A SEMI-IMPLICIT INTEGRATION FACTOR DISCONTINUOUS GALERKIN METHOD FOR THE NON-LINEAR HEAT EQUATION

by

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In this paper, a new discontinuous Galerkin method is employed to study the non-linear heat conduction equation with temperature dependent thermal conductivity. We present practical implementation of the new discontinuous Galerkin scheme with weighted flux averages. The second-order implicit integration factor for time discretization method is applied to the semi discrete form. We obtain the L^2 stability of the discontinuous Galerkin scheme. Numerical examples show that the error estimates are of second order when linear element approximations are applied. The method is applied to the non-linear heat conduction equations with source term.

Key words: *discontinuous Galerkin, implicit integration factor, non-linear heat equation*

Introduction

We consider the following fully non-linear parabolic equation [1]:

$$u_t - \nabla[\kappa(x,u)\nabla u] = f(x,u), \ (x,t) \in \Omega[0,T] \tag{1}$$

with boundary conditions:

$$\kappa(x,t)\nabla u \boldsymbol{n} = 0, \ (x,t) \in \partial\Omega[0,T] \tag{2}$$

and the initial condition:

$$u(x,0) = u_0(x), \ x \in \Omega \tag{3}$$

where Ω is a bounded domain in $\boldsymbol{R}^d$. The diffusion coefficient $\kappa(x, u)$ is assumed to be bounded uniformly from below and from above $0 < \kappa_* \le \kappa(x, u) \le \kappa^*$. We also assume that $f(x,u)$ is uniformly Lipschitz continuous with respect to the second variable.

In this work, we will construct a new discontinuous Galerkin (DG) scheme for eq. (1). The DG method was originally proposed by Reed *et al*. [2] for neutron transport equations. Then the major development of the DG method for hyperbolic conservation laws was

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made with the work of Cockburn and Shu [3]. Their schemes are termed as the Runge-Kutta discontinuous Galerkin (RKDG) method. An important component of the RKDG method is the numerical flux which is borrowed from the finite volume schemes [4]. In the current work, we propose a new weighted average in the DG Schemes and extend it to the non-linear parabolic eq. (1).

The efficient and high order temporal numerical scheme is a challenging task for the fully non-linear parabolic problem. In this paper we develop a more accurate time discretization method which is based on the implicit integration factor (IIF) methods. The new scheme is constructed by expanding the non-linear diffusion coefficient with Taylor expansion but leaving the source term on the present time level. It allows for the second-order time convergence which agrees with the spatial accuracy. In this paper we approximate the matrix-vector product directly by Krylov subspace method as in the paper [5, 6]. Our test problem consists of a set of non-linear heat transfer problem which is very important in engineering areas [7-9].

The weighted DG scheme

In order to describe the flux functions we need to introduce some notations. Denoting u_1^e and u_2^e the trace of $u(x, y)$ on face e taken from within elements Δ_1^e and Δ_2^e, *i. e.* that $u_1^e = \{u(x, y):(x, y) \in \Delta_1^e, (x, y) \to e\}$, $u_2^e = \{u(x, y):(x, y) \in \Delta_2^e, (x, y) \to e\}$. Now we define the jump of a function u on e as $[u] = u_2^e - u_1^e$. We define the DG approximation space:

$$V_h = \left\{ v \in L^2(\Omega) : v|_{\Delta_m} \in P^k(\Delta_m),\ m = 1, 2, \cdots N_e \right\} \tag{4}$$

where $P^k(\Delta_m)$ denotes the space of polynomials of total degree less than or equal to k on element, Δ_m.

The semi-discrete scheme on each computational cell is defined as follows, find $u_h \in V_h$, such that:

$$\int\limits_{\Delta_m} (u_h)_t v \mathrm{d}x\mathrm{d}y + \int\limits_{\Delta_m} \kappa(x, y, u_h) \nabla u_h \nabla v \mathrm{d}x\mathrm{d}y - \int\limits_{\partial\Delta_m} \hat{h}(u_h) v \mathrm{d}x\mathrm{d}y = \int\limits_{\Delta_m} f(x, y, u_h) v \mathrm{d}x\mathrm{d}y \tag{5}$$

The numerical flux on common edge $e \in \partial\Delta_m$ is defined:

$$\hat{h}(u_h) = \frac{\beta_e}{|e|} \alpha_e [u_h] + w_e (\kappa \nabla u_h \mathbf{n})_2^e + (1 - w_e)(\kappa \nabla u_h \mathbf{n})_1^e \tag{6}$$

In the preceding subsection we will prove the stability of the semidiscrete form. We first define the norm:

$$|\,\|u_h\|\,|^2 = \sum_{m=1}^{N_e} \int\limits_{\Delta_m} u_h^2 \mathrm{d}x\mathrm{d}y,\ \|u_h\|^2_{L^2(e)} = \int\limits_e u_h^2 \mathrm{d}s \ \text{ and } \ \|u_h\|^2_{L^2(\Delta_m)} = \int\limits_{\Delta_m} u_h^2 \mathrm{d}x\mathrm{d}y$$

For each $\Delta_m \in T_h$ and $v \in P^k(\Delta_m)$, let e be an edge of Δ_m. Then there exists a positive constant C depending only on k such that the following local inverse inequality holds:

$$\|\nabla v \mathbf{n}\|_{L^2(e)} \le C h_m^{-1/2} \|\nabla v\|_{L^2(\Delta_m)} \tag{7}$$

Assume that β_e *is* large enough in numerical flux (6) and $f(x, y, u) \in L^2[0, t, L^2(\Omega)]$. Then the solution of eq. (5) satisfies:

$$|\,\|u_h(t)\|\,| \le C\left(|\,\|u_h(0)\|\,|^2 + \int\limits_0^t |\,\|f(u_h)\|\,|^2 \mathrm{d}s \right) \tag{8}$$

To prove the correctness of the conclusion (8), one sums up for the equalities (5) over all elements and set $v = u_h$, we have the following identity:

$$\sum_{m=1}^{N_e}\int_{\Delta_m}\left[\frac{1}{2}(u_h^2)_t + \kappa(x,y,u_h)\nabla u_h \nabla u_h\right]\mathrm{d}x\mathrm{d}z + \sum_{e\in\varepsilon_h}\int_e \hat{h}(u_h)[u_h]\mathrm{d}s = \sum_{m=1}^{N_e}\int_{\Delta_m} f(x,y,u_h)u_h dxdy \tag{9}$$

We first give an estimate of the third term on the left hand side of (9). By using the definition of numerical flux (6) and trace inequality (7), we have:

$$\int_e \hat{h}(u_h)[u_h]\mathrm{d}s = \int_e \left\{\frac{\beta_e}{|e|}\alpha_e[u_h]^2 + \left[w_e(\kappa\nabla u_h n)_2^e + (1-w_e)(\kappa\nabla u_h n)_1^e\right][u_h]\right\}\mathrm{d}s \geq$$

$$\geq \int_e \frac{\beta_e}{|e|}\alpha_e[u_h]^2\,\mathrm{d}s - C\frac{\sigma_e}{\sqrt{|e|}}\kappa^*\left(\|\nabla u_h\|^2_{L^2(\Delta_2^e)} + \|\nabla u_h\|^2_{L^2(\Delta_1^e)}\right)^{1/2}\|u_h\|_{L^2(e)}$$

Throughout C is used to denote a generic positive constant, not necessarily the same at each occurrence. By the triangle inequality and Young inequality, we get:

$$\sum_{e\in\varepsilon_h}\int_e \hat{h}(u_h)[u_h]\mathrm{d}s \geq \sum_{e\in\varepsilon_h}\frac{\beta_e}{|e|}\alpha_e\|u_h\|^2_{L^2(e)} - C\kappa^*\sigma_e\left\{\sum_{e\in\varepsilon_h}\frac{1}{|e|}\|u_h\|^2_{L^2(e)}\right\}^{1/2}\left\{\sum_{m=1}^{N_e}2\|\nabla u_h\|^2_{L^2(\Delta_m)}\right\}^{1/2}$$

Replacing the estimate into (9) and by the Holder inequality and Young inequality, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||u_h(t)|||^2 + \sum_{m=1}^{N_e}(1-\delta)\|\kappa\nabla u_h\|^2_{L^2(\Delta_m)} + \sum_{e\in\varepsilon_h}\frac{\alpha_e}{|e|}\left[\beta_e - \frac{C(\kappa^*)^2\sigma_e^2}{\kappa_*^2\delta\alpha_e}\right]\|[u_h]\|^2_{L^2(e)}\,\mathrm{d}s \leq$$

$$\leq \sum_{m=1}^{N_e}\|f(u_h)\|_{L^2(\Delta_m)}\|u_h\|^2_{L2(\Delta_m)} \tag{10}$$

Multiply the eq. (10) and integrate with respect time from 0 to t:

$$|||u_h(t)|||^2 + \int_0^t\sum_{m=1}^{N_e}(1-\delta)\|\kappa\nabla u_h\|^2_{L^2(\Delta_m)}\,\mathrm{d}s + \int_0^t\sum_{e\in\varepsilon_h}\frac{\alpha_e}{|e|}\left[\beta_e - \frac{C(\kappa^*)^2\sigma_e^2}{\kappa_*^2\delta\alpha_e}\right]\|[u_h]\|^2_{L^2(e)}\,\mathrm{d}s \leq$$

$$\leq \int_0^t|||u_h(t)|||^2\,\mathrm{d}s + \int_0^t|||u_h(s)|||^2\mathrm{d}s + |||u_h(0)|||^2 \tag{11}$$

Taking $\delta < 1$ and β_e large enough so that $\beta_e > C(\kappa^*)^2\,\sigma_e^2/(\kappa_*^2\delta\alpha_e)$ we complete the proof of the L^2 stability by the Gronwall inequality.

The integration factor time discretization

Now we are ready to introduce the fully discretization of the semi-discrete form (5). The local systems (5) on each element are assembling to get the global non-linear ODE:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = A(U)U + F(U) \tag{12}$$

where $U = (U_1^T, U_2^T, ..., U_N^T)^T$ being the freedoms of u on every element, $A(U)$ is the non-linear global sparse matrix and $F(U) = (f_1^T, ..., f_N^T)^T$.

In order to apply the integration factor method, the non-linear matrix $A(U)$ need be approximated by the values of U at previous time levels. First we consider the first-order integration factor method. The non-linear diffusion coefficients are evaluated at previous time level, *i. e.* that $\kappa(x, u^n)$. We get following ODE system:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = A(U^n)U + F(U) \tag{13}$$

The matrix $A(U^n)$ is a constant matrix. Multiply by the integration factor $\mathrm{e}^{-A(Un)t}$ and integrate over one time step from t^n to t^{n+1} to obtain:

$$U^{n+1} = \mathrm{e}^{A(U^n)\Delta t}U^n + \mathrm{e}^{A(U^n)\Delta t}\int_0^{\Delta t} \mathrm{e}^{A(U^n)\tau} F[U(t^n+\tau)]\mathrm{d}\tau \tag{14}$$

The integrand is approximated by trapezium method and the semi-implicit integration factor scheme:

$$U^{n+1} = \mathrm{e}^{A(U^n)\Delta t}\left[U^n + \frac{\Delta t}{2}F\left(U^n\right)\right] + \frac{\Delta t}{2}F\left(U^{n+1}\right) \tag{15}$$

To get more accurte time discretization method, we expand the non-linear diffusion coefficient with Taylor expansion:

$$\kappa\left(x,u\right) = \kappa\left(x,u^n\right) + \kappa_u\left(x,u^n\right)\dot{u}^n\left(t-t^n\right) + O\left[\left(t-t^n\right)^2\right] \tag{16}$$

We approximate $\dot{u}^n$ with $\dot{u}^n = u^n - u^{n-1}/\Delta t$ and ignore the $O[(t - t^n)^2]$ term to get:

$$\kappa\left(x,u\right) = a + b\left(t-t^n\right) \tag{17}$$

where a and b are constants.

With the second-order approximation of $\kappa(x, u)$ at t^n, we get the new global ODE systems:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = [A + B(t-t^n)]U + F(U) \tag{18}$$

The constant matrixes A and B can be obtained from the constants a and b a in eq. (17). Multiply eq. (18) by the integration factor:

$$\mathrm{e}^{-\int_0^{t-t^n}(A+B\tau)\mathrm{d}\tau}$$

and integrate over one time step from t^n to t^{n+1} to obtain:

$$U^{n+1} = \mathrm{e}^{\int_0^{\Delta t}(A+B\tau)\mathrm{d}\tau}U^n + \mathrm{e}^{\int_0^{\Delta t}(A+B\tau)\mathrm{d}\tau}\int_{t^n}^{t^{n+1}} \mathrm{e}^{-\int_0^{t-t^n}(A+B\tau)\mathrm{d}\tau}F\left(U\right)\mathrm{d}t \tag{19}$$

Applying the second order implicit integration factor scheme:

$$U^{n+1} = \mathrm{e}^{\left(A+\frac{B}{2}\Delta t\right)\Delta t}\left[U^n + \frac{\Delta t}{2}F\left(U^n\right)\right] + \frac{\Delta t}{2}F\left(U^{n+1}\right) \tag{20}$$

The second order semi-implicit IF scheme can be written:

$$U^{n+1} = \beta^n V_{m+1}^n e^{\Delta t \bar{H}_{m+1}^n} e_1 + \frac{\Delta t}{2}F\left(U^{n+1}\right) \tag{21}$$

After the obtainment of $\beta^n V_{m+1}^n e^{\Delta t \bar{H}_{m+1}^n} e_1$ with Krylov subspace, we can solve the non-linear on each element Δ_m.

Numerical test

Example 1. Consider the following non-linear parabolic equation:

$$u_t = \nabla(u\nabla u) = f(x,y,t),\ \ (x,y)\in[0,1]^2$$

with the homogeneous Neumann boundary condition. We give the errors for three different levels of meshes in tab. 1 at t = 0.1. From this table, we can obtain the second-order accuracy of our DG scheme.

Table 1. The L_2, L_∞ error and order of convergence for second-order scheme at t = 0.1

CFL	L_2	Order	L_∞	Order
0.2	4.48e-4	–	7.32e-4	–
0.1	1.08e-4	2.05	1.86e-4	1.98
0.05	1.97e-5	2.45	4.61e-5	2.01

Example 2. The idealized dimensionless equation with the temperature dependent thermal conductivity can be written:

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \nabla\left(T^3\nabla T\right) = 1 - T^4 \tag{24}$$

The left boundary condition is defined:

$$\frac{1}{4}T + \frac{\kappa(T)}{6}\frac{\partial T}{\partial n} = 10,\ \ 0<y<8,\ \ \frac{\partial T}{\partial n} = 0,\ \ 8<y<10$$

The right boundary condition:

$$\frac{1}{4}T + \frac{\kappa(T)}{6}\frac{\partial T}{\partial n} = 0,\ \ 0<y<8,\ \ \frac{\partial T}{\partial n} = 0,\ \ 8<y<10$$

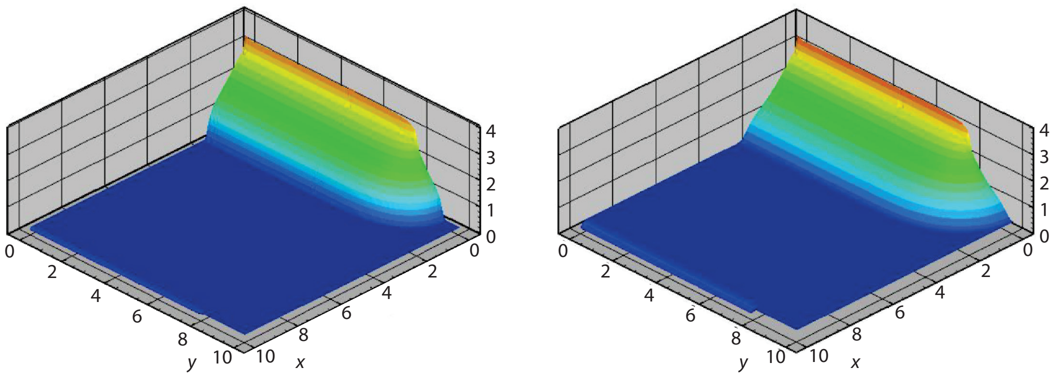

Figure 1. Conurs of temperature of *Example 2* at t = 0.25 and t = 0.5

This model shows the conduction of heat flow which is introduced from the left side, and flows out on the right side. The numerical results at t = 0.25 and t = 0.5 are plotted in fig. 1. It is seen that the numerical results agree well with those [1]. The numerical results proved that our DG scheme can effectively capture the steep temperature profiles. And also our scheme can preserve positivity of solution on triangle meshes without flux limiters

Conclusion

We have presented an efficient numerical method for the solution of non-linear parabolic problems, which is based on the space discretization by the weighted discontinuous Galerkin finite element method and the second-order implicit integration factor for time discretization. The obtained results confirm that our DG method is a powerful and reliable method for the numerical solution of non-linear diffusion problems.

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