EXTENDING OPERATOR METHOD TO LOCAL FRACTIONAL EVOLUTION EQUATIONS IN FLUIDS

by

Sheng ZHANG [a,b*], ***Sen ZHAO*** [a], ***and Yue ZHANG*** [a]

[a] School of Mathematics and Physics, Bohai University, Jinzhou 121013, China
[b] Department of Mathemtics, Hohhot Minzu College, Hohhot 010051, China

This paper is aimed to solve non-linear local fractional evolution equations in fluids by extending the operator method proposed by Zenonas Navickas. Firstly, we give the definitions of the generalized operator of local fractional differentiation and the multiplicative local fractional operator. Secondly, some properties of the defined operators are proved. Thirdly, a solution in the form of operator representation of a local fractional ordinary differential equation is obtained by the extended operator method. Finally, with the help of the obtained solution in the form of operator representation and the fractional complex transform, the local fractional Kadomtsev-Petviashvili (KP) equation and the fractional Benjamin-Bona-Mahoney (BBM) equation are solved. It is shown that the extended operator method can be used for solving some other non-linear local fractional evolution equations in fluids.

Key words: *Local fractional evolution equation, operator method,*
the generalized operator of local fractional differentiation,
the multiplicative local fractional operator,
the fractional KP equation, the fractional BBM equation

Introduction

The local fractional calculus [1] developed in recent years provide a useful mathematical tool for describing the complexity and non-differentiability of real-world problems such as vibrating string, heat transfer, and fluid mechanics. It is worth mentioning that Yang *et al.*'s meaningful contributions [1-9] are pioneering for the sound developments of the local fractional calculus. The local fractional calculus has many graceful properties, benefiting from which some existing methods originally proposed for non-linear differential equations with integer orders, for example the variational iteration approach [10, 11], have successfully been extended to some local fractional differential equations [12-14].

In 2002, Navickas proposed the operator method [15] to represent solutions of non-linear differential equations by linear operators. For such a purpose, Navickas [15] defined the linear generalized operator and the multiplicative operator, and then proved some properties of these two

* Corresponding author; e-mail: szhangchina@126.com

operators. As far as we know this operator method has not been extended to the differential equations of fractional orders. In this paper, we shall extend the operator method to construct solutions in the form of operator representation of non-linear local fractional differential equations.

The rest of this paper is organized as follows. In Section 2, we define the generalized operator of local fractional differentiation and the multiplicative local fractional operator. In Section 3, we give and prove some properties of these defined operators. In Section 4, we construct solution in the form of operator representation of a local fractional ordinary differential equation with initial conditions by the extended operator method. In Section 5, we represent solutions of the local fractional KP equation and the fractional BBM equation.

Definitions

Definition 1. The local fractional derivative is defined as [1]:

$$D_\mu^{(\alpha)}\phi(\mu)=\left.\frac{\mathrm{d}^\alpha\phi(\mu)}{\mathrm{d}\mu^\alpha}\right|_{\mu=\mu_0}=\lim_{\mu\to\mu_0}\frac{\Delta^\alpha(\phi(\mu)-\phi(\mu_0))}{(\mu-\mu_0)^\alpha} \tag{1}$$

where $0<\alpha\le 1$ and $\Delta^\alpha(\phi(\mu)-\phi(\mu_0))\cong\Gamma(1+\alpha)(\phi(\mu)-\phi(\mu_0))$ with the Euler's Gamma function:

$$\Gamma(1+\alpha)=\int_0^\infty \mu^{\alpha-1}\mathrm{e}^{-\alpha}\mathrm{d}\mu \tag{2}$$

Definition 2. Suppose $\phi(\mu)\in C_\alpha[a,b]$, then the local fractional integral of $\phi(\mu)$ of order $\alpha(0<\alpha\le 1)$ is defined as [1]:

$${}_aI_b^{(\alpha)}\phi(\mu)=\frac{1}{\Gamma(1+\alpha)}\int_a^b\phi(\mu)(\mathrm{d}\mu)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta\mu_k\to 0}\sum_{k=0}^{N-1}\phi(\mu_k)(\Delta\mu_k)^\alpha \tag{3}$$

where $\Delta\mu_k=\mu_{k+1}-\mu_k$ with $\mu_0=a<\mu_1<\cdots<\mu_{N-1}=b$.

Definition 3. If we let $A=A(x,\theta,\vartheta)$ and $B=B(x,\theta,\vartheta)$ be two polynomial of variables x, θ and ϑ, then the linear operator:

$$D_{\theta\vartheta}^{(\alpha)}=AD_\theta^{(\alpha)}+BD_\vartheta^{(\alpha)} \tag{4}$$

is called a generalized operator of local fractional differentiation.

Definition 4. The linear operator:

$$G^{(\alpha)}=G^{(\alpha)}(D_{\theta\vartheta}^{(\alpha)})=\sum_{k=0}^{+\infty}({}_0I_x^{(\alpha)}D_{\theta\vartheta}^{(\alpha)})^k \tag{5}$$

is called a multiplicative local fractional operator, here $({}_0I_x^{(\alpha)}D_{\theta\vartheta}^{(\alpha)})^0=1$ is the identity operator.

Properties

Properties 1. The local fractional operator of differentiation has some properties [1]:

$$D_x^{(\alpha)}c=0,\quad D_x^{(\alpha)}\frac{x^{l\alpha}}{\Gamma(1+l\alpha)}=\frac{x^{(l-1)\alpha}}{\Gamma(1+(l-1)\alpha)} \tag{6}$$

$$D_x^{(\alpha)}(pf(x)+qg(x)) = pD_x^{(\alpha)}f(x)+qD_x^{(\alpha)}g(x) \tag{7}$$

$$D_x^{(\alpha)}(f(x)g(x)) = (D_x^{(\alpha)}f(x))g(x)+f(x)(D_x^{(\alpha)}g(x)) \tag{8}$$

$$D_x^{(\alpha)}\frac{f(x)}{g(x)} = \frac{(D_x^{(\alpha)}f(x))g(x)-f(x)(D_x^{\alpha}g(x))}{g^2(x)} \tag{9}$$

$$D_x^{(\alpha)}E_\alpha(hx^\alpha) = hE_\alpha(x^\alpha)\,,\ \ D_x^{(\alpha)}\sin_\alpha(x^\alpha) = \cos_\alpha(x^\alpha)\,,\ \ D_x^{(\alpha)}\cos_\alpha(x^\alpha) = -\sin_\alpha(x^\alpha) \tag{10}$$

where c, l, p, q and h are all constants, and

$$E_\alpha(x^\alpha) = \sum_{k=0}^{+\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\,,\ \ \sin_\alpha(x^\alpha) = \sum_{k=0}^{+\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}\,,\ \ \cos_\alpha(x^\alpha) = \sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(1+2k\alpha)} \tag{11}$$

Properties 2. The local fractional operator of integration has some properties [1]:

$$_0I_x^{(\alpha)}c = \frac{cx^\alpha}{\Gamma(1+\alpha)}\,,\ \ _0I_x^{(\alpha)}\frac{x^{l\alpha}}{\Gamma(1+l\alpha)} = \frac{x^{(l+1)\alpha}}{\Gamma(1+(l+1)\alpha)} \tag{12}$$

$$_aI_b^{(\alpha)}(pf(x)+qg(x)) = p\,_aI_b^{\alpha}f(x)+q\,_aI_b^{\alpha}g(x) \tag{13}$$

$$_aI_b^{(\alpha)}[(D_x^{(\alpha)}f(x))g(x)] = f(x)g(x)\big|_a^b - {}_aI_b^{(\alpha)}[f(x)(D_x^{(\alpha)}g(x))] \tag{14}$$

$$_0I_x^{(\alpha)}E_\alpha(x^\alpha) = E_\alpha(x^\alpha)-1\,,\ \ _0I_x^{(\alpha)}\sin_\alpha(x^\alpha) = 1-\cos_\alpha(x^\alpha)\,,\ \ _0I_x^{(\alpha)}\cos_\alpha(x^\alpha) = \sin_\alpha(x^\alpha) \tag{15}$$

Properties 3. If let $D_\mu^{((k+1)\alpha)}\phi(\mu)\in C_\alpha(a,b)$, then $\phi(\mu)$ can be expanded as [1]:

$$\phi(\mu) = \sum_{k=0}^{+\infty}\frac{D_\mu^{(k\alpha)}\phi(\mu_0)}{\Gamma(1+k\alpha)}(\mu-\mu_0)^{k\alpha}\,,\ \ a<\mu_0<\mu<b \tag{16}$$

Properties 4. The generalized operator of local fractional differentiation has properties:

$$D_{\theta\vartheta}^{(\alpha)}\sum_{k=1}^{n}a_k f_k = \sum_{k=0}^{n}a_k D_{\theta\vartheta}^{(\alpha)}f_k\,,\ \ a_k\in R\,,\ \ f_k = f_k(x,\theta,\vartheta) \tag{17}$$

$$D_{\theta\vartheta}^{(\alpha)}(f_1 f_2) = \sum_{k=0}^{n}\binom{n}{k}(D_{\theta\vartheta}^{(k\alpha)}f_1)(D_{\theta\vartheta}^{((n-k)\alpha)}f_2)\,,\ \ n=0,1,2,\cdots \tag{18}$$

$$D_{\theta\vartheta}^{(\alpha)}f_1^{n\alpha} = f_1^{(n-1)\alpha}D_{\theta\vartheta}^{(\alpha)}f_1\,,\ \ D_{\theta\vartheta}^{(\alpha)}\frac{f_1}{f_2} = \frac{(D_{\theta\vartheta}^{(\alpha)}f_1)f_2 - f_1(D_{\theta\vartheta}^{(\alpha)}f_2)}{f_2^2} \tag{19}$$

Proof. We prove the relation (22) for $n=1$, the other ones can be proved by the similar way. In view of the definition (4), we have

$$D_{\theta\vartheta}^{(\alpha)}(f_1 f_2) = (AD_\theta^{(\alpha)}+BD_\vartheta^{(\alpha)})(f_1 f_2)$$

$$= A[(D_\theta^{(\alpha)}f_1)f_2 + f_1 D_\theta^{(\alpha)}f_2] + B[(D_\theta^{(\alpha)}f_1)f_2 + f_1 D_\theta^{(\alpha)}f_2]$$

$$= (AD_\theta^{(\alpha)}f_1)f_2 + B(D_\theta^{(\alpha)}f_1)f_2 + f_1 AD_\theta^{(\alpha)}f_2 + f_1 BD_\theta^{(\alpha)}f_2 = (D_{\theta\vartheta}^{(\alpha)}f_1)f_2 + f_1 D_{\theta\vartheta}^{(\alpha)}f_2 \tag{20}$$

Properties 5. The multiplicative local fractional operator has properties:

$$G^{(\alpha)}\sum_{k=1}^{n} a_k f_k = \sum_{k=0}^{+\infty} a_k G^{(\alpha)} f_k \tag{21}$$

$$G^{(\alpha)} f_1(\theta,\vartheta) = f_1(G^{(\alpha)}\theta, G^{(\alpha)}\vartheta) \tag{22}$$

$$G^{(\alpha)} \frac{f_1(\theta,\vartheta)}{f_2(\theta,\vartheta)} = \frac{f_1(G^{(\alpha)}\theta, G^{(\alpha)}\vartheta)}{f_2(G^{(\alpha)}\theta, G^{(\alpha)}\vartheta)}, \quad G^{(\alpha)}(\theta^{k\alpha}\vartheta^{l\alpha}) = G^{(\alpha)}(\theta^{k\alpha})G^{(\alpha)}(\vartheta^{l\alpha}) \tag{23}$$

$$G^{(\alpha)}(D_v^{(\alpha)})v^{n\alpha} = (x+v)^{n\alpha}, \quad G^{(\alpha)}(D_v^{(\alpha)}) f_1(v,\theta,\vartheta) = f_1(x+v,\theta,\vartheta) \tag{24}$$

Proof. We prove the relations (22) and (24). For the relation (22), we suppose that:

$$y_1 = y_1(x,\theta,\vartheta) = G^{(\alpha)}\theta^{\alpha}, \quad y_2 = y_2(x,\theta,\vartheta) = G^{(\alpha)}\vartheta^{\alpha} \tag{25}$$

$$z = z(x,\theta,\vartheta) = G^{(\alpha)} f_1(\theta^{\alpha},\vartheta^{\alpha}), \quad w = w(x,\theta,\vartheta) = f_1(G^{(\alpha)}\theta^{\alpha}, G^{(\alpha)}\vartheta^{\alpha}) \tag{26}$$

then following the steps in [14] yields:

$$D_x^{(\alpha)} z = D_\theta^{(\alpha)}(\sum_{k=0}^{+\infty}({}_0I_x^{(\alpha)}D_{\theta\vartheta}^{(\alpha)})^k f_1(\theta^{\alpha},\vartheta^{\alpha})) = D_{\theta\vartheta}^{(\alpha)}(\sum_{k=0}^{+\infty}({}_0I_x^{(\alpha)}D_{\theta\vartheta}^{(\alpha)})^k f_1(\theta^{\alpha},\vartheta^{\alpha}))$$
$$= PD_\theta^{(\alpha)}G^{(\alpha)} f_1(\theta^{\alpha},\vartheta^{\alpha}) + QD_\vartheta^{(\alpha)}G^{(\alpha)} f_1(\theta^{\alpha},\vartheta^{\alpha}) = PD_\theta^{(\alpha)} z + QD_\vartheta^{(\alpha)} z \tag{27}$$

Similarly, we have:

$$D_x^{(\alpha)} y_1 = PD_\theta^{(\alpha)} y_1 + QD_\vartheta^{(\alpha)} y_1, \quad D_x^{(\alpha)} y_2 = PD_\theta^{(\alpha)} y_2 + QD_\vartheta^{(\alpha)} y_2 \tag{28}$$

$$D_x^{(\alpha)} w = D_x^{(\alpha)} f_1(y_1, y_2)$$
$$= D_u^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} (PD_\theta^{(\alpha)} y_1 + QD_\vartheta^{(\alpha)} y_1) + D_v^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} (PD_\theta^{(\alpha)} y_2 + QD_\vartheta^{(\alpha)} y_2)$$
$$= P(D_u^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} D_\theta^{(\alpha)} y_1 + D_v^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} D_\theta^{(\alpha)} y_2)$$
$$+Q(D_u^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} D_\vartheta^{(\alpha)} y_1 + D_v^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} D_\vartheta^{(\alpha)} y_2) = PD_\theta^{(\alpha)} w + QD_\vartheta^{(\alpha)} w \tag{29}$$

It is easy to see from eq. (26) that:

$$z(0,\theta,\vartheta) = G^{(\alpha)} f_1(\theta^{\alpha},\vartheta^{\alpha})|_{x=0} = f_1(\theta^{\alpha},\vartheta^{\alpha}), \quad w(0,\theta,\vartheta) = f_1(G^{(\alpha)}\theta^{\alpha}, G^{(\alpha)}\vartheta^{\alpha})|_{x=0} = f_1(\theta^{\alpha},\vartheta^{\alpha}) \tag{30}$$

Thus, from eqs. (37), (29)-(30) we have $z(x,\theta,\vartheta) = w(x,\theta,\vartheta)$ which is namely the relation (22).

For the first relation in eq. (24), a direct computation shows that:

$$G^{(\alpha)}(D_v^{(\alpha)})v^{n\alpha} = v^{n\alpha} + \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)} x^{\alpha} v^{(n-1)\alpha}$$
$$+ \frac{\Gamma(1+n\alpha)}{\Gamma(1+2\alpha)\Gamma(1+(n-2)\alpha)} x^{\alpha} v^{(n-2)\alpha} + \frac{\Gamma(1+n\alpha)}{\Gamma(1+3\alpha)\Gamma(1+(n-3)\alpha)} x^{\alpha} v^{(n-3)\alpha}$$
$$+\cdots+ \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)\Gamma(1+\alpha)} x^{(n-1)\alpha} v^{\alpha} + x^{n\alpha} = \sum_{k=0}^{+\infty} \binom{n\alpha}{k\alpha} x^{k\alpha} v^{(n-k)\alpha} = (x+v)^{n\alpha} \tag{31}$$

Similarly, for the second relation in eq. (24), we have:

$$G^{(\alpha)}(D_v^{(\alpha)})f_1(v,\theta,\vartheta)=f_1(v,\theta,\vartheta)+\frac{D_v^{(\alpha)}f_1(v,\theta,\vartheta)}{\Gamma(1+\alpha)}x^{\alpha}+\frac{D_v^{(2\alpha)}f_1(v,\theta,\vartheta)}{\Gamma(1+2\alpha)}x^{2\alpha}+\frac{D_v^{(3\alpha)}f_1(v,\theta,\vartheta)}{\Gamma(1+3\alpha)}x^{3\alpha}+\cdots$$

$$=\sum_{k=0}^{+\infty}\frac{D_v^{(k\alpha)}f_1(v,\theta,\vartheta)}{\Gamma(1+k\alpha)}x^{k\alpha}=f_1(x+v,\theta,\vartheta) \tag{32}$$

Properties 5. If set

$$A=\frac{\vartheta^{\alpha}}{\Gamma(1+\alpha)},\quad B=-\frac{\theta^{\alpha}}{\Gamma(1+\alpha)} \tag{33}$$

to eq. (4), then we have:

$$G^{(\alpha)}(D_{\theta\vartheta}^{(\alpha)})\theta^{\alpha}=\theta^{\alpha}\cos_{\alpha}(x^{\alpha})+\vartheta^{\alpha}\sin_{\alpha}(x^{\alpha}) \tag{34}$$

$$G^{(\alpha)}(D_{\theta\vartheta}^{(\alpha)})\theta^{2\alpha}=(\theta^{\alpha}\cos_{\alpha}(x^{\alpha})+\vartheta^{\alpha}\sin_{\alpha}(x^{\alpha}))^2,\quad D_x^{(\alpha)}(G^{(\alpha)}(D_{\theta\vartheta}^{(\alpha)})\theta^{\alpha})=G^{(\alpha)}(D_{\theta\vartheta}^{(\alpha)})\vartheta^{\alpha} \tag{35}$$

Proof. As an example, we prove the relation (34). In view of eqs. (4) and (33), we have:

$$G^{(\alpha)}(D_{\theta\vartheta}^{(\alpha)})\theta^{\alpha}=\theta^{\alpha}+\vartheta^{\alpha}\frac{x^{\alpha}}{\Gamma(1+\alpha)}-\theta^{\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}-\vartheta^{\alpha}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\theta^{\alpha}\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\vartheta^{\alpha}\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}-\cdots$$

$$=\theta^{\alpha}\cos_{\alpha}(x^{\alpha})+\vartheta^{\alpha}\sin_{\alpha}(x^{\alpha}) \tag{36}$$

Properties 6. If let:

$$D_{v\theta\vartheta}^{(\alpha)}=D_v^{(\alpha)}+AD_{\theta}^{(\alpha)}+BD_{\vartheta}^{(\alpha)} \tag{37}$$

then we have:

$$G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})f_1(v)=G^{(\alpha)}(D_v^{(\alpha)})f_1(v)=f_1(x+v) \tag{38}$$

$$G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})f_1(v,\theta,\vartheta)=f_1(G^{(\alpha)}(D_v^{(\alpha)})v,G^{(\alpha)}(D_v^{(\alpha)})\theta,G^{(\alpha)}(D_v^{(\alpha)})\vartheta) \tag{39}$$

Proof. We can see that the relation (38) is obvious. The proof of the relation (39) is similar to that of the relation (22) and we omit it here for simplification.

Theorem

Theorem 1. If a local fractional ordinary differential equation is given by:

$$D_x^{(2\alpha)}u=P(x,u,D_x^{(\alpha)}u),\quad u=u(x,\theta,\vartheta),\quad u(v,\theta,\vartheta)=\theta^{\alpha},\quad D_x^{(2\alpha)}u|_{x=v}=\vartheta^{\alpha} \tag{40}$$

where $P(x,\theta,\vartheta)$ is a polynomial of variables x, θ and ϑ, then eq. (40) has a solution of the operator representation:

$$u=\sum_{k=0}^{+\infty}\frac{(x-v)^{k\alpha}}{\Gamma(1+\alpha)}(D_v^{(\alpha)}+\vartheta^{\alpha}D_{\theta}^{(\alpha)}-P(v,\theta,\vartheta)D_{\vartheta}^{(\alpha)})^k\theta^{\alpha},\quad v\in R \tag{41}$$

Proof. Let

$$D_{v\theta\vartheta}^{(\alpha)}=\frac{D_v^{(\alpha)}+\theta^\alpha D_\theta^{(\alpha)}+P(v,\theta,\vartheta)D_\vartheta^{(\alpha)}}{\Gamma(1+\alpha)},\quad z(x,\theta,\vartheta,v)=G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\theta^\alpha \tag{42}$$

we have:

$$D_x^{(\alpha)}z(x,\theta,\vartheta,v)=D_x^{(\alpha)}G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\theta^\alpha=D_{v\theta\vartheta}^{(\alpha)}G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\theta^\alpha=G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})D_{v\theta\vartheta}^{(\alpha)}\theta^\alpha=G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\vartheta^\alpha \tag{43}$$

$$D_x^{(2\alpha)}z(x,\theta,\vartheta,v)=D_x^{(\alpha)}G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\vartheta^\alpha=G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})P(v,\theta,\vartheta)=P(x+`v,G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\theta^\alpha,G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\vartheta^\alpha) \tag{44}$$

and then arrive at:

$$D_x^{(2\alpha)}z(x-v,\theta,\vartheta,v)=P(x,z(x-v,\theta,\vartheta,v),D_x^{(\alpha)}z(x-v,\theta,\vartheta,v)) \tag{45}$$

Thus, from eqs. (17) and (45) we obtain:

$$u=z(x-v,\theta,\vartheta,v)=G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\theta^\alpha\,|_{x=x-v}=\sum_{k=0}^{+\infty}\frac{(x-v)^{k\alpha}}{\Gamma(1+\alpha)}[D_v^{(\alpha)}+\vartheta^\alpha D_\theta^{(\alpha)}+P(v,\theta,\vartheta)D_\vartheta^{(\alpha)}]^k\theta^\alpha \tag{46}$$

Applications

Example 1. Application to the KP equation:

$$D_x^{(\alpha)}(D_t^{(\alpha)}u+6uD_x^{(\alpha)}+D_x^{(3\alpha)}u)+D_y^{(\alpha)}u=0 \tag{47}$$

Firstly, we take the travelling-wave transform [4]:

$$\xi=x^\alpha+y^\alpha+t^\alpha \tag{48}$$

then eq. (47) is reduced into:

$$D_\xi^{(\alpha)}(D_\xi^{(\alpha)}u+6uD_\xi^{(\alpha)}u+D_\xi^{(3\alpha)}u)+D_\xi^{(2\alpha)}u=0 \tag{49}$$

Secondly, we integrate eq. (49 with respective to ξ twice, then eq. (49) becomes:

$$2u+3u^2+D_\xi^{(2\alpha)}u=0 \tag{50}$$

Finally, we suppose:

$$P(\xi,u,D_\xi^{(2\alpha)}u)=-2u-3u^2,\quad u(v)=\theta^\alpha,\quad D_\xi^{(\alpha)}u(\xi)\,|_{\xi=v}=\vartheta^\alpha \tag{51}$$

and then obtain a solution of operator representation by using *Theorem 1*:

$$u=\sum_{k=0}^{+\infty}\frac{(\xi-v)^{k\alpha}}{\Gamma(1+\alpha)}[D_v^{(\alpha)}+\vartheta^\alpha D_\theta^{(\alpha)}+(3\theta^{2\alpha}+2\theta^\alpha)D_\vartheta^{(\alpha)}]^k\theta^\alpha \tag{52}$$

Example 2. Application to the BBM equation:

$$D_t^{(\alpha)}u + 2\omega u D_x^{(\alpha)}u + 3uD_x^{(\alpha)}u - D_{xxt}^{(3\alpha)}u = 0\,,\quad \omega = \text{const.} \tag{53}$$

Similarly, we take the fractional complex transform [15]:

$$\xi = \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{54}$$

then eq. (53) is converted into:

$$D_\xi^{(\alpha)}u + 2\omega u D_\xi^{(\alpha)}u + 3uD_\xi^{(\alpha)}u - D_{\xi\xi\xi}^{(3\alpha)}u = 0 \tag{55}$$

Integrating eq. (55) with respective to ξ once, we have:

$$(1+2\omega)u + \frac{3}{2}u^2 - D_\xi^{(2\alpha)}u = 0 \tag{56}$$

Supposing

$$P(\xi, u, D_\xi^{(2\alpha)}u) = (1+2\omega)u + \frac{3}{2}u^2\,,\quad u(v) = \theta^\alpha\,,\quad D_\xi^{(\alpha)}u(\xi)\,|_{\xi=v} = \vartheta^\alpha \tag{57}$$

and using *Theorem 1*, then we obtain a solution of operator representation:

$$u = \sum_{k=0}^{+\infty} \frac{(\xi - v)^{k\alpha}}{\Gamma(1+\alpha)} \{D_v^{(\alpha)} + \vartheta^\alpha D_\theta^{(\alpha)} - [(1+2\omega)\theta + \frac{3}{2}\theta^2]D_\vartheta^{(\alpha)}\}^k \theta^\alpha \tag{58}$$

Conclusion

In summary, the extended operator method has been established for constructing solutions of operator representation of non-linear local fractional evolution equations in fluids. When the fractional-order tends to 1, the extended operator method, the generalized operator of local fractional differentiation (4), the multiplicative local fractional operator (5), the properties (17)-(19), (21)-(24) and (34)-(35), and the obtained solution (46) degenerates into those of Navickas' [14]. To the best of our knowledge, solutions (52) and (58) of the local fractional KP and BBM equations are novel.

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Nomenclature

t – time co-ordinate, [s]	α – fractional order, [–]

x, y, v, μ, θ, ϑ – space co-ordinates, [m] a, b – real numbers, [–]
$\mathrm{d}^{\alpha}/\mathrm{d}^{\alpha}x$ – the first local fractional derivative [–]

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