APPLICATION OF DGJ METHOD FOR SOLVING NONLINEAR LOCAL FRACTIONAL HEAT EQUATIONS

by

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In this paper, the initial value problem for a new nonlinear local fractional heat equation is considered. The fractional complex transform method and the DGJ decomposition method are used to solve the problem, and the approximate analytical solutions are also obtained.

Key words: nonlinear local fractional heat equation, fractional complex transform method, the DGJ decomposition method

Introduction

On a continuous medium, the following classical heat equation

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \] (1)

describes the evolution in time of the density \( u \) of some quantity such as heat, chemical concentration, etc, where \( k \) is the thermal diffusivity coefficient [1]

But for fractal media, Eq. (1) has to be modified. The local fractional calculus have been as an alternative approach proposed to study the fractal heat conduction problem [2-7]. The linear heat equation involving the local fractional derivative operators have been intensively studied over the last decade. Recent, several authors have investigated the nonlinear local fractional heat equation, which can be used to model the anomalous diffusion on a fractal media [8-13]. Motivated by these results, our interest here is to solve the following nonlinear local fractional heat equation, given as follows:

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = d(x,t) \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} + N(u) \] (2)

with the conditions

\[ u(x,0) = \varphi(x), \] (3)

where \( \frac{\partial^\alpha u}{\partial t^\alpha} \) and \( \frac{\partial^{2\beta} u}{\partial x^{2\beta}} \) are the local fractional derivatives [7-10] \( 0 < \alpha \leq 1, 0 < \beta \leq 1 \), \( d(x,t), \varphi(x) \) and \( N(u) \) are the given functions.

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The solution of the local fractional differential equation is much involved. Some numerical and analytical methods for solving local fractional differential equations were presented [5-10,16-20], such as the fractional complex transform method and the DGJ method. Fractional complex transform can convert the fractional differential equation into the ordinary differential equation [11-12]. The DGJ method (DGJM) was proposed by Daftardar-Gejji, Varsha, and Hossein Jafari in [13-18]. DGJM is a powerful tool for obtaining solutions of nonlinear problems. Recently, Daftardar-Gejji, Varsha and Sachin Bhalekar[15] found the exact solution and approximate solution of the fractional differential equations by using DGJM.

The main aim of this work is to solve the problems (1)-(2) by using the complex transform and DGJM. The structure of this paper is as follows. In Section 2, we introduce the basic concept of the local fractional derivative and the ideal of DGJM. In Section 3, we solve the Eq. (1) by using fractional complex transform and DGJM. Finally, the conclusion is given in Section 4.

Preliminaries

Local fractional derivative

In this section, we recall some definitions and properties of the local fractional derivative (for more details, see [8-13]).

Definition 1. For arbitrary $\varepsilon > 0$, we give the relation as follows:
\[ |f(x) - f(x_0)| < \varepsilon^\alpha \] (4)
with $|x - x_0| < \delta$. Then $f(x)$ is so-called local fractional continuous at $x_0$, which is denoted by
\[ \lim_{x \to x_0} f(x) = f(x_0). \] If $f(x)$ is so-called local fractional continuous on the interval $(a, b)$, an it is denoted as
\[ f(x) \in C_\alpha (a, b). \]

Definition 2. Let $f(x) \in C_\alpha (a, b)$. In fractal space, the local fractional derivative of $f(x)$ of order $\alpha$ at the point $x = x_0$ is given as
\[ D^\alpha_x f(x_0) = \left. \frac{d^\alpha}{dx^\alpha} f(x) \right|_{x = x_0} = f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \] (5)
where $\Delta(f(x) - f(x_0)) \equiv \Gamma(\alpha + 1)(f(x) - f(x_0))$.

Definition 3. The local fractional partial derivative of high order is defined as
\[ \frac{\partial^k u(x,t)}{\partial x^{k\alpha}} = \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha} u(x,t), \] (6)
The following property
\[ \frac{\partial^\alpha u(g(x,t))}{\partial x^\alpha} = u'(g(x,t)) \frac{\partial^\alpha g(x,t)}{\partial x^\alpha}, \] (7)
holds true, where there exist \( u'(g(x,t)) \) and \( \frac{\partial^{\alpha} g(x,t)}{\partial x^{\alpha}} \).

**DGJ method**

To illustrate the DGJ method (DGJM) [13-16], we consider the following general function equation as follows:

\[
    u = T(u) + \Phi(u) + Y,
\]

where \( T \) is a linear operator, \( \Phi \) is a nonlinear operator from a Banach space \( \Psi \rightarrow \Psi \) and \( Y \) is a known function.

We assume that the solution of Eq. (8) is of the form

\[
    u = \sum_{i=0}^{\infty} u_i, \tag{9}
\]

The nonlinear operator \( \Phi \) can be decomposed as:

\[
    \Phi\left(\sum_{i=0}^{\infty} u_i\right) = \Phi(u_0) + \sum_{i=1}^{\infty} \left\{ \Phi\left(\sum_{j=0}^{i} u_j\right) - \Phi\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \tag{10}
\]

From Eqs. (9) and (10), Eq. (8) is equivalent to

\[
    \sum_{i=0}^{\infty} u_i = Y + \sum_{i=0}^{\infty} T(u_i) + \Phi(u_0) + \sum_{i=1}^{\infty} \left\{ \Phi\left(\sum_{j=0}^{i} u_j\right) - \Phi\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \tag{11}
\]

We define the recurrence relation as follows:

\[
    u_0 = Y(x), \quad u_m = T(u_m) + M_m, \quad m = 1, 2, \ldots. \tag{12}
\]

where

\[
    M_0 = \Phi(u_0), \tag{13}
\]

\[
    M_m = \Phi\left(\sum_{i=0}^{m} u_i\right) - \Phi\left(\sum_{i=0}^{m-1} u_i\right), \quad m = 1, 2, \ldots. \tag{14}
\]

Then k-term approximate solution of (8) is given by

\[
    u = u_0 + u_1 + \cdots + u_{k-1}. \tag{15}
\]

**Solution of the problem (2) - (3)**

Consider the following initial value problem for the nonlinear local fractional heat equation, given as follows:
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = d(x,t) \frac{\partial^2 \beta u}{\partial x^{2\beta}} + N(u),
\]

where \( u(x,0) = \varphi(x) \).

By using the fractional complex transform \([14-15]\)
\[
X = \frac{x^\beta}{\Gamma(1 + \beta)}, \quad T = \frac{t^\alpha}{\Gamma(1 + \alpha)},
\]
the problem (16) becomes
\[
\begin{align*}
\frac{\partial U}{\partial T} &= d(X,T) \frac{\partial^2 U}{\partial X^2} + N(U), \\
U(X,0) &= \Phi(X).
\end{align*}
\]

We rewrite Eq. (18) as:
\[
U(X,T) = \Phi(X) + \int_0^T d(X,T) \frac{\partial^2 U}{\partial X^2} + N(U) \,dT.
\]

Suppose that the solution of (19) takes the form:
\[
U(X,T) = \sum_{l=0}^\infty U_l(X,T) = U_0(X,T) + U_1(X,T) + U_2(X,T) + \cdots,
\]
and the nonlinear term in Eq. (18) is decomposed as:
\[
N(U) = \sum_{l=0}^\infty N_l = N_0 + N_1 + N_2 + \cdots,
\]

where
\[
N_0 = N(U_0),
\]
and
\[
N_p = N\left(\sum_{l=0}^p U_l\right) - N\left(\sum_{l=0}^{p-1} U_{l-1}\right), \quad p = 1, 2, \ldots.
\]

Then, according to the DGJM, we obtain:
\[
U_0(X,T) = \Phi(X),
\]
\[
U_1(X,T) = \int_0^T \left( d(X,T) \frac{\partial^2 U_0}{\partial X^2} + N_0 \right) \,dT,
\]
\[
U_{p+1}(X,T) = \int_0^T \left( d(X,T) \frac{\partial^2 U}{\partial X^2} + N_p \right) \,dT, \quad (p = 1, 2, \cdots)
\]
Thus, the $p$-term approximate solution of Eq. (18) is given by
\begin{equation}
U(X, T) = U_0(X, T) + U_1(X, T) + U_2(X, T) + \cdots + U_p(X, T).
\end{equation}

From (14), we can get the solution of Eq. (16) as follows:
\begin{equation}
u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots + u_n(x, t) + \cdots.
\end{equation}

Consider Eq. (2) in the form:
\begin{equation}
\begin{cases}
\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - 2u(u+1)^2, \\
u(x, 0) = \left(1 - \frac{x^\beta}{\Gamma(1 + \beta)}\right)\left(\frac{x^\beta}{\Gamma(1 + \beta)} + \exp\left(\frac{x^\beta}{\Gamma(1 + \beta)}\right)\right)^{-1}.
\end{cases}
\end{equation}

By using the relations (17), we obtain:
\begin{equation}
\begin{cases}
\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2} - 2U(U + 1)^2, \\
U(x, 0) = \frac{1 - X}{X + e^x}.
\end{cases}
\end{equation}

Thus, from the (22), we have
\begin{align*}
U_0(X, T) &= \frac{1 - X}{X + e^x}, \\
U_1(X, T) &= \frac{2 + e^x + Xe^x}{(X + e^x)^2} T, \\
&\vdots
\end{align*}
and so on.

Hence, by (17), we obtain
\begin{align*}
u_0(x, t) &= \left(1 - \frac{x^\beta}{\Gamma(1 + \beta)}\right)\left(\frac{x^\beta}{\Gamma(1 + \beta)} + E\right)^{-1}, \\
u_1(x, t) &= \frac{2\Gamma^2(1 + \beta) + E\Gamma^2(1 + \beta) + x^\beta E\Gamma(1 + \beta)}{(x^\beta + E\Gamma(1 + \beta))^2}\Gamma(1 + \alpha) t^\alpha, \\
&\vdots
\end{align*}
where
\begin{equation}
E = \exp\left(\frac{x^\beta}{\Gamma(1 + \beta)}\right), \quad G = 8 + 3E\frac{x^\beta}{\Gamma(1 + \beta)}\left(1 + \frac{x^\beta}{\Gamma(1 + \beta)}\right) - E^2\left(3 + \frac{x^\beta}{\Gamma(1 + \beta)}\right).
\end{equation}

Finally, the solution of (23) is given as follows:
\begin{equation}
u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + u_4(x, t) + \cdots.
\end{equation}

When $d(x, t) = 1, \alpha = \beta = 1$, we have
which is close to the exact solution [24]
\[ u(x,t) = \frac{1-x+2t}{x-2t+e^{xt}}. \]

**Conclusion**

In our task, we studied a nonlinear local fractional heat equation on fractals. The fractional complex transform method and DGJM had been successfully applied to find the approximate analytical solutions of the equation. It is shown that DGJM is a powerful and efficient technique in finding the analytical solutions for the nonlinear differential equation defined on Cantor sets.

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