

# A PERIODIC SOLUTION FOR THE LOCAL FRACTIONAL BOUSSINESQ EQUATION ON CANTOR SETS

by

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*In this paper, the periodic solution for the local fractional Boussinesq equation can be obtained in the sense of the local fractional derivative. It's given by applying direct integration with symmetry condition. Meanwhile, the periodic solution of the non-differentiable type with generalized functions defined on Cantor sets is analyzed. As a result, we have a new point to look the local fractional Boussinesq equation through the local fractional derivative theory.*

*Key words: local fractional derivative, local fractional Boussinesq equation, periodic solution*

## Introduction

It is known that fractional derivative can be better describing things than integral order derivative from natural science, such as particle physics, electrical systems, mathematical physics (see[1-5]). Many researchers proposed some effective methods to deal with them, such as the homotopy analysis method [6], the Lie group method [7-9], and others. However, the local fractional derivative only have a litter results to be known. For example, the local fractional Laplace series expansion method [12], the local fractional variational iteration method [13], and the local fractional Riccati differential equation method [14], and so on.

In this paper, we consider the local fractional Boussinesq equation of fractal long water waves of (1+1)-fractal dimensional space [15], e.g.,

$$\frac{\partial^{2\varepsilon}\Omega(\chi, \tau)}{\partial \tau^{2\varepsilon}} - \alpha \frac{\partial^{4\varepsilon}\Omega(\chi, \tau)}{\partial \chi^{2\varepsilon} \partial \tau^{2\varepsilon}} - \beta \frac{\partial^{2\varepsilon}\Omega(\chi, \tau)}{\partial \chi^{2\varepsilon}} = 0, \quad (1)$$

where  $\Omega(\chi, \tau)$  is the fractal wave function on space variable  $\chi$  and time variable  $\tau$ ,

while  $\alpha$  and  $\beta$  are two constants, and  $\varepsilon$  is the fractal dimension.

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To the best of our knowledge, it have not been reported in other places. The aim of this paper is to obtain the periodic solution of local fractional Boussinesq equation with symmetry condition.

### Preliminaries of local fractional calculus

In this section, some definitions and prosperities of the local fractional derivative and integral for the non-differential functions can be given by [15-24]

*Definition 1* Suppose that  $\Theta(\zeta) \in C_\varepsilon$  and  $0 < \varepsilon \leq 1$ . For  $\delta > 0$  and  $0 < |\zeta - \zeta_0| < \delta$ , the limit (see [15])

$$D^{(\varepsilon)}\Theta(\zeta_0) = \frac{d^{(\varepsilon)}\Theta(\zeta)}{d\zeta^\varepsilon} \Big|_{\zeta=\zeta_0} = \lim_{\zeta \rightarrow \zeta_0} \frac{\Delta^\varepsilon[\Theta(\zeta) - \Theta(\zeta_0)]}{(\zeta - \zeta_0)^\varepsilon}, \quad (2)$$

exists and is finite, where  $\Delta^\varepsilon[\Theta(\zeta) - \Theta(\zeta_0)] \cong \Gamma(1 + \varepsilon)[\Theta(\zeta) - \Theta(\zeta_0)]$  and  $C_\varepsilon$  is a set of the non-differentiable functions.

Some special functions defined on Cantor sets are listed as follows: [15]

$$[\sin_\varepsilon(\mu^\varepsilon) = \frac{E_\varepsilon(i^\varepsilon \mu^\varepsilon) - E_\varepsilon(-i^\varepsilon \mu^\varepsilon)}{2i^\varepsilon} = \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{(2k+1)\varepsilon}}{\Gamma(1 + (2k+1)\varepsilon)}, \quad (7)$$

$$\cos_\varepsilon(\mu^\varepsilon) = \frac{E_\varepsilon(i^\varepsilon \mu^\varepsilon) + E_\varepsilon(-i^\varepsilon \mu^\varepsilon)}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{2k\varepsilon}}{\Gamma(1 + 2k\varepsilon)}, \quad (8)$$

where  $i^\varepsilon$  is an imaginary unit of a fractal set.

Suppose that  $\varphi(\zeta) \in C_\varepsilon(a, b)$ . Then, we define the local fractional integral of  $\varphi(\zeta)$  of order  $\varepsilon$  ( $0 < \varepsilon \leq 1$ ) by (see[15])

$${}_a I_b^{(\varepsilon)}\varphi(\zeta) = \frac{1}{\Gamma(1 + \varepsilon)} \int_a^b \varphi(\zeta) (d\zeta)^\varepsilon = \frac{1}{\Gamma(1 + \varepsilon)} \lim_{\Delta\zeta_k \rightarrow 0} \sum_{k=0}^{N-1} \varphi(\zeta_k) (\Delta\zeta_k)^\varepsilon, \quad (3)$$

where  $\Delta\zeta_k = \zeta_{k+1} - \zeta_k$ , ( $k = 0, 1, \dots, N$ ) with  $a = \zeta_0 < \zeta_1 < \dots < \zeta_{N-1} < \zeta_N = b$ .

For the details of the local fractional derivatives integrals, see[1].

### Periodic solutions of the non-differentiability

In this section, we will search for periodic wave solutions of the non-differentiable of Eq.(1) with symmetry condition.

First, using the following fractal wave transformation [20]

$$\Phi(\mathcal{G}) = \Omega(\chi, \tau), \mathcal{G}^\varepsilon = \chi^\varepsilon + \varpi^\varepsilon \tau^\varepsilon, \quad (14)$$

where  $\lim_{\varepsilon \rightarrow 1} \mathcal{G} = \chi + \varpi\tau$  with the wave speed  $\varpi$ .

As a result, Eq.(1) leads to

$$\varpi^{2\varepsilon} \frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} - \beta \frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} + \alpha \varpi^{2\varepsilon} \frac{\partial^{4\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{4\varepsilon}} = 0. \quad (16)$$

Meanwhile, there is a symmetry condition:  $\Phi(\mathcal{G}) = \Phi(-\mathcal{G}), \mathcal{G} \in (-\infty, +\infty)$ .

Let  $\mathcal{G} \geq 0$ . Finding the local fractional integral of Eq.(16) with respect to  $\mathcal{G}$ , we have

$$\alpha \varpi^{2\varepsilon} \frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} + \varpi^{2\varepsilon} \Phi(\mathcal{G}) - \beta \Phi(\mathcal{G}) = C_1, \mathcal{G} \geq 0, \quad (18)$$

where  $C_1$  is a constant. When  $C_1 = 0$ , Eq.(18) becomes

$$\alpha \varpi^{2\varepsilon} \frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} + \varpi^{2\varepsilon} \Phi(\mathcal{G}) - \beta \Phi(\mathcal{G}) = 0, \mathcal{G} \geq 0.$$

Thus, we have  $\frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} + \left( \frac{1}{\alpha} - \frac{\beta}{\alpha \varpi^{2\varepsilon}} \right) \Phi(\mathcal{G}) = 0, \mathcal{G} \geq 0$ , which can be rewritten in the form

$$\frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} + k^2 \Phi(\mathcal{G}) = 0, \mathcal{G} \geq 0, \quad (21)$$

where  $k^2 = \frac{1}{\alpha} - \frac{\beta}{\alpha \varpi^{2\varepsilon}}$  and it need to satisfy the constraint condition:  $1/\alpha - \beta/(\alpha \varpi^{2\varepsilon}) \geq 0$ .

In what following, we construct the periodic solutions of the non-differentiable type under the constraint condition (22).

Making use of

$$\Phi(\mathcal{G}) = \cos_{\varepsilon}(k\mathcal{G}^{\varepsilon}), \quad \frac{\partial^{\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{\varepsilon}} = -k \sin_{\varepsilon}(k\mathcal{G}^{\varepsilon}), \quad \frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} = -k^2 \cos_{\varepsilon}(k\mathcal{G}^{\varepsilon}). \quad (23)$$

we find a periodic solution of the non-differentiable type of Eq.(1), that is

$$\Phi(\mathcal{G}) = \cos_{\varepsilon}(k\mathcal{G}^{\varepsilon}), \mathcal{G} \geq 0. \quad (26)$$

Using the symmetry condition (15), yields

$$\Phi(\mathcal{G}) = \cos_{\varepsilon}(k|\mathcal{G}^{\varepsilon}|), \mathcal{G} \in (-\infty, +\infty). \quad (27)$$

in the whole interval.

Hence, the periodic solution of the non-differentiable type of the LBE in the form

$$\Omega(\chi, \tau) = \cos_{\varepsilon} \left[ \left( \frac{1}{\alpha} - \frac{\beta}{\alpha \varpi^{2\varepsilon}} \right) |\chi^{\varepsilon} + \varpi^{\varepsilon} \tau^{\varepsilon}| \right]. \quad (28)$$

In a similar way, the periodic function solution of the non-differentiable type under the constraint condition (22) was also constructed as follows:

First of all, we suppose

$$\Phi(\mathcal{G}) = \sin_{\varepsilon}(k\mathcal{G}^{\varepsilon}), \quad \frac{\partial^{\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{\varepsilon}} = -k \cos_{\varepsilon}(k\mathcal{G}^{\varepsilon})$$

and

$$\frac{\partial^{2\varepsilon} \Phi(\mathcal{G})}{\partial \mathcal{G}^{2\varepsilon}} = -k^2 \sin_{\varepsilon}(k\mathcal{G}^{\varepsilon}). \quad (31)$$

Therefore, we find a periodic solution of the non-differentiable type of Eq.(1), that is

$$\Phi(\mathcal{G}) = \sin_{\varepsilon}(k\mathcal{G}^{\varepsilon}), \mathcal{G} \geq 0. \quad (32)$$

Using the symmetry condition (22), it follows that

$$\Phi(\mathcal{G}) = \sin_{\varepsilon}(k|\mathcal{G}^{\varepsilon}|), \mathcal{G} \in (-\infty, +\infty). \quad (33)$$

Therefore, the periodic solution of the non-differentiable type of the LBE in the form

$$\Omega(\chi, \tau) = \sin_{\varepsilon} \left[ \left( \frac{1}{\alpha} - \frac{\beta}{\alpha \varpi^{2\varepsilon}} \right) |\chi^{\varepsilon} + \varpi^{\varepsilon} \tau^{\varepsilon}| \right]. \quad (34)$$

As a result, the general periodic solution of combination of trigonometric functions of the non-differentiable type of the LBE in the form

$$\Omega(\chi, \tau) = \cos_{\varepsilon} \left[ \left( \frac{1}{\alpha} - \frac{\beta}{\alpha \varpi^{2\varepsilon}} \right) |\chi^{\varepsilon} + \varpi^{\varepsilon} \tau^{\varepsilon}| \right] + \sin_{\varepsilon} \left[ \left( \frac{1}{\alpha} - \frac{\beta}{\alpha \varpi^{2\varepsilon}} \right) |\chi^{\varepsilon} + \varpi^{\varepsilon} \tau^{\varepsilon}| \right]. \quad (35)$$

## Conclusion

In our task, with the help of the local fractional derivatives and symmetry condition, a periodic wave solution of the local fractional Boussinesq equation was obtained on Cantor sets. It can help us better understand the physical phenomena and give a new perspective to look to wave theory. For other types solutions, such as rational solutions, solitary wave solutions and traveling wave solutions of the non-differentiable type, we will further discuss them in the forthcoming day's future work.

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## Nomenclature

$\chi$ - spatial variable, [m]	$\tau$ -time variable, [s]
$\alpha$ -a free constant, [-]	$\beta$ -a free constant, [-]
$\varepsilon$ -fractal dimension, [-]	$C_{\varepsilon}(a, b)$ -cantor set, [-]
$\varpi$ -wave speed, [m/s]	

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