

AN ANALYTICAL SOLUTION FOR SOLVING A NEW WAVE EQUATION WITHIN LORENZO-HARTLEY KERNEL

by

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In this article we investigate the general fractional-order derivatives of the Riemann-Liouville type via Lorenzo-Hartley kernel, general fractional-order integrals and the new general fractional-order wave equation defined on the definite domain with the analytical solution.

Keywords: *general fractional-order derivative, general fractional-order integral, general fractional-order wave equation, analytical solution*

Introduction

Theory of general fractional-order calculus, as new mathematical tool, has successfully used to investigate the mathematical model in mathematical physics. The general fractional derivatives via the special kernels have the properties, such as memory and integro-differential representations with the non-singular behaviors, were developed in [1]. For example, these kernels, proposed by Caputo and Fabrizio[2], Mittag-Leffler[3], Rabotnov[4], Miller and Ross [5], and so on, have been investigated as follows:

$$\Omega(\lambda t^\alpha) = \begin{cases} e^{-\lambda t}, & [2] \\ E_\alpha(\lambda t^\alpha) = \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa t^{\kappa\alpha}}{\Gamma(\kappa\alpha + 1)}, & [3] \\ G_{\alpha, \nu}(\lambda t^\alpha) = \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa t^{\kappa\alpha - 1 + \nu}}{\Gamma(\kappa\alpha + \nu)}, & [4] \\ \mathfrak{M}_\alpha(\lambda t) = t^\alpha \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa t^\kappa}{\Gamma(\kappa + 1 + \alpha)}, & [5] \end{cases}$$

The tasks have been paid attention to many researchers in the mathematical physics in [6,7].

Recently, the Lorenzo and Hartley have proposed a special function (called the Lorenzo-Hartley function, see[8]). The general fractional-order calculus operators via the Lorenzo-Hartley function were investigated in [1]. The main goal of the article is to model the wave

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equation within general fractional derivative via the Lorenzo-Hartley kernel. The structure of the article is designed as follows. In Section 2, we give the basic theory of the general fractional calculus via Lorenzo-Hartley kernel. In Section 3, we propose the wave equation within general fractional derivative of the Riemann-Liouville type via Lorenzo-Hartley kernel and investigate the analytical solution with the special function. Finally, the conclusion is given in Section 4.

Basic theory of the general fractional calculus via Lorenzo-Hartley kernel

In this section, we give the concepts and properties of the general fractional-order derivatives and general fractional integrals. In order to show the properties of them, we start with the Lorenzo-Hartley function.

The Lorenzo-Hartley function can be expressed as follows [1]:

$$\Theta_{\gamma}(vt^{\gamma}) = \sum_{\kappa=0}^{\infty} \frac{v^{\kappa} t^{(\kappa+1)\gamma-1}}{\Gamma((\kappa+1)\gamma)}, \quad (1)$$

and its Laplace transform as follows[1,8]:

$$\mathfrak{L}\{\Theta_{\gamma}(vt^{\gamma})\} = \sum_{\kappa=0}^{\infty} \frac{v^{\kappa}}{s^{(\kappa+1)\gamma}} = s^{-\gamma} (1 - vs^{-\gamma})^{-1} \quad (|vs^{-\gamma}| < 1), \quad (2)$$

where the Laplace transform operator is denoted as [1]

$$\mathfrak{L}[l(t)] = l(s) = \int_0^{\infty} e^{-st} l(t) dt. \quad (3)$$

General fractional-order derivatives of the Riemann-Liouville type within Lorenzo-Hartley kernel

The left-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel is given as [1]

$${}_{RLY}D_{a+}^{\gamma, v} h(t) = \frac{d}{dt} \int_a^t \Theta_{\gamma}(-v(t-\tau)^{\gamma}) h(\tau) d\tau, \quad (4)$$

and the right-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel as

$${}_{RLY}D_{b-}^{\gamma, v} h(t) = -\frac{d}{dt} \int_t^b \Theta_{\gamma}(-v(\tau-t)^{\gamma}) h(\tau) d\tau. \quad (5)$$

The left-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel is given as [1]

$${}_{RLY}D_{a+}^{\gamma, n, v} h(t) = \frac{d^n}{dt^n} \int_a^t \Theta_{\gamma}(-v(\tau-t)^{\gamma}) h(\tau) d\tau \quad (6)$$

and the right-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel as

$${}_{RLY}D_{b-}^{\gamma, n, v} h(t) = (-1)^n \frac{d^n}{dt^n} \int_t^{\infty} \Theta_{\gamma}(-v(t-\tau)^{\gamma}) h(\tau) d\tau \quad (7)$$

where n is positive integer numbers.

When $a = 0$, the general fractional-order derivatives of the Riemann-Liouville type within the Lorenzo-Hartley kernel is given as [1]

$${}_{RLY}D_{0+}^{\gamma,\nu}h(t) = \frac{d}{dt} \int_0^t \Theta_\gamma \left(-\nu(t-\tau)^\gamma \right) h(\tau) d\tau \quad (8)$$

and

$${}_{RLY}D_{0+}^{\gamma,n,\nu}h(t) = \frac{d^n}{dt^n} \int_0^t \Theta_\gamma \left(-\nu(\tau-t)^\gamma \right) h(\tau) d\tau. \quad (9)$$

General fractional-order integrals

The left-sided general fractional-order integral is defined as[1]

$${}_RI_{a+}^{\gamma,n,\nu}g(t) = \int_a^t (t-\tau)^{n-\gamma-1} E_{\gamma,n-\gamma}^{-1} \left(-\nu(t-\tau)^\gamma \right) g(\tau) d\tau \quad (10)$$

and the right-sided general fractional-order integral as

$${}_RI_{b-}^{\gamma,n,\nu}g(t) = (-1)^n \int_t^b (\tau-t)^{n-\gamma-1} E_{\gamma,n-\gamma}^{-1} \left(-\nu(\tau-t)^\gamma \right) g(\tau) d\tau. \quad (11)$$

For $n = 1$, the left-sided general fractional-order integral is defined as[1]

$${}_RI_{a+}^{\gamma,\nu}g(t) = {}_RI_{a+}^{\gamma,1,\nu}g(t) = \int_a^t (t-\tau)^{-\gamma} E_{\gamma,1-\gamma}^{-1} \left(-\nu(t-\tau)^\gamma \right) g(\tau) d\tau \quad (12)$$

and the right-sided general fractional-order integral as

$${}_RI_{b-}^{\gamma,\nu}g(t) = {}_RI_{b-}^{\gamma,1,\nu}g(t) = -\int_t^b (\tau-t)^{-\gamma} E_{\gamma,1-\gamma}^{-1} \left(-\nu(\tau-t)^\gamma \right) g(\tau) d\tau. \quad (13)$$

For $a = 0$, the general fractional-order integrals are defined as [1]

$${}_RI_{0+}^{\gamma,n,\nu}g(t) = \int_0^t (t-\tau)^{n-\gamma-1} E_{\gamma,n-\gamma}^{-1} \left(-\nu(t-\tau)^\gamma \right) g(\tau) d\tau \quad (14)$$

and

$${}_RI_{0+}^{\gamma,\nu}g(t) = {}_RI_{0+}^{\gamma,1,\nu}g(t) = \int_0^t (t-\tau)^{-\gamma} E_{\gamma,1-\gamma}^{-1} \left(-\nu(t-\tau)^\gamma \right) g(\tau) d\tau. \quad (15)$$

The Laplace transforms of the general fractional calculus operators

The Laplace transforms of the general fractional derivatives and general fractional integrals are given as follows [1]:

$$L\left\{{}_{RLY}D_{0+}^{\gamma,n,\nu}g(t)\right\} = s^{n-\gamma} \left(1 + \nu s^{-\gamma}\right)^{-1} g(s) - \sum_{\mu=0}^{n-1} s^{n-\mu-1} \left(D^\mu \left[{}_RI_{0+}^{\gamma,n,\nu}g(+0)\right]\right), \quad (16)$$

$$L\left\{{}_{RLY}D_{0+}^{\gamma,\nu}g(t)\right\} = s^{1-\gamma} \left(1 + \nu s^{-\gamma}\right)^{-1} g(s) - {}_RI_{0+}^{\gamma,\nu}g(+0), \quad (17)$$

$$\mathfrak{I}\left\{{}_R I_{0+}^{\gamma,n,\nu} g(t)\right\} = s^{\gamma-n} (1 + \nu s^{-\gamma}) g(s) \quad (18)$$

and

$$\mathfrak{I}\left\{{}_R I_{0+}^{\gamma,\nu} g(t)\right\} = s^{\gamma-1} (1 + \nu s^{-\gamma}) g(s). \quad (19)$$

A new wave equation within general fractional derivative via Lorenzo-Hartley kernel

Let us consider the new wave equation in the one-dimensional cases containing the general fractional derivative via Lorenzo-Hartley kernel defined on the definite domain

$${}_{RLY} \partial_{0+}^{\gamma,2,\nu} \mathfrak{R}(x,t) = \frac{\partial^2}{\partial x^2} \mathfrak{R}(x,t) \quad (2 > x > 0, t > 0) \quad (20)$$

with the initial and boundary conditions as follows:

$$\mathfrak{R}(0,t) = 0, \quad \mathfrak{R}(2,t) = 0, \quad {}_R I_{0+}^{\gamma,2,\nu} \mathfrak{R}(x,+0) = 0, \quad {}_R I_{0+}^{\gamma,\nu} \mathfrak{R}(x,+0) = \sin(\pi x), \quad (21)$$

where

$${}_{RLY} \partial_{0+}^{\gamma,2,\nu} \mathfrak{R}(x,t) = \frac{\partial^2}{\partial t^2} \int_0^t \Theta_\gamma(-\nu(\tau-t)^\gamma) \mathfrak{R}(x,\tau) d\tau \quad (22)$$

and

$${}_R I_{0+}^{\gamma,\nu} \mathfrak{R}(x,+0) = {}_R I_{0+}^{\gamma,1,\nu} \mathfrak{R}(x,+0). \quad (23)$$

With use of the Laplace transform, we easily obtain

$$\frac{\partial^2}{\partial x^2} \mathfrak{R}(x,s) = s^{1-\gamma} (1 + \nu s^{-\gamma})^{-1} \mathfrak{R}(x,s) - \sin(\pi x). \quad (24)$$

Setting the general solution of (24) given as

$$\mathfrak{R}(x,s) = \aleph_1 \sin(\pi x) + \aleph_2 \cos(\pi x). \quad (25)$$

where \aleph_1 and \aleph_2 are the constants, we have

$$\left(-\pi^2 - s^{1-\gamma} (1 + \nu s^{-\gamma})^{-1}\right) (\aleph_1 \sin(\pi x) + \aleph_2 \cos(\pi x)) = -\sin(\pi x) \quad (26)$$

which leads to

$$\aleph_2 = 0 \quad \text{and} \quad \aleph_1 = \frac{1}{\pi^2 + s^{1-\gamma} (1 + \nu s^{-\gamma})^{-1}}. \quad (27)$$

Thus, we have

$$\mathfrak{R}(x,s) = \frac{\sin(\pi x)}{\pi^2 + s^{1-\gamma} (1 + \nu s^{-\gamma})^{-1}} = \frac{\sin(\pi x)}{\pi^2} \left(\frac{1}{1 + \frac{1}{\pi^2} s^{1-\gamma} (1 + \nu s^{-\gamma})^{-1}} \right). \quad (28)$$

Thus, we have

$$\Re(x, s) = \frac{\sin(\pi x)}{\pi^2} \sum_{i=0}^n \left(-\frac{1}{\pi^2}\right)^i s^{(1-\gamma)i} (1 + \nu s^{-\gamma})^{-i}, \left| \frac{1}{\pi^2} s^{1-\gamma} (1 + \nu s^{-\gamma})^{-1} \right| < 1. \quad (29)$$

Thus, we get the analytical solution for the new wave equation in the one-dimensional cases containing the general fractional derivative via Lorenzo-Hartley kernel defined on the definite domain as follows:

$$\Re(x, t) = \frac{\sin(\pi x)}{\pi^2} \sum_{i=0}^n \left(-\frac{1}{\pi^2}\right)^i t^{\gamma i - i - 1} E_{\gamma, \gamma i - i}^i(-\nu t^\gamma). \quad (30)$$

Conclusion

In the present work, we have investigated the theory of the general fractional calculus via Lorenzo-Hartley kernel. Moreover, the general fractional-order model for the wave defined on the definite domain within general fractional derivative via Lorenzo-Hartley kernel was proposed and its solution were presented with the aid of the Laplace transforms and the special functions. The obtained results are presented to describe the general fractional-order models arising in mathematical physics.

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Nomenclature

γ - fractional order, [-]

x - space coordinate, [m]

t - time coordinate, [s]

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