., Callebaut, D.K., Seadawy, A. R., Nonlinear dispersive instabilities in Kelvin-Helmhooltz magnetohynamic flows, *Phys. Scrip., 67* (2003), pp. 340-349

[9] Khater, A. H., Helal, M. A., Seadawy, A. R., General soliton solutions of an n-dimensional nonlinear Schrödinger equation, *IL Nuovo Cimento(B), 115* (2000), 11, pp. 1303-1312

[10] Biswas, A., Mirzazadeh, M., *et al.,* Solitons in optical materials by functional variable method and first integral approach, *Frequenz, 68* (2014), 11-12, pp. 525-530

[11] Mirzazadeh, M., Eslami, M., *et al.,* Optical solitons in nonlinear directional couplers by sine-cosine function method and Bernoulli's equation approach, *Nonl. Dyn., 81* (2015), 4, pp. 1933-1949

[12] Yıldırım, Y., *et al.*, Lie Symmetry analysis and exact solutions to N-coupled nonlinear Schrödinger's Equations with kerr and parabolic law nonlinearities, *Rom. J. of Phys., 63* (2018), pp. 103-115

[13] Bansal, A., *et al.,* Optical soliton perturbation with Radhaskrishnan-Kundu-Lakshmanan equation by Lie group analysis, *Optik, 163* (2017), pp. 137-141

[14] Biswas, A., *et al.,* Conservation laws for Gerdjikov-Ivanov equation in nonlinear fiber optics and PCF, *Optik*, *148* (2017), pp. 209-214

[15] Bansal, A., *et al.,* Optical solitons and group invariant solutions to Lakshmanan–Porsezian–Daniel model in optical fibers and PCF, *Optik, 160* (2018), pp. 86-91

[16] Morris, R., *et al.,* On symmetries, reductions, conservation laws and conserved quantities of optical solitons with inter-modal dispersion, *Optik, 124* (2013), 21, pp. 5116-5123

[17] Fakhar, K., *et al.,* Symmetry reductions and conservation laws of the short pulse equation, *Optik, 127* (2016), 21, pp. 10201-10207

[18] Hashemi, M. S., *et al.,* Lie symmetry analysis of steady-state fractional reaction-convection-difusion equation, *Optik, 138* (2017), pp. 240-249

[19] Yuzbasi, Z.K., *et al.,* Lie symmetry analysis and exact solutions of Tzitzeica surpaces PDE in Galilean space, *J. Adv. Phys., 7* (2018), 1, pp. 88-91

[20] Pavkovic, B. J., *et al.,* The Equiform Differential Geometry of Curves in The Galilean Space G_3, *Glasnik Math., 22* (2008), 42, pp. 449-457

[21] Lipschutz, M., *Theory and problems of differential geometry*, Schaum's outline series, 1969

[22] Sahin, T., Intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean space, *Acta Math. Sci., 33* (2013), 3, pp. 701-711

[23] Baldwin, D., *et al.,* Symbolic computation of exact solutions expressible in hyperbolic and eliptic functions for nonlinear PDEs, *J. Symb. Comput., 37* (2004), 6, pp. 669-705

[24] Olver, P. J., *Applications of Lie groups to differential equations*, Second Edition,G.T.M, Springer-Verlag. New York, 107, 1993

[25] Bluman, G., *et al., Symmetry and integration methods for differential equations*, Springer Science & Business Media, 154, 2008