A NEW GENERAL FRACTIONAL-ORDER WAVE MODEL INVOLVING MILLER-ROSS KERNEL

by

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In the paper we consider a general fractional-order wave model with the general fractional-order derivative involving the Miller-Ross kernel for the first time. The analytical solution for the general fractional-order wave model is investigated in detail. The obtained result is given to explore the complex processes in the mining rock.

Key words: *fractional-order wave model, general fractional-order derivative, Miller-Ross kernel, mining rock*

Introduction

The mathematical model for the wave propagation in the mining rock has been investigated by many scientists, see [1-4] and references cited therein. For example, the linear model for the wave propagation:

$$\frac{\partial^2 \Re(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[\phi_1 \frac{\partial \Re(x,t)}{\partial x}\right] \tag{1}$$

where ϕ_1 is a constant and $\Re(x, t)$ – the wave function, was proposed in [5]. As a special case of (1), the linear model for the wave propagation:

$$\frac{\partial^2 \Re(x,t)}{\partial t^2} = \phi_2 \frac{\partial^2 \Re(x,t)}{\partial x^2} \tag{2}$$

where ϕ_2 is a constant and $\Re(x, t)$ is the wave function, was proposed in [6]. The models can be used to describe 1-D wave propagation in the mining rock. Recently, a general fractional-order derivative within the Miller-Ross kernel [7]. The main aim of the article is to propose the general fractional-order derivative model for the wave propagation based on the general fractional-order derivative involving the Miller-Ross kernel [8] and to investigate its analytical solution.

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A general fractional-order calculus involving the Miller-Ross kernel

In this section, we introduce the general fractional-order derivative involving the special function, which proposed in by Miller and Ross, see [8], from the point of view of the general fractional-order calculus application.

The Miller-Ross function and its Laplace transform

For the given real constant, λ, the Miller-Ross function with one-parameter constant λ is defined [7, 8]:

$$\wp_\alpha\left(\lambda t^\alpha\right)=t^\alpha\sum_{\kappa=0}^{\infty}\frac{\lambda^\kappa t^\kappa}{\Gamma\left(\kappa+1+\alpha\right)}=\sum_{\kappa=0}^{\infty}\frac{\lambda^\kappa t^{\kappa+\alpha}}{\Gamma\left(\kappa+1+\alpha\right)} \tag{1}$$

with the Laplace transform [8]:

$$L\left\{\wp_\alpha\left(\lambda t\right)\right\}=L\left\{\sum_{\kappa=0}^{\infty}\frac{\lambda^\kappa t^{\kappa+\alpha}}{\Gamma\left(\kappa+1+\alpha\right)}\right\}=\sum_{\kappa=0}^{\infty}\frac{\lambda^\kappa}{s^{\kappa+\alpha+1}}=s^{-(\alpha+1)}\left(1-\lambda s^{-1}\right)^{-1}\quad\left(\left|\lambda s^{-1}\right|<1\right) \tag{2}$$

where the Laplace transform operator of the function $u(t)$ is represented [7]:

$$\Im\left[u\left(t\right)\right]=u\left(s\right)=\int_0^\infty \mathrm{e}^{-st}u\left(t\right)\mathrm{d}t \tag{3}$$

A general fractional-order integral operators involving the Miller-Ross kernel

The left-sided general fractional-order integral operator involving the Miller-Ross kernel is defined [7]:

$$_{\mathrm{MR}}I_{a+}^{\alpha,\lambda}j\left(t\right)=\int_a^t\wp_\alpha\left[-\lambda\left(t-\tau\right)^\alpha\right]j\left(\tau\right)\mathrm{d}\tau \tag{4}$$

and the right-sided general fractional-order integral operator involving the Miller-Ross kernel:

$$_{\mathrm{MR}}I_{b-}^{\alpha,\lambda}j\left(t\right)=\int_t^b\wp_\alpha\left[-\lambda\left(\tau-t\right)^\alpha\right]j\left(\tau\right)\mathrm{d}\tau \tag{5}$$

When a = 0, the general fractional-order integral operator involving the Miller-Ross kernel become:

$$_{\mathrm{MR}}I_{0+}^{\alpha,\lambda}j\left(t\right)=\int_0^t\wp_\alpha\left[-\lambda\left(t-\tau\right)^\alpha\right]j\left(\tau\right)\mathrm{d}\tau \tag{6}$$

with its Laplace transform:

$$\Im\left[{}_{\mathrm{MR}}I_{0+}^{\alpha,\lambda}j\left(t\right)\right]=s^{\alpha+1}\left(1+\lambda s^{-1}\right)j\left(s\right) \tag{7}$$

General fractional-order derivatives involving the Miller-Ross kernel

The left-sided general fractional-order derivative of the Riemann-Liouville type involving the Miller-Ross kernel is defined [7]:

$${}_{\mathrm{MR}}^{\mathrm{RL}}\mathrm{D}_{a+}^{\alpha,\kappa,\lambda}j\left(t\right)=\frac{\mathrm{d}^\kappa}{\mathrm{d}t^\kappa}\left[{}_{\mathrm{MR}}I_{a+}^{\alpha,\lambda}j\left(t\right)\right]=\frac{\mathrm{d}^\kappa}{\mathrm{d}t^\kappa}\int_a^t\wp_\alpha\left[-\lambda\left(t-\tau\right)^\alpha\right]j\left(\tau\right)\mathrm{d}\tau \tag{8}$$

and the right-sided general fractional-order derivative of the Riemann-Liouville type involving the Miller-Ross kernel:

$$ {}^{\mathrm{RL}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\kappa,\lambda}_{b-} j(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\kappa} \left[{}_{\mathrm{MR}}I^{\alpha,\lambda}_{b-} j(t)\right] = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\kappa} \int_t^b \wp_\alpha \left[-\lambda(\tau - t)^\alpha\right] j(\tau)\,\mathrm{d}\tau \tag{9} $$

where κ is the positive integer numbers.

The left-sided general fractional-order derivative of the Liouville-Sonine type involving the Miller-Ross kernel is defined [7]:

$$ {}^{\mathrm{LS}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\lambda}_{a+} j(t) = {}_{\mathrm{MR}}I^{\alpha,\lambda}_{a+}\left[j^{(1)}(t)\right] = \int_a^t \wp_\alpha \left[-\lambda(t-\tau)^\alpha\right] j^{(1)}(\tau)\,\mathrm{d}\tau \tag{10} $$

and the right-sided general fractional-order derivative of the Liouville-Sonine type within the Miller-Ross kernel:

$$ {}^{\mathrm{LS}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\lambda}_{b-} j(t) = {}_{\mathrm{MR}}I^{\alpha,\lambda}_{b-}\left[-j^{(1)}(t)\right] = -\int_t^b \wp_\alpha \left[-\lambda(\tau-t)^\alpha\right] j^{(1)}(\tau)\,\mathrm{d}\tau \tag{11} $$

The left-sided general fractional-order derivative of the Liouville-Sonine-Caputo type involving the Miller-Ross kernel is defined [7]:

$$ {}^{\mathrm{LS}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\kappa,\lambda}_{a+} j(t) = {}_{\mathrm{MR}}I^{\alpha,\lambda}_{a+}\left[j^{(\kappa)}(t)\right] = \int_a^t \wp_\alpha \left[-\lambda(t-\tau)^\alpha\right] j^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{12} $$

and the right-sided general fractional-order derivative of the Liouville-Sonine-Caputo type within the Miller-Ross kernel:

$$ {}^{\mathrm{LSC}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\kappa,\lambda}_{b-} j(t) = {}_{\mathrm{MR}}I^{\alpha,\lambda}_{b-}\left[(-1)^\kappa j^{(\kappa)}(t)\right] = (-1)^\kappa \int_t^b \wp_\alpha \left[-\lambda(\tau-t)^\alpha\right] j^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{13} $$

The relation between the general fractional-order derivative of the Riemann-Liouville and Liouville-Sonine types is given [7]:

$$ {}^{\mathrm{LSC}}_{\mathrm{MR}}D^{\alpha,\lambda}_{0+} j(t) = {}^{\mathrm{RL}}_{\mathrm{MR}}D^{\alpha,1,\lambda}_{0+} j(t) - \wp_\alpha\left(-\lambda t^\alpha\right) j(0) \tag{14} $$

The Laplace transforms of the general fractional-order derivatives can be given [7]:

$$ \Im\left[{}^{\mathrm{RL}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\kappa,\lambda}_{0+} j(t)\right] = s^{\kappa-\alpha-1}\left(1+\lambda s^{-1}\right)^{-1} j(s) \tag{15} $$

$$ \Im\left[{}^{\mathrm{LS}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\lambda}_{0+} j(t)\right] = s^{-\alpha-1}\left(1+\lambda s^{-1}\right)^{-1} \left[sj(s) - j(0)\right] \tag{16} $$

and

$$ \Im\left[{}^{\mathrm{LSC}}_{\mathrm{MR}}\mathrm{D}^{\alpha,\lambda}_{0+} j(t)\right] = s^{-\alpha-1}\left(1+\lambda s^{-1}\right)^{-1} \left(s^\kappa j(s) - \sum_{j=1}^{\kappa} s^{\kappa-j} j^{(j-1)}(0)\right) \tag{17} $$

A new general fractional-order wave model

We now consider a new general fractional-order wave model containing the general fractional-order derivative of the Liouville-Sonine type within the Miller-Ross kernel:

$$ {}^{\mathrm{LSC}}_{\mathrm{MR}}\partial^{\alpha,2,\lambda}_{0+} \Re(x,t) = \frac{\partial^2 \Re(x,t)}{\partial x^2} \quad (x>0,\ t>0) \tag{18} $$

subjected to the initial and boundary conditions:

$$ \Re^{(1)}(x,0) = 0,\ \Re(x,0)=0,\ \Re(0,t)=0,\ \Re(+\infty,t)=0 \tag{19} $$

where the general fractional-order partial derivatives of orders 2 and 1 are defined:

$$ {}_{\mathrm{MR}}^{\mathrm{LSC}}\partial_{0+}^{\alpha,2,\lambda}\Re(x,t)=\int_0^t \wp_\alpha\left[-\lambda(t-\tau)^\alpha\right]\Re^{(2)}(x,\tau)\,\mathrm{d}\tau \tag{20} $$

and

$$ {}_{\mathrm{MR}}^{\mathrm{LSC}}\partial_{0+}^{\alpha,\lambda}\Re(x,t)=\int_0^t \wp_\alpha\left[-\lambda(t-\tau)^\alpha\right]\Re^{(1)}(x,\tau)\,\mathrm{d}\tau \tag{21} $$

respectively.

With the use of the Laplace transform, we present:

$$ \frac{\partial^2\Re(x,s)}{\partial x^2}=s^{1-\alpha}\left(1+\lambda s^{-1}\right)^{-1}\Re(x,s) \tag{22} $$

with the general solution, given:

$$ \Re(x,s)=\Lambda_1\mathrm{e}^{-x\sqrt{s^{1-\alpha}\left(1+\lambda s^{-1}\right)^{-1}}}+\Lambda_2\mathrm{e}^{x\sqrt{s^{1-\alpha}\left(1+\lambda s^{-1}\right)^{-1}}} \tag{23} $$

where Λ_1 and Λ_2 are the constants.

Finally, we have $\Lambda_2 = 0$ and $\Lambda_1 = 0$ and such that:

$$ \Re(x,s)=\mathrm{e}^{-x\sqrt{s^{1-\alpha}\left(1+\lambda s^{-1}\right)^{-1}}}=\mathrm{e}^{-xs^{\frac{1-\alpha}{2}}\left(1+\lambda s^{-1}\right)^{-1/2}} \tag{24} $$

Thus, we have:

$$ \Re(x,s)=\sum_{n=0}^{\infty}\frac{(-x)^n}{\Gamma(n+1)}s^{-\frac{(\alpha-1)n}{2}}\left(1+\lambda s^{-1}\right)^{-n/2} \tag{25} $$

which leads to:

$$ \Re(x,t)=\sum_{n=0}^{\infty}\frac{(-x)^n}{\Gamma(n+1)}t^{\frac{(\alpha-1)n}{2}-1}E_{1,\frac{(\alpha-1)n}{2}}^{n/2}(-\lambda t) \tag{26} $$

where the Laplace transform of the generalized Prabhakar function is written [7]:

$$ \Im\left[t^{\frac{(\alpha-1)n}{2}-1}E_{1,\frac{(\alpha-1)n}{2}}^{n/2}(-\lambda t)\right]=\sum_{n=0}^{\infty}\frac{(-x)^n}{\Gamma(n+1)}s^{-\frac{(\alpha-1)n}{2}}\left(1+\lambda s^{-1}\right)^{-n/2} \tag{27} $$

Conclusion

In our task, we investigate the new general fractional-order wave model with the general fractional-order derivative involving the Miller-Ross kernel. With the aid of the Laplace transform, we obtain the analytical solution. The special functions are accurate and efficient for descriptions of the mining rock.

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Nomenclature

α – fractional order, [–]
t – time co-ordinate, [m]
x – space co-ordinate, [m]

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