NEW GENERAL CALCULI WITH RESPECT TO ANOTHER FUNCTIONS APPLIED TO DESCRIBE THE NEWTON-LIKE DASHPOT MODELS IN ANOMALOUS VISCOELASTICITY

by

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In this article, we address the general derivatives and integrals with respect to another function for the first time. We consider the new perspective in anomalous viscoelasticity containing the general derivatives with respect to another functions containing the power-law, exponential, and logarithmic functions. The results are accurate and efficient in the descriptions of the complex behaviors of the materials.

Key words: *general derivatives, general integrals, general calculi, anomalous viscoelasticity, dashpot*

Introduction

The classical calculus (the well-known Newton-Leibniz calculus) of the functions with the integer order and the variable can be extended since the order becomes (I) any fractional-order order and (II) any function order, and the variables of the functions can suggested as (III) the functions, or both are considered (see [1,2]). When the condition (I) is valid, the fractional calculus of constant order has been presented in [3,4]. When the condition (II) is given, the fractional calculus of variable order has been proposed in [5]. When (I) and (3) are considered, the fractional calculus of constant order with respect to another function have been reported in [6]. When (2) and (3) are employed, the fractional calculus of variable order with respect to another function has been presented in [7]. For more information other definitions of the calculi, see [8].

As applications of the Newton-Leibniz calculus, Newton proposed the dashpot element (called the Newtonian dashpot element) is given as [9]

$$\sigma(\tau) = \gamma \mathbb{D}^{(1)} \varepsilon(\tau),$$

where γ is the material parameter, ε is the strain, σ is the stress, and τ is the time. As a generalization of the Newtonian dashpot element, the fractional-order models in rheology we re proposed in [10,11] due to the Nutting behaviors in materials (see[12]).

The idea that new calculi consist of the Newton-Leibniz calculus and (III) are has been not considerable. The goals of the paper are to present the general calculi with respect to another functions containing the power-law, exponential, and logarithmic functions, and to present the new

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applications to the Newton-like dashpot models in anomalous viscoelasticity. The structure of the paper is as follows. In Section 2, we define the general derivatives and integrals with respect to another functions. In Section 3, we give the Newton-like dashpot models in anomalous viscoelasticity. Finally, we present the conclusion in Section 4.

The general calculi with respect to another functions

Classical calculus

The classical derivative (the well-known Newton-Leibniz derivative) is defined as

$$\mathbb{D}^{(1)}\Theta(\tau)=\frac{d\Theta(\tau)}{d\tau}=\lim_{\Delta\tau\to 0}\frac{\Theta(\tau+\Delta\tau)-\Theta(\tau)}{\Delta\tau}. \tag{1}$$

The classical integral (the well-known Newton-Leibniz integral) is defined as

$$\mathbb{I}^{(1)}\Phi(\tau)=\int_0^\tau \Phi(\tau)d\tau. \tag{2}$$

The relations between (1) and (2) are given as

$$\Theta(\tau)=\frac{d}{d\tau}\int_0^\tau \Theta(\tau)d\tau \tag{3}$$

and

$$\Theta(\tau)=\int_0^\tau \frac{d}{d\tau}\Theta(\tau)d\tau+\Theta(0). \tag{4}$$

The general calculus with respect to another function

Let $h^{(1)}(\tau)>0$. The general derivatives and integrals with respect to another functions are presented as follows.

The general derivative with respect to another function is defined as

$$\mathbb{D}_{\tau,h}^{(1)}\Theta(\tau)=\left(\frac{1}{\frac{dh(\tau)}{d\tau}}\right)\frac{d}{d\tau}\Theta(\tau)=\left(\frac{d\tau}{dh(\tau)}\right)\frac{d}{d\tau}\Theta(\tau)=\frac{d}{dh(\tau)}\Theta(\tau)=\frac{1}{h^{(1)}(\tau)}\frac{d\Theta(\tau)}{d\tau}. \tag{5}$$

The general integral with respect to another function is defined as

$${}_0\mathbb{I}_{\tau,h}^{(1)}\Phi(\tau)=\int_0^\tau \Phi(\tau)h^{(1)}(\tau)d\tau. \tag{6}$$

The general derivative of higher order with respect to another function is defined as

$$\mathbb{D}_{\tau,h}^{(n)}\Theta(\tau)=\left(\frac{1}{h^{(1)}(\tau)}\frac{d}{d\tau}\right)^n\Theta(\tau)\cdot \tag{7}$$

Their relationships between (5) and (6) are presented as

$$\Theta(\tau)=\frac{1}{h^{(1)}(\tau)}\frac{d}{d\tau}\int_0^\tau \Theta(\tau)h^{(1)}(\tau)d\tau \tag{8}$$

and

$$\Theta(\tau)=\int_0^\tau \left[\left(\frac{1}{h^{(1)}(\tau)}\frac{d}{d\tau}\right)\Theta(\tau)\right]h^{(1)}(\tau)d\tau+\Theta(0). \tag{9}$$

More generally, the general integral with respect to another function is defined as

$${}_a\mathbb{I}_{\tau,h}^{(1)}\Phi(\tau)=\int_a^\tau \Phi(\tau)h^{(1)}(\tau)d\tau\,. \tag{10}$$

In this case, we have their relationships as follows

$$\Theta(\tau)=\frac{1}{h^{(1)}(\tau)}\frac{d}{d\tau}\int_a^\tau \Theta(\tau)h^{(1)}(\tau)d\tau \tag{11}$$

and

$$\Theta(\tau)=\int_a^\tau \left[\left(\frac{1}{h^{(1)}(\tau)}\frac{d}{d\tau}\right)\Theta(\tau)\right]h^{(1)}(\tau)d\tau+\Theta(a)=\int_a^\tau \frac{d\Theta(\tau)}{d\tau}d\tau+\Theta(a). \tag{12}$$

Remark that for $h(\tau)=\tau$, (5) and (6) become (1) and (2), respectively.

The general calculus with respect to power-law function

The general derivative with respect to power-law function, denoted as $h(\tau)=\tau^\alpha$ $(\tau\neq 0)$, is defined as

$$\mathbb{D}_{\tau,\tau^\alpha}^{(1)}\Theta(\tau)=\frac{1}{\alpha\tau^{\alpha-1}}\frac{d\Theta(\tau)}{d\tau}, \tag{13}$$

where α are any real numbers.

The general integral with respect to power-law function is defined as

$${}_0\mathbb{I}_{\tau,\tau^\alpha}^{(1)}\Phi(\tau)=\alpha\int_0^\tau \Phi(\tau)\tau^{\alpha-1}d\tau\,. \tag{14}$$

The general derivative of higher order with respect to power-law function is defined as

$$\mathbb{D}_{\tau,\tau^\alpha}^{(n)}\Theta(\tau)=\left(\frac{1}{\alpha\tau^{\alpha-1}}\frac{d}{d\tau}\right)^n\Theta(\tau)\cdot \tag{15}$$

Their relationships between (13) and (14) are presented as

$$\Theta(\tau)=\frac{1}{\tau^{\alpha-1}}\frac{d}{d\tau}\int_0^\tau \Theta(\tau)\tau^{\alpha-1}d\tau \tag{16}$$

and

$$\Theta(\tau)=\int_0^\tau \frac{d\Theta(\tau)}{d\tau}d\tau+\Theta(0). \tag{17}$$

More generally,

$$\Theta(\tau)=\frac{1}{\tau^{\alpha-1}}\frac{d}{d\tau}\int_a^\tau \Theta(\tau)\tau^{\alpha-1}d\tau \tag{18}$$

and

$$\Theta(\tau)=\int_a^\tau\left[\left(\frac{1}{\tau^{\alpha-1}}\frac{d}{d\tau}\right)\Theta(\tau)\right]\tau^{\alpha-1}d\tau+\Theta(a)=\int_a^\tau\frac{d\Theta(\tau)}{d\tau}d\tau+\Theta(a). \tag{19}$$

Remarek that Chen sugggested that the power-law function as the Hausdorff measure the with the aid of the hypotheses of fractal invariance and fractal equivalence[13].

The general calculus with respect to exponential function

The general derivative with respect to exponential function, denoted as $h(\tau)=e^{\lambda\tau}$ with real number λ, is defined as

$$\mathbb{D}^{(1)}_{\tau,e^{\lambda\tau}}\Theta(\tau)=\frac{1}{\lambda e^{\lambda\tau}}\frac{d\Theta(\tau)}{d\tau}. \tag{20}$$

The general integral with respect to exponential function is defined as

$${}_0\mathbb{I}^{(1)}_{\tau,e^{\lambda\tau}}\Theta(\tau)=\int_0^\tau\Theta(\tau)\lambda e^{\lambda\tau}d\tau. \tag{21}$$

The general derivative of higher order with respect to another function is defined as

$$\mathbb{D}^{(n)}_{\tau,e^{\lambda\tau}}\Theta(\tau)=\left(\frac{1}{\lambda e^{\lambda\tau}}\frac{d}{d\tau}\right)^n\Theta(\tau)\cdot \tag{22}$$

Their relationships between (20) and (21) are given as follows:

$$\Theta(\tau)=\frac{1}{e^{\lambda\tau}}\frac{d}{d\tau}\int_0^\tau\Theta(\tau)e^{\lambda\tau}d\tau \tag{23}$$

and

$$\Theta(\tau)=\int_0^\tau\frac{d}{d\tau}\Theta(\tau)d\tau+\Theta(0). \tag{24}$$

Generally, the general integral with respect to exponential function is defined as

$${}_a\mathbb{I}^{(1)}_{\tau,e^{\lambda\tau}}\Theta(\tau)=\int_a^\tau\Theta(\tau)\lambda e^{\lambda\tau}d\tau, \tag{25}$$

$$\Theta(\tau)=\frac{1}{e^{\lambda\tau}}\frac{d}{d\tau}\int_a^\tau\Theta(\tau)e^{\lambda\tau}d\tau \tag{26}$$

and

$$\Theta(\tau)=\int_a^\tau\left[\left(\frac{1}{e^{\lambda\tau}}\frac{d}{d\tau}\right)\Theta(\tau)\right]e^{\lambda\tau}d\tau+\Theta(a)=\int_a^\tau\frac{d}{d\tau}\Theta(\tau)d\tau+\Theta(a). \tag{27}$$

The general calculus with respect to logarithmic function

The general derivative with respect to logarithmic function is defined as

$$\mathbb{D}_{\tau,\ln\tau}^{(1)}\Theta(\tau)=\tau\frac{d\Theta(\tau)}{d\tau}. \tag{28}$$

The general integral with respect to logarithmic function is defined as

$${}_0\mathbb{I}_{\tau,\ln\tau}^{(1)}\Theta(\tau)=\int_0^\tau\frac{1}{\tau}\Theta(\tau)d\tau. \tag{29}$$

The general derivative of higher order with respect to logarithmic function is defined as

$$\mathbb{D}_{\tau,\ln\tau}^{(n)}\Theta(\tau)=\left(\tau\frac{d}{d\tau}\right)^n\Theta(\tau)\cdot \tag{30}$$

Their relationships between (28) and (29) can be written as follows:

$$\Theta(\tau)=\left(\tau\frac{d}{d\tau}\right)\int_0^\tau\frac{1}{\tau}\Theta(\tau)d\tau=\tau\frac{d}{d\tau}\int_0^\tau\frac{1}{\tau}\Theta(\tau)d\tau \tag{31}$$

and

$$\Theta(\tau)=\int_0^\tau\left(\tau\frac{d}{d\tau}\right)\Theta(\tau)\frac{1}{\tau}d\tau+\Theta(0)=\int_0^\tau\frac{d}{d\tau}\Theta(\tau)d\tau+\Theta(0). \tag{32}$$

More generally,

$${}_a\mathbb{I}_{\tau,\ln\tau}^{(1)}\Theta(\tau)=\int_a^\tau\frac{1}{\tau}\Theta(\tau)d\tau, \tag{33}$$

$$\Theta(\tau)=\left(\tau\frac{d}{d\tau}\right)\int_a^\tau\frac{1}{\tau}\Theta(\tau)d\tau=\tau\frac{d}{d\tau}\int_a^\tau\frac{1}{\tau}\Theta(\tau)d\tau, \tag{34}$$

and

$$\Theta(\tau)=\int_a^\tau\left(\tau\frac{d}{d\tau}\right)\Theta(\tau)\frac{1}{\tau}d\tau+\Theta(a)=\int_a^\tau\frac{d}{d\tau}\Theta(\tau)d\tau+\Theta(a). \tag{35}$$

The Newton-like dashpot elements containing the general derivatives with respect to another functions

Model 1 The Newton-like dashpot element containing the general derivative with respect to another function is given as

$$\sigma(\tau)=\frac{\gamma}{h^{(1)}(\tau)}\frac{d\varepsilon(\tau)}{d\tau}=\gamma\mathbb{D}_{\tau,h}^{(1)}\varepsilon(\tau), \tag{36}$$

where γ is the material parameter, ε is the strain, σ is the stress, and τ is the time.

When $\sigma(0)=\sigma_0$, (36) can be presented as follows:

$$\sigma_0=\gamma\mathbb{D}_{\tau,h}^{(1)}\varepsilon(\tau) \tag{37}$$

with the solution

$$\varepsilon(\tau)=\int_0^\tau\frac{\sigma_0}{\gamma}h^{(1)}(\tau)d\tau. \tag{38}$$

Model 2 The Newton-like dashpot element containing the general derivative with respect to power-law function is given as [14]

$$\sigma(\tau)=\gamma\mathbb{D}_{\tau,\tau^{\alpha}}^{(1)}\varepsilon(\tau), \tag{39}$$

where γ is the material parameter, ε is the strain, σ is the stress, and τ is the time.
When $\sigma(0)=\sigma_0$, (39) can be written as follows:

$$\sigma_0=\gamma\mathbb{D}_{\tau,\tau^{\alpha}}^{(1)}\varepsilon(\tau) \tag{40}$$

with the solution

$$\varepsilon(\tau)=\frac{\sigma_0}{\gamma}\tau^{\alpha}. \tag{41}$$

Model 3 The Newton-like dashpot element containing the general derivative with respect to exponential function is given as

$$\sigma(\tau)=\gamma\mathbb{D}_{\tau,e^{\lambda\tau}}^{(1)}\varepsilon(\tau), \tag{42}$$

where γ is the material parameter, ε is the strain, σ is the stress, and τ is the time.
When $\sigma(0)=\sigma_0$, (42) can be given as follows:

$$\sigma_0=\gamma\mathbb{D}_{\tau,e^{\lambda\tau}}^{(1)}\varepsilon(\tau) \tag{43}$$

with the solution

$$\varepsilon(\tau)=\frac{\sigma_0}{\gamma}e^{\lambda\tau}. \tag{44}$$

Model 4 The Newton-like dashpot element containing the general derivative with respect to logarithmic function is considered as follows:

$$\sigma(\tau)=\gamma\mathbb{D}_{\tau,\ln\tau}^{(1)}\varepsilon(\tau), \tag{45}$$

where γ is the material parameter, ε is the strain, σ is the stress, and τ is the time.
When $\sigma(0)=\sigma_0$, (45) can be presented as follows:

$$\sigma_0=\gamma\mathbb{D}_{\tau,\ln\tau}^{(1)}\varepsilon(\tau) \tag{46}$$

with the solution

$$\varepsilon(\tau)=\frac{\sigma_0}{\gamma}\ln\tau. \tag{47}$$

Conclusion

In this work, as the extended version of the well-known Newton-Leibniz calculus, we have proposed the general calculi with respect to another functions, containing the general derivative and integral with respect to another functions, such as power-law, exponential, and logarithmic functions for the first time. The Newton-like dashpot models in anomalous viscoelasticity were considered in detail. The formulae could be used to model the complex problems for the solid materials in the deep underground engineering.

Acknowledgement

This work was supported by the financial support of the 333 Project of Jiangsu Province, People's Republic of China (Grant No. BRA2018320), and the Yue-Qi Scholar of the China University of Mining and Technology (Grant No. 102504180004). This paper is devoted to the 110^{th}

Anniversary of the 1909 Founding of China University of Mining and Technology.

Nomenclature

γ - material parameter, [$Pa \cdot s$]
τ - time, [s]
$\varepsilon(\tau)$ - strain, [-]
x - space coordinate, [m]
$\sigma(\tau)$ - stress, [Pa]

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Paper submitted: September 21, 2018
Paper revised: January 13, 2019
Paper accepted: February 18, 2019