A NEW GENERAL FRACTIONAL-ORDER WAVE MODEL INVOLVING MILLER-ROSS KERNEL

by

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In the paper we consider a general fractional-order wave model with the general fractional-order derivative involving the Miller-Ross kernel for the first time. The analytical solution for the general fractional-order wave model is investigated in detail. The obtained result is given to explore the complex processes in the mining rock.

Key words: fractional-order wave model, general fractional-order derivative, Miller-Ross kernel, mining rock

Introduction

The mathematical model for the wave propagation in the mining rock has been investigated by many scientists (see[1-4] and references cited therein). For example, the linear model for the wave propagation

\[
\frac{\partial^2 \mathcal{R}(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left( \phi_1 \frac{\partial \mathcal{R}(x,t)}{\partial x} \right),
\]

(1)

where $\phi_1$ is a constant and $\mathcal{R}(x,t)$ is the wave function, was proposed in [5]. As a special case of (1), the linear model for the wave propagation

\[
\frac{\partial^2 \mathcal{R}(x,t)}{\partial t^2} = \frac{\partial^2}{\partial x^2} \mathcal{R}(x,t),
\]

(2)

where $\phi_2$ is a constant and $\mathcal{R}(x,t)$ is the wave function, was proposed in [6]. The models can be used to describe one-dimensional wave propagation in the mining rock. Recently, a general fractional-order derivative within the Miller-Ross kernel [7]. The main aim of the article is to propose the general fractional-order derivative model for the wave propagation based on the general

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fractional-order derivative involving the Miller-Ross kernel [8] and to investigate its analytical solution. The structure of the paper is given as follows. In Section 2, we investigate the Miller-Ross function, general fractional-order integral involving the Miller-Ross kernel, and general fractional-order derivative involving the Miller-Ross kernel. In Section 3, the general fractional-order wave model and its analytical solution are proposed. Finally, the conclusion is given in Section 4.

A general fractional-order calculus involving the Miller-Ross kernel

In this section, we introduce the general fractional-order derivative involving the special function, which proposed in by Miller and Ross (see[8]) from the point of view of the general fractional-order calculus application.

The Miller-Ross function and its Laplace transform

For the given real constant \( \lambda \), the Miller-Ross function with one-parameter constant \( \lambda \) is defined as [7,8]:

\[
\phi_a \left( \lambda t^\alpha \right) = t^\alpha \sum_{k=0}^{\infty} \frac{\lambda^k t^{k+\alpha}}{\Gamma(k+1+\alpha)} = \sum_{k=0}^{\infty} \frac{\lambda^k t^{k+\alpha}}{\Gamma(k+1+\alpha)}
\]

with the Laplace transform [8]

\[
L \{ \phi_a \left( \lambda t^\alpha \right) \} = L \left\{ \sum_{k=0}^{\infty} \frac{\lambda^k t^{k+\alpha}}{\Gamma(k+1+\alpha)} \right\} = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k+1+\alpha)} = s^{-(\alpha+1)} \left( 1 - \lambda s^{-1} \right)^{-1} \left( |\lambda s^{-1}| < 1 \right),
\]

where the Laplace transform operator of the function \( u(t) \) is represented as follows [7]:

\[
\mathcal{L} \left[ u(t) \right] = u(s) = \int_{0}^{\infty} e^{-st} u(t) \, dt.
\]

A general fractional-order integral operators involving the Miller-Ross kernel

The left-sided general fractional-order integral operator involving the Miller-Ross kernel is defined as [7]

\[
_{a}^{\alpha} I_{MR}^{a,\lambda} j(t) = \int_{a}^{t} \phi_a \left( -\lambda \left( t - \tau \right)^\alpha \right) j(\tau) \, d\tau
\]

and the right-sided general fractional-order integral operator involving the Miller-Ross kernel as

\[
_{b}^{\alpha} I_{MR}^{a,\lambda} j(t) = \int_{b}^{t} \phi_a \left( -\lambda \left( \tau - t \right)^\alpha \right) j(\tau) \, d\tau.
\]

When \( a = 0 \), the general fractional-order integral operator involving the Miller-Ross kernel become

\[
_{0}^{\alpha} I_{MR}^{0,\lambda} j(t) = \int_{0}^{t} \phi_a \left( -\lambda \left( t - \tau \right)^\alpha \right) j(\tau) \, d\tau
\]

with its Laplace transform
General fractional-order derivatives involving the Miller-Ross kernel

The left-sided general fractional-order derivative of the Riemann-Liouville type involving the Miller-Ross kernel is defined as [7]

\[ \frac{d^{x}}{dt^{x}} (MR I_{a}^{\alpha,\lambda} j(t)) = \frac{d^{x}}{dt^{x}} \int_{a}^{t} \varphi(t-\tau) \left(-\lambda (t-\tau)^{\alpha}\right) j(\tau) d\tau, \quad \alpha \in \mathbb{R}, \quad \lambda > 0, \quad a < t. \]  

and the right-sided general fractional-order derivative of the Riemann-Liouville type involving the Miller-Ross kernel as

\[ \frac{d^{x}}{dt^{x}} (MR D_{b}^{\alpha,\lambda} j(t)) = \left(-\frac{d}{dt}\right)^{x} \int_{a}^{b} \varphi(t-\tau) \left(-\lambda (t-\tau)^{\alpha}\right) j(\tau) d\tau, \quad \alpha \in \mathbb{R}, \quad \lambda > 0, \quad a < b. \]  

where \( \kappa \) is the positive integer numbers.

The left-sided general fractional-order derivative of the Liouville-Sonine type involving the Miller-Ross kernel is defined as [7]

\[ \frac{d^{x}}{dt^{x}} (LS I_{a}^{\alpha,\lambda} j(t)) = \int_{a}^{t} \varphi(t-\tau) \left(-\lambda (t-\tau)^{\alpha}\right) j^{(1)}(\tau) d\tau, \quad \alpha \in \mathbb{R}, \quad \lambda > 0, \quad a < t. \]  

and the right-sided general fractional-order derivative of the Liouville-Sonine type within the Miller-Ross kernel as

\[ \frac{d^{x}}{dt^{x}} (LS D_{b}^{\alpha,\lambda} j(t)) = -\int_{t}^{b} \varphi(t-\tau) \left(-\lambda (t-\tau)^{\alpha}\right) j^{(1)}(\tau) d\tau, \quad \alpha \in \mathbb{R}, \quad \lambda > 0, \quad a < b. \]  

The left-sided general fractional-order derivative of the Liouville-Sonine-Caputo type involving the Miller-Ross kernel is defined as [7]

\[ \frac{d^{x}}{dt^{x}} (LSC I_{a}^{\alpha,\lambda} j(t)) = \int_{a}^{t} \varphi(t-\tau) \left(-\lambda (t-\tau)^{\alpha}\right) j^{(1)}(\tau) d\tau, \quad \alpha \in \mathbb{R}, \quad \lambda > 0, \quad a < t. \]  

and the right-sided general fractional-order derivative of the Liouville-Sonine-Caputo type within the Miller-Ross kernel as

\[ \frac{d^{x}}{dt^{x}} (LSC D_{b}^{\alpha,\lambda} j(t)) = \int_{t}^{b} \varphi(t-\tau) \left(-\lambda (t-\tau)^{\alpha}\right) j^{(1)}(\tau) d\tau, \quad \alpha \in \mathbb{R}, \quad \lambda > 0, \quad a < b. \]  

The relation between the general fractional-order derivative of the Riemann-Liouville and Liouville-Sonine types is given as follows [7]:

\[ \frac{d^{x}}{dt^{x}} (LS D_{0}^{\alpha,\lambda} j(t)) = \frac{d^{x}}{dt^{x}} (RL D_{0}^{\alpha,\lambda} j(t)) = \frac{d^{x}}{dt^{x}} \int_{0}^{t} \varphi(t-\tau) \left(-\lambda t^{\alpha}\right) j(\tau) d\tau. \]

The Laplace transforms of the general fractional-order derivatives can be given as follows [7]:

\[ \mathcal{L} \left[ \frac{d^{x}}{dt^{x}} (RL D_{0}^{\alpha,\lambda} j(t)) \right] = s^{\alpha-1} \left(1 + \lambda s^{-1}\right) j(s), \quad \alpha \in \mathbb{R}, \quad \lambda > 0. \]
\[ \mathfrak{A}\left[ \mathcal{L}_{\text{MR}}^{\alpha,\beta} j(t) \right] = s^{-\alpha-1} \left( 1 + \lambda s^{-1} \right)^{-1} \left( sj(s) - j(0) \right), \quad (16) \]

and

\[ \mathfrak{A}\left[ \mathcal{L}_{\text{SC}}^{\alpha,\beta} j(t) \right] = s^{-\alpha-1} \left( 1 + \lambda s^{-1} \right)^{-1} \left( s^\alpha j(s) - \sum_{j=1}^{\infty} s^{\alpha-j} j^{(j-1)}(0) \right). \quad (17) \]

**A new general fractional-order wave model**

We now consider a new general fractional-order wave model containing the general fractional-order derivative of the Liouville-Soneine type within the Miller-Ross kernel

\[ \mathcal{L}_{\text{MR}}^{\alpha,\beta} \mathcal{R}(x,t) = \frac{\partial^2 \mathcal{R}(x,t)}{\partial x^2} \quad (x > 0, t > 0) \quad (18) \]

subjected to the initial and boundary conditions:

\[ \mathcal{R}(x,0) = 0, \quad \mathcal{R}(x,0) = 0, \quad \mathcal{R}(0,t) = 0, \quad \text{and} \quad \mathcal{R}(+\infty,t) = 0, \quad (19) \]

where the general fractional-order partial derivatives of orders 2 and 1 are defined as

\[ \mathcal{L}_{\text{MR}}^{\alpha,\beta} \mathcal{R}(x,t) = \int_{0}^{t} \theta_{\alpha} \left( -\lambda (t-\tau)^{\alpha} \right) \mathcal{R}^{(1)}(x,\tau) d\tau \quad (20) \]

and

\[ \mathcal{L}_{\text{MR}}^{\alpha,\beta} \mathcal{R}(x,t) = \int_{0}^{t} \theta_{\alpha} \left( -\lambda (t-\tau)^{\alpha} \right) \mathcal{R}^{(2)}(x,\tau) d\tau, \quad (21) \]

respectively.

With the use of the Laplace transform, we present

\[ \frac{\partial^2 \mathcal{R}(x,s)}{\partial x^2} = s^{-\alpha} \left( 1 + \lambda s^{-1} \right)^{-1} \mathcal{R}(x,s) \quad (22) \]

with the general solution, given as follows:

\[ \mathcal{R}(x,s) = \Lambda_1 e^{-x^{1-\alpha(1+\lambda s^{-1})^{-1}}} + \Lambda_2 e^{-x^{1-\alpha(1+\lambda s^{-1})^{-1}}}, \quad (23) \]

where \( \Lambda_1 \) and \( \Lambda_1 \) are the constants.

Finally, we have \( \Lambda_2 = 0 \) and \( \Lambda_1 = 1 \) such that

\[ \mathcal{R}(x,s) = e^{-x^{1-\alpha(1+\lambda s^{-1})^{-1}}} = e^{-\frac{x^{1-\alpha}}{\lambda} (1+\lambda s^{-1})^{\frac{1}{\alpha}}}. \quad (24) \]

Thus, we have

\[ \mathcal{R}(x,s) = \sum_{n=0}^{\infty} \left( -x \right)^{n} \frac{\Gamma(n+1)}{\Gamma(n+1)} s^{-\frac{\alpha-1}{2}} (1 + \lambda s^{-1})^{-\frac{\alpha}{2}} \quad (25) \]

which leads to
\[ \mathcal{R}(x,t) = \sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma(n+1)} t^{\frac{1}{2}(\alpha-1)n-1} E_{\frac{1}{2},\frac{1}{2}}^n \left( -\lambda t \right). \]  

(26)

where the Laplace transform of the generalized Prabhakar function is written as [7]

\[ \mathcal{L} \left[ t^{\frac{1}{2}(\alpha-1)n-1} E_{\frac{1}{2},\frac{1}{2}}^n \left( -\lambda t \right) \right] = \sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma(n+1)} s^{\frac{1}{2}(\alpha-1)n} \left( 1 + \lambda s^{-1} \right)^{-\frac{n}{2}}. \]  

(27)

Conclusion

In our task, we investigate the new general fractional-order wave model with the general fractional-order derivative involving the Miller-Ross kernel. With the aid of the Laplace transform, we obtain the analytical solution. The special functions are accurate and efficient for descriptions of the mining rock.

Acknowledgement

The work is supported by the Fundamental Research Funds for the Central Universities (2017CXNL01).

Nomenclature

\( \alpha \) - fractional order, [-]  
\( x \) - space coordinate, [m]  
\( t \) - time coordinate, [m]

References


Paper submitted: September 23, 2018  
Paper revised: January 11, 2019  
Paper accepted: January 28, 2019