APPLICATION OF DGJ METHOD FOR SOLVING NON-LINEAR LOCAL FRACTIONAL HEAT EQUATIONS

by

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In this paper, the initial value problem for a new non-linear local fractional heat equation is considered. The fractional complex transform method and the DGJ decomposition method are used to solve the problem, and the approximate analytical solutions are also obtained.

Key words: *non-linear local fractional heat equation, DGJ decomposition method, fractional complex transform method*

Introduction

On a continuous medium, the following classical heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{1}$$

describes the evolution in time of the density u of some quantity such as heat, chemical concentration, *etc*., where k is the thermal diffusivity coefficient [1].

But for fractal media, eq. (1) has to be modified. The local fractional calculus have been as an alternative approach proposed to study the fractal heat conduction problem [2-7]. The linear heat equation involving the local fractional derivative operators have been intensively studied over the last decade. Recent, several authors have investigated the non-linear local fractional heat equation, which can be used to model the anomalous diffusion on a fractal media [8-13]. Motivated by these results, our interest here is to solve the following non-linear local fractional heat equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = d(x,t)\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} + N(u) \tag{2}$$

with the conditions:

$$u(x,0) = \varphi(x) \tag{3}$$

where $\partial^\alpha u/\partial t^\alpha$ and $\partial^{2\beta} u/\partial x^{2\beta}$ are the local fractional derivatives [7-10] ($0 < \alpha \le 1$, $0 < \beta \le 1$), $d(x, t)$, $\varphi(x)$, and $N(u)$ are the given functions.

The solution of the local fractional differential equation is much involved. Some numerical and analytical methods for solving local fractional differential equations were presented [5-10, 14-18], such as the fractional complex transform method and the DGJ method. Fractional complex transform can convert the fractional differential equation into the ODE [11, 12]. The

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DGJ method (DGJM) was proposed by Daftardar-Gejji *et al*. [13-16, 19, 20]. The DGJM is a powerful tool for obtaining solutions of non-linear problems. Recently, Daftardar-Gejji *et al*. [20] found the exact solution and approximate solution of the fractional differential equations by using DGJM.

The main aim of this work is to solve the problems (1)-(2) by using the complex transform and DGJM.

Preliminaries

Local fractional derivative

In this section, we recall some definitions and properties of the lcal fractional derivative, see [8-13].

Definition 1. For arbitrary $\varepsilon > 0$, we give the relation as:

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{4}$$

with $|x-x_0|<\delta$. Then $f(x)$ is so-caled local fractional continuous at x_0, which is denoted by $\lim_{x\to x_0} f(x)=f(x_0)$. If $f(x)$ is so-caled local fractional continuous on the interval (a, b) an it is denoted as $f(x)\in C_\alpha(a,b)$.

Definition 2. Let $f(x)\in C_\alpha(a,b)$. In fractal space, the local fractional derivative of $f(x)$ of order α at the point $x=x_0$ is given:

$$\mathrm{D}_x^{\alpha} f(x_0)=\left.\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} f(x)\right|_{x=x_0}=f^{(\alpha)}(x_0)=\lim_{x\to x_0}\frac{\Delta^{\alpha}[f(x)-f(x_0)]}{(x-x_0)^{\alpha}} \tag{5}$$

where $\Delta[f(x)-f(x_0)]\cong\Gamma(\alpha+1)[f(x)-f(x_0)]$.

Definition 3. The local fractional partial derivative of high order is defined:

$$\frac{\partial^{k\alpha}u(x,t)}{\partial x^{k\alpha}}=\overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k\ \text{times}}u(x,t) \tag{6}$$

The following property:

$$\frac{\partial^{\alpha}u[g(x,t)]}{\partial x^{\alpha}}=u'[g(x,t)]\frac{\partial^{\alpha}g(x,t)}{\partial x^{\alpha}} \tag{7}$$

holds true, where there exist $u'[g(x,t)]$ and $\partial^{\alpha}g(x,t)/\partial x^{\alpha}$.

The DGJ method

To illustrate the DGJM [13, 14, 19, 20], we consider the following general function equation:

$$u=T(u)+\Phi(u)+Y \tag{8}$$

where T is a linear operator, Φ – the non-linear operator from a Banach space $\Psi\to\Psi$ and Y – the known function.

We assume that the solution of eq. (8) is of the form:

$$u=\sum_{i=0}^{\infty}u_i \tag{9}$$

The non-linear operator Φ can be decomposed:

$$\Phi\left(\sum_{i=0}^{\infty}u_i\right)=\Phi(u_0)+\sum_{i=1}^{\infty}\left[\Phi\left(\sum_{j=0}^{i}u_j\right)-\Phi\left(\sum_{j=0}^{i-1}u_j\right)\right] \tag{10}$$

From eqs. (9) and (10), eq. (8) is equivalent to:

$$\sum_{i=0}^{\infty} u_i = Y + \sum_{i=0}^{\infty} T(u_i) + \Phi(u_0) + \sum_{i=1}^{\infty}\left[\Phi\left(\sum_{j=0}^{i} u_j\right) - \Phi\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{11}$$

We define the recurrence relation:

$$u_0 = Y(x), \quad u_m = T(u_m) + M_m, \quad m = 1, 2, \cdots \tag{12}$$

where:

$$M_0 = \Phi(u_0) \tag{13}$$

$$M_m = \Phi\left(\sum_{i=0}^{m} u_i\right) - \Phi\left(\sum_{i=0}^{m-1} u_i\right), \quad m = 1, 2, ... \tag{14}$$

Then k-term approximate solution of (8) is given:

$$u = u_0 + u_1 + \cdots + u_{k-1} \tag{15}$$

Solution of the problem (2)-(3)

Consider the following initial value problem for the non-linear local fractional heat equation, given:

$$\begin{cases} \dfrac{\partial^{\alpha} u}{\partial t^{\alpha}} = d(x,t)\dfrac{\partial^{2\beta} u}{\partial x^{2\beta}} + N(u) \\ \\ u(x,0) = \varphi(x) \end{cases} \tag{16}$$

where we assume that the functions $d(x, t)$ and $\varphi(x)$ are local fractional continuous.

By using the fractional complex transform [19, 20]:

$$X = \frac{x^{\beta}}{\Gamma(1+\beta)}, \quad T = \frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{17}$$

the problem (16) becomes:

$$\begin{cases} \dfrac{\partial U}{\partial T} = d(X,T)\dfrac{\partial^2 U}{\partial X^2} + N(U) \\ U(X,0) = \Phi(X) \end{cases} \tag{18}$$

We rewrite eq. (18):

$$U(X,T) = \Phi(X) + \int_0^T \left[d(X,T)\frac{\partial^2 U}{\partial X^2} + N(U) \right] \mathrm{d}T \tag{19}$$

Suppose that the solution of (19) takes the form:

$$U(X,T) = \sum_{l=0}^{\infty} U_l(X,T) = U_0(X,T) + U_1(X,T) + U_2(X,T) + ... \tag{20}$$

and the non-linear term in eq. (18) is decomposed as:

$$N(U) = \sum_{l=0}^{\infty} N_l = N_0 + N_1 + N_2 + ... \tag{21}$$

where $N_0 = N(U_0)$ and

$$N_p = N(\sum_{l=0}^{p} U_l) - N(\sum_{l=0}^{p-1} U_{l-1}), \quad p = 1,2,...$$

Then, according to the DGJM, we obtain:

$$\begin{gathered} U_0(X,T) = \Phi(X), \quad U_1(X,T) = \int_0^T \left[d(X,T)\frac{\partial^2 U_0}{\partial X^2} + N_0 \right] \mathrm{d}T \\ U_{p+1}(X,T) = \int_0^T \left[d(X,T)\frac{\partial^2 U}{\partial X^2} + N_p \right] \mathrm{d}T, \quad (p = 1,2,\cdots) \end{gathered} \tag{22}$$

Thus, the p-term approximate solution of eq. (18) is given:

$$U(X,T) = U_0(X,T) + U_1(X,T) + U_2(X,T) + \cdots + U_p(X,T)$$

From (14), we can get the solution of eq. (16):

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots u_n(x,t) + ...$$

Consider eq. (2) in the form:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = \dfrac{\partial^{2\beta} u}{\partial x^{2\beta}} - 2u\left(u+1\right)^2 \\ u(x,0) = \left[1 - \dfrac{x^\beta}{\Gamma(1+\beta)}\right]\left\{\dfrac{x^\beta}{\Gamma(1+\beta)} + \exp\left[\dfrac{x^\beta}{\Gamma(1+\beta)}\right]\right\}^{-1} \end{cases} \tag{23}$$

By using the relations (17), we obtain:

$$\begin{cases} \dfrac{\partial U}{\partial T} = \dfrac{\partial^2 U}{\partial X^2} - 2U\left(U+1\right)^2 \\ U(x,0) = \dfrac{1-X}{X+\mathrm{e}^X} \end{cases}$$

Thus, from the (22), we have:

$$\begin{gathered} U_0(X,T) = \frac{1-X}{X+\mathrm{e}^X} \\ U_1\left(X,T\right) = \frac{2+\mathrm{e}^X + X\mathrm{e}^X}{\left(X+\mathrm{e}^X\right)^2} T \\ \vdots \end{gathered}$$

and so on.

Hence, by (17), we obtain:

$$\begin{gathered} u_0(x,t) = \left[1 - \frac{x^\beta}{\Gamma(1+\beta)}\right]\left[\frac{x^\beta}{\Gamma(1+\beta)} + E\right]^{-1} \\ u_1(x,t) = \frac{2\Gamma^2(1+\beta) + E\,\Gamma^2(1+\beta) + x^\beta E\,\Gamma(1+\beta)}{[x^\beta + E\,\Gamma(1+\beta)]^2\,\Gamma(1+\alpha)} t^\alpha \\ \vdots \end{gathered}$$

where:

$$E=\exp\left[\frac{x^{\beta}}{\Gamma(1+\beta)}\right],\ G=8+3E\frac{x^{\beta}}{\Gamma(1+\beta)}\left[1+\frac{x^{\beta}}{\Gamma(1+\beta)}\right]-E^{2}\left[3+\frac{x^{\beta}}{\Gamma(1+\beta)}\right]$$

Finally, the solution of eq. (23) is given:

$$u(x,t)=u_{0}(x,t)+u_{1}(x,t)+u_{2}(x,t)+u_{3}(x,t)+u_{4}(x,t)+...$$

When $d(x, t) = 1$, $\alpha = \beta = 1$, we have:

$$u(x,t)=\frac{1-x}{x+\mathrm{e}^{x}}+\frac{2+(x+1)}{(x+\mathrm{e}^{x})^{2}}t+\frac{8+(3x+x^{2})\mathrm{e}^{x}-(3+x)\mathrm{e}^{2x}}{2(x+\mathrm{e}^{x})^{2}}t^{2}+\cdots$$

which is close to the exact solution [21]:

$$u(x,t)=\frac{1-x+2t}{x-2t+e^{x+t}}$$

Conclusion

In our task, we studied a non-linear local fractional heat equation on fractals. The fractional complex transform method and DGJM had been successfully applied to find the approximate analytical solutions of the equation. It is shown that DGJM is a powerful and efficient technique in finding the analytical solutions for the non-linear differential equation defined on Cantor sets.

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