A modified Fourier-Fick analysis for modeling non-Newtonian mixed convective flow considering heat generation

Muhammad Imran Anwar\(^1\), Muhammad Saqlain\(^1\), Muhammad Mudassar Gulzar\(^2\) and Muhammad Waqas\(^2,*\)

\(^1\)Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan
\(^2\)NUTECH School of Applied Sciences and Humanities, National University of Technology, Islamabad, 44000, Pakistan

*Corresponding author E-mail: mw_qau88@yahoo.com, muhammadwaqas@nutech.edu.pk (M. Waqas)

Abstract: Homotopic solutions for Jeffrey material in frames of buoyancy forces are constructed in this research. The improved Fourier-Fick laws are considered for formulation. In addition, variable liquid aspects (thermal conductivity, mass diffusivity) along with heat source are accounted. Prandtl's boundary-layer idea is utilized to model the problem. Involvement of similarity variables resulted into nonlinear system of coupled equations. The well-known homotopic scheme is employed for nonlinear analysis. Besides, a comprehensive discussion is reported for arising dimensionless variables versus significant profiles. Our results indicate a rise in thermal and solutal fields when variable conductivity and mass diffusivity parameters are increased.

Keywords: Improved Fourier-Fick laws; heat generation; Jeffrey material; Mixed convective flow; Variable liquid aspects.

1 Introduction

Fluids featuring non-Newtonian characteristics have vital contribution in distinct industrial utilizations due to their manifold attributes in nature. Numerous engineering structures comprise liquid crystals, polycrystalline materials, fibrous materials, course grain structures or colloidal suspensions elaborate features of non-Newtonian types. The classical Navier-Stokes concept is not able to predict their salient aspects. Therefore distinct models elaborating non-Newtonian characteristics have been introduced. The considered model (Jeffrey liquid) is subcategory of rate type materials and has prospective to scrutinize the characteristics of relaxation/retardation
aspects. Researchers considered Jeffrey model subjected to various configurations. For illustration, Chemically reacting Jeffrey material flow subjected to nonlinear radiation is scrutinized by Raju et al. [1]. Hayat et al. [2, 3] formulated hydromagnetic characteristics for Jeffrey material stretched flow under heat sink/source aspects. Thermo diffusion impact in fractional hydromagnetic Jeffrey material in frames of radiation is explored by Imran et al. [4]. Waqas et al. [5] formulated thermally radiating stratified Jeffrey nanoliquid subjected to buoyancy forces.

There are extremely effective utilizations of heat transportation mechanism for illustration chilling of nuclear reactor, medical utilizations like drug targeting and conduction of heat in tissues. Heat transport approach is firstly communicated through well-known Fourier law [6]. The thermal expression subjected to Fourier law has parabolic nature and so initial disruption is recognized continually throughout. To regulate such unrealistic characteristic in the thermal inertia factor, thermal expression is incorporated via stable heat-conduction. As a result, the innovative model revises the behavior of temporal solution and yields the heat-conduction expression into damped hyperbolic one [7]. Tibullo and Zampoli [8] established uniqueness results for incompressible nature problems subjected to modified Fourier law. Haddad [9] scrutinized thermal volatility in the permeable Brinkman medium employing modified Fourier law. A modified Fourier analysis considering magneto-Casson material under hydromagnetic characteristics is communicated by Malik et al. [10]. Waqas et al. [11] analytically addressed forced convective stratified Burgers material flow under modified Fourier law. Characteristics of modified Fourier law for computational analysis of non-Newtonian (Carreau and Jeffrey) material are communicated by Khan et al. [12, 13]. Waqas et al. [14] introduced heat source concept in mixed convective Burgers material subjected to improved Fourier-Fick expression. Variable liquid aspects in stratified Carreau nanoliquid subject to improved Fourier law are examined by Khan et al. [15].

Keeping above mentioned research attempts at mind, we found that improved Fourier law is less scrutinized in comparison to traditional Fourier law. Thus our focus here is to formulate and examine the non-Newtonian (Jeffrey) fluid in frames of the improved Fourier law. In addition, mixed convection, variable type conductivity/diffusivity and heat source characteristics are considered. Nonlinear systems are tackled through homotopy algorithm [16-25]. The non-dimensional quantities are exhibited and deliberated.
2 Modeling

An incompressible Jeffrey material mixed convective flow by stretchable surface is formulated. The improved Fourier-Fick laws are considered for formulation. In addition, variable liquid aspects (thermal conductivity, mass diffusivity) along with heat source are accounted. Viscous dissipation along with radiation is ignored. The governing two-dimensional expressions are:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\nu}{1 + \lambda_1} \left[ \frac{\partial^2 u}{\partial y^2} + \lambda_2 \left( \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + u \frac{\partial^3 u}{\partial x \partial y^3} - \frac{\partial u}{\partial x} \frac{\partial^3 u}{\partial y^3} + v \frac{\partial^3 u}{\partial y^3} \right) \right] + g \left( \Lambda_1 (T - T_\infty) + \Lambda_2 (C - C_\infty) \right), \quad (2)
\]

\[
u \frac{\partial T}{\partial y} + \frac{\partial T}{\partial y} + \lambda_T \left( \frac{u}{\rho c_p} \frac{\partial T}{\partial x} + \frac{v}{\rho c_p} \frac{\partial T}{\partial y} + \frac{\nu}{\rho c_p} \frac{\partial T}{\partial x} \right) + 2uv \frac{\partial T}{\partial y} + u^2 \frac{\partial^2 T}{\partial y^2} + v^2 \frac{\partial^2 T}{\partial y^2} - \frac{Q}{\rho c_p} \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{1}{\rho c_p} \frac{\partial}{\partial y} \left( K(T) \frac{\partial T}{\partial y} \right) + \frac{Q}{\rho c_p} (T - T_\infty), \quad (3)
\]

\[
u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + \lambda_c \left( \frac{u}{\rho c_p} \frac{\partial C}{\partial x} + \frac{v}{\rho c_p} \frac{\partial C}{\partial y} + \frac{\nu}{\rho c_p} \frac{\partial C}{\partial x} \right) + 2uv \frac{\partial C}{\partial y} + u^2 \frac{\partial^2 C}{\partial y^2} + v^2 \frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y} \left( D(C) \frac{\partial C}{\partial y} \right), \quad (4)
\]

subject to conditions [14]:

\[
u = U_w(x) = cx, \quad v = 0, \quad T = T_w, \quad C = C_w \text{ at } y = 0,
\]

\[
u \rightarrow 0, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty \text{ when } y \rightarrow \infty. \quad (5)
\]

Here \( u \) and \( v \) elucidate velocities of fluid in horizontal and vertical directions respectively, \( \rho \) liquid density, \( \nu \) kinematic viscosity, \( \lambda_2 \) relaxation time, \( \lambda_1 \) relation between relaxation/retardation times, \( \Lambda_1 \) and \( \Lambda_2 \) thermal and solutal expansion coefficients, \( g \) gravitational acceleration, \( T \) and \( C \) fluid temperature and concentration, \( \lambda_r \) and \( \lambda_c \) heat and mass flux relaxation times, \( T_\infty \) and \( C_\infty \) ambient fluid temperature and concentration, \( Q \) heat source/sink coefficient and \( c \) stretching rate. Mathematical forms of variable conductivity
\( (K(T)) \) and variable mass diffusivity \( (D(C)) \) is [26]:

\[
K(T) = K_\infty \left( 1 + \varepsilon_1 \frac{T - T_\infty}{T_w - T_\infty} \right),
\]

\[
D(C) = D_\infty \left( 1 + \varepsilon_2 \frac{C - C_\infty}{C_w - C_\infty} \right),
\]

in which \( (K_\infty, D_\infty) \) illustrate ambient liquid (thermal conductivity, mass diffusivity) and \( (\varepsilon_1, \varepsilon_2) \) small parameters which elaborates the characteristics of temperature and concentration for thermal and solutal dependent conductivity and diffusivity.

Employing [14]:

\[
\eta = y \sqrt{\frac{c}{v}}, \quad u = c x f'(\eta), \quad v = -\sqrt{ev} f'(\eta),
\]

\[
\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty},
\]

Eq. (1) is validated automatically while Eqs. (2) – (5) are:

\[
f'''(1 + \lambda_1)\left(ff'' - f'^2\right) + \beta\left(f''^2 - ff''\right) + \lambda (\theta + N\phi) = 0,
\]

\[
\left(1 + \varepsilon_1 \theta\right)\theta'' + \varepsilon_1 \theta'^2 + Pr f\theta' + Pr \delta\theta - Pr \gamma_1 f\theta'
- Pr \gamma_1 \left(ff'\theta' + f^2\theta''\right) = 0,
\]

\[
(1 + \varepsilon_2 \phi)\phi'' + \varepsilon_2 \phi'^2 + Scf \phi' - Sc\gamma_2 \left(ff'\phi' + f^2\phi''\right) = 0,
\]

\[
f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) \to 0,
\]

\[
\theta(0) = 1, \quad \theta(\infty) \to 0,
\]

\[
\phi(0) = 1, \quad \phi(\infty) \to 0.
\]

Here \( \beta = \lambda_2 c, \quad \lambda = \frac{Gr_\infty}{Re_x^2}, \quad N = \frac{Gr^*_x}{Gr}, \quad Gr^*_x = \frac{g \Lambda_1 \left(T_w - T_\infty\right) x^3}{v^2}, \quad \gamma_1 = \lambda_1 c, \quad \gamma_2 = \lambda_2 c \) and \( Sc = \frac{v}{D_\infty} \) illustrate Deborah number, thermal buoyancy factor, ratio of solutal to thermal buoyancy, thermal Grashof number, solutal Grashof number, Reynolds number, Prandtl number, heat generation factor, thermal...
relaxation factor, solutal relaxation factor and Schmidt number respectively.

3 Solution scheme and convergence

We employed homotopy algorithm \([16-25]\) for nonlinear analysis of Eqs. (9)-(11) subject to Eqs. (12)-(14). Undoubtedly h-curves are vital to certify convergence of Eqs. (9)-(11). Thus we revealed h-curves in Fig. 1 for such purpose. Flat portions in Fig. 1 help to attain allowable values of \(h_f, h_\theta\) and \(h_\phi\). We noticed \(-1.35 \leq h_f \leq -0.30\), \(-1.50 \leq h_\theta \leq -0.40\) and \(-1.40 \leq h_\phi \leq -0.40\) with \(\beta = 0.4\), \(\lambda_1 = 0.5\), \(\lambda = 0.3\), \(N = \gamma_1 = 0.2\), \(S = \gamma_2 = \varepsilon_1 = \varepsilon_2 = 0.1\), \(Sc = 0.8\) and \(Pr = 1.0\). Additionally, convergence is confirmed numerically via Table 1. Clearly Eq. (8) converges at 20th order approximation while Eqs. (9) and (10) converge at 25th order approximation respectively.

![Fig. 1. h-curves for \(f, \theta\) and \(\phi\).](image)

<table>
<thead>
<tr>
<th>Order of approximations</th>
<th>(-f''(0))</th>
<th>(-\theta'(0))</th>
<th>(-\phi'(0))</th>
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<tr>
<td>1</td>
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<td>0.7080</td>
<td>0.7400</td>
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<td>5</td>
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<td>0.4529</td>
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<td>0.5106</td>
</tr>
<tr>
<td>15</td>
<td>0.8003</td>
<td>0.4270</td>
<td>0.5110</td>
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</tbody>
</table>
4 Analysis

This section emphasizes the noteworthy characteristics of arising variables against thermal field $(\theta(\eta))$ and solutal field $(\phi(\eta))$. Thus, Figs. 2–8 are sketched and explained in detail. Fig. 2 reports $S$ characteristics on $\theta(\eta)$. Clearly $\theta(\eta)$ increases when $S$ is augmented. Heat transports promptly via higher $S$ estimations. Outcome of $Pr$ versus $\theta(\eta)$ is evaluated via Fig. 3. Larger $Pr$ values yields less diffusivity which accordingly reduces $\theta(\eta)$. Fig. 4 interprets variation in $\theta(\eta)$ for $\varepsilon_1$. As anticipated, larger $\varepsilon_1$ corresponds to $\theta(\eta)$ enhancement. In fact material’s conductivity upsurges when $\varepsilon_1$ is incremented. Consequently extra heat quantity is switched from surface towards material and as a result $\theta(\eta)$ is boosted. Effects of $\gamma_1$ and $\gamma_2$ versus $\theta(\eta)$ and $\phi(\eta)$ is explained in Figs. 5 and 6. Larger $\gamma_1$ and $\gamma_2$ signify non-conducting trend due to which $\theta(\eta)$ and $\phi(\eta)$ dwindles. Further, $\theta(\eta)$ and $\phi(\eta)$ are large for $\gamma_1 = 0 = \gamma_2$ when compared with $\gamma_1 > 0$ and $\gamma_2 > 0$. Fig. 7 addresses $Sc$ influence on $\phi(\eta)$. Here $\phi(\eta)$ reduces for higher $Sc$ estimation. Physically, the $Sc$ expression involves Brownian diffusivity which dwindles subject to higher $Sc$ values. Characteristics of $\varepsilon_2$ versus $\phi(\eta)$ are expressed via Fig. 8. Clearly $\phi(\eta)$ enhances subject to higher $\varepsilon_2$. Undoubtedly liquids having higher mass diffusivity corresponds to higher concentration. In fact liquid mass diffusivity is increased via larger $\varepsilon_2$. 

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<td>30</td>
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<td>0.5114</td>
</tr>
</tbody>
</table>
Fig. 2. $S$ versus $\theta(\eta)$.

Fig. 3. $Pr$ versus $\theta(\eta)$.

Fig. 4. $\epsilon_i$ versus $\theta(\eta)$.
Fig. 5. $\gamma_1$ versus $\theta(\eta)$.

Fig. 6. $\gamma_2$ versus $\phi(\eta)$.

Fig. 7. $Sc$ versus $\phi(\eta)$.
5 Final remarks

This communication reports heat source, variable (thermal conductivity, mass diffusivity) and buoyancy forces aspects in non-Newtonian (Jeffrey) material flow by moving vertical surface. Improved Fourier-Fick laws are utilized for formulation of energy and concentration equations. We acquired following noteworthy points through abovementioned investigation:

- Thermal field $\theta(\eta)$ upsurges when variable conductivity $(\varepsilon_i)$ heat source $(S)$ factors are augmented.
- Larger thermal relaxation $(\gamma_1)$ and Prandtl number $(Pr)$ correspond to $\theta(\eta)$ decline.
- A rise in solutal relaxation $(\gamma_2)$ and Schmidt number $(Sc)$ yield lower solutal field $\phi(\eta)$.
- The well-known Fourier-Fick laws can be recovered by letting $\gamma_1 = 0 = \gamma_2$ in Eqs. (10) and (11).
- Jeffrey liquid model yields viscous liquid outcomes when $\lambda_1 = 0 = \beta$.

References

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