

LEGENDRE WAVELET OPERATIONAL MATRIX METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS IN SOME SPECIAL CONDITIONS

Aydin Secer¹, Selvi Altun¹ and Mustafa Bayram²

¹Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey

²Faculty of Engineering and Architecture, Istanbul Gelisim University, Istanbul, Turkey

Corresponding author; E-mail: aydinsecer@gmail.com

This paper proposes a new technique which rests upon Legendre wavelets for solving linear and non-linear forms of fractional order initial and boundary value problems. In some particular circumstances, a new operational matrix of fractional derivative is generated by utilizing some significant properties of wavelets and orthogonal polynomials. We approached the solution in a finite series with respect to Legendre wavelets and then by using these operational matrices, we reduced the FDEs into a system of algebraic equations. Finally, the introduced technique is tested on several illustrative examples. The obtained results demonstrate that this technique is a very impressive and applicable mathematical tool for solving FDEs.

Key words: *Operational matrix, Legendre wavelets, Caputo fractional derivative, Fractional differential equations.*

1. Introduction

Fractional calculus is the study of any real-order or complex-order derivative and integral composed of combining and extending the concepts of multiple integral and integer order derivative.

Let $D = \frac{d}{dt}$ be a differential operator and n be a positive integer. It is well known that, the meaning of the $D^n x(t)$ is the n^{th} derivative of the function $x(t)$. But if n is not a positive integer, it is difficult to comment the meaning of the $D^{-\alpha}$ or D^α for $\text{Re}(\alpha) > 0$. A variety of definitions that satisfy the idea of fractional derivative have been found by several great mathematician. But Riemann-Liouville and Caputo fractional derivatives are most commonly utilized definements in the world of fractional calculus.

For three centuries, analysis of fractional calculus has been restraint to the discipline of pure theoretical mathematics. Over the past several decades, FDEs have considerable attention, because of their suitability for the explanation of a great many scientific phenomena such as thermal science especially bioheat transfer, fluid mechanics; image processing, fluid-dynamic traffic model, earthquake engineering, damping laws, biomedical engineering, diffusion process, solid mechanics, colored noise, control theory, viscoelasticity, electromagnetic etc. We can list some well-established techniques for the numerical solutions of FDEs as operational matrix methods improved in [12-14], [17-21] by using Bernstein, Genocchi, Legendre, Jacobi polynomials and Genocchi, Legendre, Chebyshev wavelets, perturbation-iteration method [16], sinc-Galerkin method [29], computational matrix method [15], homotopy analysis method [31], variational iteration method, a hybrid technique [30], Adomian decomposition method [23].

The orthogonal functions and polynomial series are very important field in science and engineering. They are basis of several numerical methods developed for the solution of differential equations. The reason is that the use of orthogonal polynomials is easy. Because they have good convergence properties and they properly represent the weight distribution of a function on a definite network. Mentioned equations are solved by truncating series of orthogonal basis functions. Block-pulse functions, Legendre, Hermite, Jacobi, Laguerre and Chebyshev polynomials are the most commonly utilized among these functions. What makes these functions important is that they permit the

undertaking problem to be reduced to system of algebraic equations and the approximation of analytic functions.

Wavelet theory is very significant in science, engineering and technology and in recent years wavelets have achieved to attract an enormous attention in many fields of investigation, such as spectroscopy, signal analysis, feature detection in earth science, time-frequency analysis, image manipulation etc. Many scholars have contributed to the development of wavelets. Especially, Daubechies, Belkin, Meyer and Mallat are some of them. Thanks to their contribution, there has been a substantial increment in the number of studies on wavelets. There are a wide variety of wavelet functions such as Daubechies, Haar, Laguerre, Legendre, Shannon, Lagrange, Hermitian and Chebyshev wavelets available. Among them, we chose Legendre wavelets in this study because of their orthonormality and explicitness. Many applications of Legendre wavelets can be viewed from [1-6]. In addition, in these papers [26,27], applications of Legendre wavelets related to thermal sciences can be reached.

This paper touches on the numerical analysis of FDEs by utilizing Legendre wavelet operational matrix method (LWOMM). First of all, we start by presenting some requirement definitions, notations and mathematical preliminaries of fractional calculus theory and Legendre wavelets in Section 2. Then we continue by introducing function approximations and the operational matrix of fractional derivative in some particular circumstances. Section 3 and Section 4 generalizes these operational matrices to fractional differential equations of the linear and non-linear forms. In Section 5, we implement the introduced method to illustrative examples. Finally, we finished the paper with a conclusion part in Section 6.

2. Preliminaries and notations

It is necessary to know some mathematical definitions to understand the applications of fractional analysis, shifted Legendre polynomials and wavelets required for this paper. Some of these definitions and theorems are presented below.

2.1. The fractional derivative in the Caputo sense

Definition 1. The Caputo fractional derivative is defined as [7]

$${}^c D^\beta x(t) = \frac{1}{\Gamma(i-\beta)} \int_0^t \frac{x^{(i)}(\zeta)}{(t-\zeta)^{\beta+1-i}} d\zeta, \quad i-1 < \beta \leq i, \quad i \in \mathbb{N} \quad (1)$$

This derivative has the following characteristics

$${}^c D^\alpha S = 0 \quad (2)$$

where, S is a constant.

$$D^\beta t^\omega = \begin{cases} 0, & \omega \in \mathbb{N}_0 \text{ and } \omega < \lceil \beta \rceil \\ \frac{\Gamma(\omega+1)}{\Gamma(\omega+1-\beta)} t^{\omega-\beta}, & \omega \in \mathbb{N}_0 \text{ and } \omega \geq \lceil \beta \rceil \text{ or } \omega \notin \mathbb{N} \text{ and } \omega > \lfloor \beta \rfloor \end{cases} \quad (3)$$

Here, $\lfloor \beta \rfloor$ implies largest integer less than or equal to β and $\lceil \beta \rceil$ indicates the smallest integer greater than or equal to β .

2.2. Legendre wavelets

There are a wide variety of wavelet functions such as Daubechies, Haar, Laguerre, Legendre, Shannon, Lagrange, Hermitian and Chebyshev wavelets available. Among them, we chose Legendre wavelets in this study because of their orthonormality and explicitness. This wavelet function can be defined as follows:

Define

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k+1}{2}} \sqrt{m+\frac{1}{2}} P_m(2^k t - n), & \frac{n}{2^k} \leq t \leq \frac{n+1}{2^k} \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

where $n = 0, 1, \dots, (2^k - 1)$; $m = 0, 1, \dots, M$. The coefficient $\sqrt{\frac{m+1}{2}}$ is for orthonormality.

Shifted Legendre polynomials $P_m(t)$ are defined as follows [1]:

$$P_m(t) = \sum_{k=0}^m (-1)^{m+k} \frac{(m+k)!}{(m-k)! (k!)^2} t^k \quad 0 \leq t \leq 1 \quad (5)$$

2.3. Function approximation

A function $x(t)$ defined over $[0, 1]$ can be expanded in the terms of Legendre wavelet as

$$x(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} w_{nm} \psi_{nm}(t) \quad (6)$$

where $w_{nm} = (x(t), \psi_{nm}(t))$ in which (\cdot, \cdot) indicates the inner product. Truncating infinite series in (6), we get

$$x(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^M w_{nm} \psi_{nm}(t) = W^T \psi(t) \quad (7)$$

where W and $\psi(t)$ are matrices given by [1]

$$W = \begin{bmatrix} w_{0,0}, w_{0,1}, \dots, w_{0,M}, \dots, w_{2^k-1,0}, w_{2^k-1,1}, \dots, w_{2^k-1,M} \end{bmatrix}^T \quad (8)$$

$$\psi = \begin{bmatrix} \psi_{0,0}, \psi_{0,1}, \dots, \psi_{0,M}, \dots, \psi_{2^k-1,0}, \psi_{2^k-1,1}, \dots, \psi_{2^k-1,M} \end{bmatrix}^T$$

2.4. The operational matrix of derivative and fractional derivative

Theorem 1. Let $\psi(t)$ be the Legendre wavelets vector presented in Equation (4), then $\psi(t)$ can be written as [1]

$$\frac{d\psi(t)}{dt} \cong D\psi(t) \quad (9)$$

where D is the $2^k(M+1)$ operational matrix of derivative expressed as:

$$D = \begin{bmatrix} X & O & \cdots & O \\ O & X & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & X \end{bmatrix} \quad (10)$$

where X is an $(M+1)(M+1)$ matrix and its (i, j) th element is expressed as [1]:

$$X_{i,j} = \begin{cases} 2^{k+1} \sqrt{(2i-1)(2j-1)}, & i = 2, \dots, (M+1), j = 1, \dots, i-1 \text{ and } (i+j) \text{ odd} \\ 0, & \text{otherwise} \end{cases} \quad (11)$$

Corollary 1. If we use Equation (9), then the operational matrix for the n th derivative can be defined as

$$\frac{d^n \psi(t)}{dt^n} \cong D^n \psi(t) \quad (12)$$

where D^n is the n th power of matrix D [14].

Lemma 1. Let $\psi(t)$ be the Legendre wavelets vector given in Equation (4) and assume that $k = 0$ then

$$D^\beta \psi_r(t) = 0, \quad r = 0, 1, \dots, \lceil \beta \rceil - 1, \quad \beta > 0 \quad (13)$$

Proof. Please see [28].

Theorem 2. Let $\psi(t)$ be the Legendre wavelets vector given in Equation (5). Assume that $k = 0$ and $\beta > 0$, then

$$D^\beta \psi(t) \cong D^{(\beta)} \psi(t) \quad (14)$$

where $D^{(\beta)}$ is the $(M+1) \times (M+1)$ the operational matrix of fractional derivative of the order $\beta > 0$, $N-1 < \beta \leq N$ in the Caputo sense and is expressed as:

$$D^{(\beta)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{h=\lceil \beta \rceil}^{\lceil \beta \rceil} \varpi_{\lceil \beta \rceil, 0, h} & \sum_{h=\lceil \beta \rceil}^{\lceil \beta \rceil} \varpi_{\lceil \beta \rceil, 1, h} & \cdots & \sum_{h=\lceil \beta \rceil}^{\lceil \beta \rceil} \varpi_{\lceil \beta \rceil, m, h} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{h=\lceil \beta \rceil}^r \varpi_{r, 0, h} & \sum_{h=\lceil \beta \rceil}^r \varpi_{r, 1, h} & \cdots & \sum_{h=\lceil \beta \rceil}^r \varpi_{r, m, h} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{h=\lceil \beta \rceil}^m \varpi_{m, 0, h} & \sum_{h=\lceil \beta \rceil}^m \varpi_{m, 1, h} & \cdots & \sum_{h=\lceil \beta \rceil}^m \varpi_{m, m, h} \end{pmatrix} \quad (15)$$

where $\varpi_{r,s,h}$ is presented by

$$\varpi_{r,s,h} = \sqrt{2r+1} \sqrt{2s+1} \sum_{p=0}^s \frac{(-1)^{r+s+h+p} (r+h)! (s+p)!}{(r-h)! h! \Gamma(h-\beta+1) (s-p)! (p!)^2 (h+p-\beta+1)}. \quad (16)$$

Take in consideration in $D^{(\beta)}$, the first $\lceil \beta \rceil$ rows are all zero.

Proof. Please see [28].

3. Solving linear fractional differential equations

This section introduces the application of LWOM of fractional derivative for solving linear FDEs. Consider the following equation

$$D^\beta x(t) = v_o(t) D^{\zeta_0} x(t) + \dots + v_{k-1}(t) D^{\zeta_{k-1}} x(t) + v_k(t) x(t) + f(t) \quad (17)$$

with these initial conditions

$$x(t_0) = x_0, \quad \frac{dx}{dt}(t_0) = x_1, \quad \frac{d^2x}{dt^2}(t_0) = x_2, \dots, \frac{d^{n-1}x}{dt^{n-1}}(t_0) = x_{n-1} \quad (18)$$

where $v_0(t), v_1(t), \dots, v_k(t)$ can be any function and $0 < \zeta_0 < \zeta_1 < \dots < \zeta_{k-1} < \beta, n < \beta \leq n+1$, and D^β indicates the Caputo fractional derivative of order β .

First approximating $x(t), f(t)$ and $v_0(t), v_1(t), \dots, v_k(t)$ by the Legendre wavelets, then we obtain

$$x(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^M w_{n,m} \psi_{n,m} = W^T \psi(t) \quad (19)$$

$$f(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^M f_{n,m} \psi_{n,m} = F^T \psi(t) \quad (20)$$

$$v_0(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^M v_{0n,m} \psi_{n,m} = V_0^T \psi(t), \dots, v_k(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^M v_{kn,m} \psi_{n,m} = V_k^T \psi(t) \quad (21)$$

where F and V_i ($i = 0, 1, \dots, k$) are known vectors but W is an unknown vector and $\psi(t)$ is the vector given in Equation (4). By utilizing Equation (14) and (19), we get

$$D^\beta x(t) \cong W^T D^\beta \psi(t) \cong W^T D^{(\beta)} \psi(t) \quad (22)$$

$$D^{\zeta_i} x(t) \cong W^T D^{\zeta_i} \psi(t) \cong W^T D^{(\zeta_i)} \psi(t), \quad i = 0, \dots, (k-1). \quad (23)$$

Substituting Equations (19)-(23) in Equation (17) the residual $R(t)$ can be expressed as

$$R(t) \cong W^T D^{(\beta)} \psi(t) - (V_0^T \psi(t)) (W^T D^{(\zeta_0)} \psi(t)) - \dots - (V_{k-1}^T \psi(t)) (W^T D^{(\zeta_{k-1})} \psi(t)) - (V_k^T \psi(t)) (W^T D^{(\zeta_k)} \psi(t)) - (V_{k+1}^T \psi(t)) (F^T \psi(t)). \quad (24)$$

We get $2^k(M+1) - n$ linear equations by employing

$$\langle R(t), \psi_r(t) \rangle = \int_0^1 \psi_r(t) R(t) dt = 0, \quad r = 0, \dots, 2^k(M+1) - n. \quad (25)$$

If we substitute Equation (19) in Equation (18) then we have

$$x(t_0) = W^T \psi(t_0) = x_0, \quad \frac{dx}{dt}(t_0) = W^T D \psi(t_0) = x_1$$

$$\frac{d^2 x}{dt^2}(t_0) = W^T D^2 \psi(t_0) = x_2, \dots, \frac{d^n x}{dt^n}(t_0) = W^T D^{(n)} \psi(t_0) = x_n \quad (26)$$

$2^k(M+1)$ set of linear equations are obtained Equations (25) and (26). We can solve these linear equations for unknown coefficients of the vector W . Accordingly, $x(t)$ presented in Equation (18) can be computed.

4. Solving non-linear fractional differential equations

This section introduces the application of LWOM of fractional derivative for solving non-linear FDEs. Consider the following equation

$$D^\beta x(t) = F(t, x(t), D^{\zeta_1} x(t), \dots, D^{\zeta_k} x(t)) \quad (27)$$

with these initial conditions

$$x(t_0) = x_0, \quad \frac{dx}{dt}(t_0) = x_1, \quad \frac{d^2 x}{dt^2}(t_0) = x_2, \dots, \frac{d^{n-1} x}{dt^{n-1}}(t_0) = x_{n-1} \quad (28)$$

where $n < \beta \leq n+1$, $0 < \zeta_1 < \zeta_2 < \dots < \zeta_k < \beta$ and D^β indicates the Caputo fractional derivative of order β .

First approximating $x(t)$, $D^\beta x(t)$ and for $i=1,2,\dots,k$ $D^{\zeta_i} x(t)$ by the Legendre wavelets as Equations (19), (22) and (23) respectively and substituting these equations in Equation (26), then we obtain

$$W^T D^{(\beta)} \psi(t) \cong F\left(t, W^T \psi(t), W^T D^{(\zeta_1)} \psi(t), \dots, W^T D^{(\zeta_k)} \psi(t)\right). \quad (29)$$

Also, if we substitute Equation (20) in Equation (29) then we have

$$x(t_0) = W^T \psi(t_0) = x_0, \quad \frac{dx}{dt}(t_0) = W^T D \psi(t_0) = x_1 \quad (30)$$

$$\frac{d^2 x}{dt^2}(t_0) = W^T D^2 \psi(t_0) = x_2, \dots, \frac{d^n x}{dt^n}(t_0) = W^T D^{(n)} \psi(t_0) = x_n.$$

First collocating Equation (29) at $2^k(M+1)-n$ points, then we can obtain the solution $x(t)$. We should use the first $2^k(M+1)-n$ roots of shifted Legendre $P_{2^k(M+1)}(t)$ to get a better result. Utilizing these equations together with Equation (30), then we have $2^k(M+1)$ non-linear equations. We can solve these non-linear equations for unknown coefficients of the vector W . Accordingly, $x(t)$ presented in Equation (27) can be computed.

5. Illustrative examples

Example 1. We first consider the following FDE of the linear form [20]

$$4(t+1)D^{\frac{5}{2}}x(t) + 4D^{\frac{3}{2}}x(t) + \frac{1}{\sqrt{t+1}}x(t) = \sqrt{t} + \sqrt{\pi}$$

subject to

$$x(0) = \sqrt{\pi}, \quad \frac{dx}{dt}(0) = \frac{\sqrt{\pi}}{2}, \quad x(1) = \sqrt{2\pi}.$$

The exact solution of the previous system is

$$x(t) = \sqrt{\pi(t+1)}.$$

If we implement the method illustrated in Section 3 with $M=3$, $k=0$ then we get

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 \end{pmatrix}, \quad D^{\left(\frac{3}{2}\right)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{16\sqrt{5}}{\sqrt{\pi}} & \frac{16\sqrt{5}\sqrt{3}}{5\sqrt{\pi}} & -\frac{16}{7\sqrt{\pi}} & \frac{16\sqrt{7}\sqrt{5}}{105\sqrt{\pi}} \\ -\frac{16\sqrt{7}}{\sqrt{\pi}} & \frac{80\sqrt{7}\sqrt{3}}{7\sqrt{\pi}} & \frac{16\sqrt{7}\sqrt{5}}{3\sqrt{\pi}} & \frac{-80}{11\sqrt{\pi}} \end{pmatrix},$$

$$D^{\left(\frac{5}{2}\right)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{160\sqrt{7}}{\sqrt{\pi}} & \frac{32\sqrt{7}\sqrt{3}}{\sqrt{\pi}} & -\frac{32\sqrt{7}\sqrt{5}}{7\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} \end{pmatrix}$$

The approximate solution obtained by using LWOMM and the exact solution are displayed in Table 1.

Table 1. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for Example 1.

t	$x(t)$	$x_{LWOMM}(t)$	Absolute Error
0.0	1.772455923	1.772453851	$0.2072 \cdot 10^{-5}$
0.1	1.858967455	1.859556019	-0.000588564
0.2	1.941628183	1.942759272	-0.001131089
0.3	2.020910686	2.022385308	-0.001474622
0.4	2.097198131	2.098755828	-0.001557697
0.5	2.170806302	2.172192531	-0.001386229
0.6	2.241999108	2.243017118	-0.001018010
0.7	2.310999784	2.311551286	-0.000551502
0.8	2.377999159	2.378116734	-0.000117575
0.9	2.443161886	2.443035165	0.000126721
1.0	2.506631205	2.506628274	$0.2931 \cdot 10^{-5}$

Example 2. Consider the following fractional Bagley-Torvik differential equation of the linear form with the initial conditions [13]

$$\frac{d^2 x(t)}{dt^2} + D^{\frac{3}{2}} x(t) + x(t) = 1 + t$$

$$x(0) = 0, \quad \frac{dx}{dt}(0) = 1.$$

The exact solution of this problem is known as

$$x(t) = 1 + t.$$

If we apply the method introduced in Section 3 with $M = 2$, $k = 0$ then we get

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 \end{pmatrix}, \quad D^{\left(\frac{3}{2}\right)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{16\sqrt{5}}{\sqrt{\pi}} & \frac{16\sqrt{3}\sqrt{5}}{5\sqrt{\pi}} & -\frac{16}{7\sqrt{\pi}} \end{pmatrix}$$

The numerical results which are attained by utilizing the LWOMM are shown in Table 2.

Table 2. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for Example 2.

t	$x(t)$	$x_{LWOMM}(t)$	Absolute Error
0.0	1.000000000	0.999999999	$0.1 \cdot 10^{-8}$
0.1	1.100000000	1.099999999	$0.1 \cdot 10^{-8}$
0.2	1.200000000	1.199999999	$0.1 \cdot 10^{-8}$
0.3	1.300000000	1.299999999	$0.1 \cdot 10^{-8}$
0.4	1.400000000	1.399999999	$0.1 \cdot 10^{-8}$
0.5	1.500000000	1.499999999	$0.1 \cdot 10^{-8}$
0.6	1.600000000	1.599999999	$0.1 \cdot 10^{-8}$
0.7	1.700000000	1.699999999	$0.1 \cdot 10^{-8}$
0.8	1.800000000	1.799999999	$0.1 \cdot 10^{-8}$
0.9	1.900000000	1.899999999	$0.1 \cdot 10^{-8}$
1.0	2.000000000	1.999999999	$0.1 \cdot 10^{-8}$

Example 3. Consider the following FDE of the non-linear form with the initial conditions [21]

$$\frac{d^3x(t)}{dt^3} + D^{\frac{5}{2}}x(t) + x^2(t) = t^4$$

$$x(0) = 0, \quad \frac{dx}{dt}(0) = 0, \quad \frac{d^2x}{dt^2}(0) = 2,$$

The exact solution of the previous system is

$$x(t) = t^2.$$

If we apply the method introduced in Section 4 with $M = 3$, $k = 0$ then we get

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 & 0 \\ 0 & 4\sqrt{15}\sqrt{35} & 0 & 0 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8\sqrt{3}\sqrt{15}\sqrt{35} & 0 & 0 & 0 \end{pmatrix}, \quad D^{\left(\frac{5}{2}\right)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{160\sqrt{7}}{\sqrt{\pi}} & \frac{32\sqrt{7}\sqrt{3}}{\sqrt{\pi}} & -\frac{32\sqrt{7}\sqrt{5}}{7\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} \end{pmatrix}$$

The numerical results of the exact solution and the approximate solution are displayed in Table 3.

Table 3. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for Example 3.

t	$x(t)$	$x_{LWOMM}(t)$	Absolute Error
0.0	0.00	0.8410661107 10^{-12}	-0.8410661107 10^{-12}
0.1	0.01	0.01000000006	-0.6 10^{-10}
0.2	0.04	0.04000000011	-0.11 10^{-9}
0.3	0.09	0.09000000014	-0.14 10^{-9}
0.4	0.16	0.16000000020	-0.2 10^{-9}
0.5	0.25	0.25000000020	-0.2 10^{-9}
0.6	0.36	0.36000000010	-0.1 10^{-9}
0.7	0.49	0.49000000010	-0.1 10^{-9}
0.8	0.64	0.64000000010	-0.1 10^{-9}
0.9	0.81	0.81000000000	0
1.0	1.00	0.99999999999	0.1 10^{-9}

Example 4. Consider the following FDE of the non-linear form with the boundary conditions [19]

$$\frac{d^2x(t)}{dt^2} + \Gamma\left(\frac{4}{5}\right)t^{\frac{6}{5}}D^{\frac{6}{5}}x(t) + \frac{11}{9}\Gamma\left(\frac{5}{6}\right)t^{\frac{1}{6}}D^{\frac{1}{6}}x(t) - \left(\frac{dx(t)}{dt}\right)^2 = 2 + \frac{1}{10}t^2$$

subject to

$$x(0) = 1, \quad x(1) = 2.$$

The exact solution of the previous system is

$$x(t) = 1 + t^2.$$

If we apply the method illustrated in Section 4 with $M = 2$, $k = 0$ then we get

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}, \quad D^{(\frac{1}{6})} = \begin{pmatrix} 0 & 0 & 0 \\ 2.008717540 & 1.023294363 & -0.05743771053 \\ -2.288145774 & 0.5858729181 & 1.235012279 \end{pmatrix}$$

$$D^{(\frac{6}{5})} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16.00534636 & 7.920592311 & -0.5381810900 \end{pmatrix}$$

The approximate solution obtained by using introduced method and the exact solution are shown in Table 4.

Table 4. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for Example 4.

t	$x(t)$	$x_{LWOMM}(t)$	Absolute Error
0.0	1.00	0.9999999998	$0.2 \cdot 10^{-9}$
0.1	1.01	1.009995734	$0.4266 \cdot 10^{-5}$
0.2	1.04	1.039992417	$0.7583 \cdot 10^{-5}$
0.3	1.09	1.089990047	$0.9953 \cdot 10^{-5}$
0.4	1.16	1.159988625	0.000011375
0.5	1.25	1.249988151	0.000011849
0.6	1.36	1.359988625	0.000011375
0.7	1.49	1.489990047	$0.9953 \cdot 10^{-5}$
0.8	1.64	1.639992416	$0.7584 \cdot 10^{-5}$
0.9	1.81	1.809995734	$0.4266 \cdot 10^{-5}$
1.0	2.00	2.000000000	0.000000000

Example 5. Consider the following FDE of the non-linear form with the initial conditions

$$D^{1.3}x(t) + x^2(t) = \frac{20}{7} \frac{t^{0.7}}{\Gamma(0.7)} + t^4$$

subject to

$$x(0) = 0, \quad \frac{dx}{dt}(0) = 0.$$

The exact solution of the previous system is

$$x(t) = t^2.$$

If we apply the method illustrated in Section 4 with $M = 2$, $k = 0$ then we get

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}, \quad D^{(1.3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 17.37179094 & 7.800806362 & -0.8165511655 \end{pmatrix}$$

The numerical results of the approximate solution and the exact solution are shown in Table 5.

Table 5. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for Example 5.

t	$x(t)$	$x_{LWOMM}(t)$	Absolute Error
0.0	0.00	$-0.1 \cdot 10^{-9}$	$0.1 \cdot 10^{-9}$
0.1	0.01	0.01009304147	0.00009304147
0.2	0.04	0.04037216618	0.00037216618
0.3	0.09	0.09083737403	0.00083737403
0.4	0.16	0.16148866500	0.00148866500
0.5	0.25	0.25232603910	0.00232603910
0.6	0.36	0.36334949640	0.00334949640
0.7	0.49	0.49455903680	0.00455903680
0.8	0.64	0.64595466040	0.00595466040
0.9	0.81	0.81753636710	0.00753636710
1.0	1.00	1.00930415700	0.00930415700

6. Conclusion

In this study, we first introduced LWOMM shortly, and then utilized it to attain approximate solutions of FDEs. These approximate solutions were clearly written and were matched against the exact solutions of the presented problems. A very effective algorithm is also formulated to obtain the numerical solution of these problems on the Maple. We produced all graphical representations and numerical results via Maple. The results illustrate that the LWOMM can solve FDEs efficaciously and simply.

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