THE SOURCE FUNCTION OF FRACTIONAL HEAT-LIKE SYSTEMS

by

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In this paper, we study the inverse problem for seeking an unknown source function of the linear fractional heat systems with variable coefficient using Adomian decomposition method. The results prove that Adomian decomposition method is very effective and simple for the inverse problem of finding the source function of the heat systems.

Key words: *fractional heat systems, Adomian decomposition method, inverse problems, temperature*

Introduction

For modeling of a lot of phenomena, especial in the modeling of options in financial mathematics, there appears the heat equation [1-4]. The heat equation governs heat diffusion, as well as other diffusive processes, such as particle diffusion or the propagation of action potential in nerve cells. They are not diffusive in nature, but some quantum mechanics problems are also controlled by a mathematical analog of the heat equation [1-3]. It also can be used to model some phenomena arising in finance, like the Black-Scholes or the Ornstein-Uhlenbeck processes [4]. Non-linear vibration problems of heat exchanges have been studied very recently [5].

The heat equation is an important PDE. With certain data, solving an equation in a specified condition is called direct problem. On the contrary, when solving an unknown input by the given output, the method is called an inverse problem. These unknown inputs may be some coefficients, or some source functions of an equation. In general, the inverse problems are very sensitive to the errors coming from the measured input. In order to overcome this fault, the method has been invented and boosted. The method for finding the inverse problems have a lot of practical applications, such as the quantum mechanics, geophysics, optics, astronomy, medical imaging, and photoelasticity. On the other hand, as a mathematical analysis method, the fractional calculus is a generalization of integer orders integrals and derivatives [6, 7]. Now, it has been used successful to the modeling of memory dependent phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and porous media [8, 9]. It emerged as an vital and efficient tool for the study of the dynamical systems [10-16] where classical methods having strong limitations. The theory and applications of the inverse problems for the fractional differential equation were studied [17-20]. However, the invese problem for the fractional linear heat-like heat system have been not considered.

In this paper, we shall investigate the following fractional linear heat-like heat systems:

$$\mathrm{D}_t^{\lambda} u(x,t) = v_{xx}(x,t) + g(x),\ \mathrm{D}_t^{\lambda} v(x,t) = u_{xx}(x,t),\ x>0,\ t>0,\ 0<\lambda\le 1$$

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which could be used to model the heat conduction in a rod. By the Adomian decomposition method [9], we resolve the source functions. The structure of the paper is as follows.

Preliminaries

In this section we present the following definitions.

Definition 1. For any order $\lambda > 0$, the Riemann-Liouville fractional integral operator for a function f: $(0, +\infty) \rightarrow R$ is given [8]:

$$I^{\lambda} f(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-s)^{\lambda-1} f(s) \mathrm{d}s \tag{1}$$

where $\Gamma(\cdot)$ is Gamma function [8]:

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} \mathrm{e}^{-t} \mathrm{d}t, \quad \xi > 0$$

Definition 2. For any order $\lambda > 0$, the Caputo fractional derivative of a continuous function f: $(0, +\infty) \rightarrow R$ is given [8]:

$$\mathrm{D}^{\lambda} f(x) = I^{m-\lambda} \mathrm{D}^m f(x) = \frac{1}{\Gamma(m-\lambda)} \int_0^x (x-s)^{m-\lambda-1} f^{(m)}(s) \mathrm{d}s \tag{2}$$

where $m - 1 < \lambda \leq m$, and m is an integer number.

Definition 3. For any order $\lambda > 0$, the Caputo time fractional derivative of a continuous function f: $(0, +\infty) \rightarrow R$ is given [8]:

$$\mathrm{D}_t^{\lambda} u(x,t) = \frac{\partial^{\lambda} u(x,t)}{\partial t^{\lambda}} = \frac{1}{\Gamma(m-\lambda)} \int_0^t (t-s)^{m-\lambda-1} \frac{\partial^m u(x,t)}{\partial t^m}(s) \mathrm{d}s, \ m-1 < \lambda \leq m \tag{3}$$

where $m - 1 < \lambda \leq m$, and m is an integer number.

Anlaysis of the method applied

We now consider the linear fractional heat systems with variable coefficient:

$$\mathrm{D}_t^{\lambda} u(x,t) = v_x(x,t) + g(x), \ \mathrm{D}_t^{\lambda} v(x,t) = u_x(x,t), \ x > 0, \ t > 0, \ 0 < \lambda \leq 1 \tag{4}$$

where $u(x, 0)$, $f_1(x)$, $v(x,0) = f_2(x)$, $u(0, t) = h_1(t)$, and $u_x(0, t) = \mathrm{d}_1(t)$.

It follows from eq. (4):

$$I_t^{\lambda} \mathrm{D}_t^{\lambda} v(x,t) = I_t^{\lambda} \left[u_x(x,t) \right] \tag{5}$$

we have:

$$v(x,t) = v(x,0) + I_t^{\lambda} \left[u_x(x,t) \right] \tag{6}$$

and

$$v_x(x,t) = v_x(x,0) + I_t^{\lambda} \left[u_{xx}(x,t) \right] \tag{7}$$

In this case, we have:

$$\mathrm{D}_t^{\lambda} u(x,t) = f_2'(x) + I_t^{\lambda} \left[u_{xx}(x,t) \right] + \mathrm{g} \tag{8}$$

$$I_t^{\lambda} \mathrm{D}_t^{\lambda} u(x,t) = I_t^{\lambda} \left\{ f_2'(x) + I_t^{\lambda} \left[u_{xx}(x,t) \right] + g(x) \right\} \tag{9}$$

such that:

$$u(x,t) = u(x,0) + I_t^{\lambda} \left\{ f_2'(x) + I_t^{\lambda} \left[u_{xx}(x,t) \right] + g(x) \right\} \tag{10}$$

Making use of the ADM, the solution is written in series form [9]:

$$u(x,t)=\sum_{0}^{\infty}u_n(x,t) \tag{11}$$

where u and u_n, $n \in N$, are defined in $C^{\infty}[0, \infty) \times C^1 [0, \infty)$.

Substituting the decomposition eq. (11) into eq. (10) and setting the recurrence scheme:

$$u_0(x,t)=u(x,0)+I_t^{\lambda}\left[f_2'(x)+g(x)\right],\ u_{n+1}(x,t)=I_t^{\lambda}\left[(u_n)_{xx}(x,t)\right] \tag{12}$$

By the computation in eq. (12), we have the ADM polynomials:

$$u_0(x,t)=f_1(x)+\left[f_2'(x)+g(x)\right]\frac{t^{\lambda}}{\Gamma(\lambda+1)}$$

$$u_1(x,t)=I_t^{\lambda}\left[(u_0)_{xx}(x,t)\right]=f_1''(x)\frac{t^{\lambda}}{\Gamma(\lambda+1)}+\left[f_2'''(x)+g''(x)\right]\frac{t^{2\lambda}}{\Gamma(2\lambda+1)}$$

$$u_2(x,t)=I_t^{\lambda}\left[(u_1)_{xx}(x,t)\right]=f_1^{(4)}(x)\frac{t^{2\lambda}}{\Gamma(2\lambda+1)}+\left[f_2^{(5)}(x)+g^{(4)}(x)\right]\frac{t^{3\lambda}}{\Gamma(3\lambda+1)} \tag{13}$$

After writing these polynomials in eq. (10), the solution $u(x, t)$ of eq. (4) is given:

$$u(x,t)=f_1(x)+\left[f_2'(x)+g(x)+f_1''(x)\right]\frac{t^{\lambda}}{\Gamma(\lambda+1)}+$$

$$+\left[f_2'''(x)+g''(x)+f_1^{(4)}(x)\right]\frac{t^{2\lambda}}{\Gamma(2\lambda+1)}+\cdots \tag{14}$$

For seeking the unknown source function, we expand the boundary functions $h_1(t)$ and $d_1(t)$ into the following series for the space whose bases are $\{t^{n\lambda}/\Gamma(n\lambda+1)\}_{n=0}^{\infty}$, $0<\lambda\le 1$:

$$h_1(t)=h_1(0)+h_1'(0)\frac{t^{\lambda}}{\Gamma(\lambda+1)}+h_1''(0)\frac{t^{\lambda}}{\Gamma(\lambda+1)}+\cdots \tag{15}$$

$$d_1(t)=d_1(0)+d_1'(0)\frac{t^{\lambda}}{\Gamma(\lambda+1)}+d_1''(0)\frac{t^{\lambda}}{\Gamma(\lambda+1)}+\cdots \tag{16}$$

Meanwhile, with use of eq. (14), we have:

$$h_1(t)=f_1(0)+\left[f_2'(0)+g(0)+f_1''(0)\right]\frac{t^{\lambda}}{\Gamma(\lambda+1)}+$$

$$+\left[f_2'''(0)+g''(0)+f_1^{(4)}(0)\right]\frac{t^{2\lambda}}{\Gamma(2\lambda+1)}+\left[f_2^{(5)}(0)+g^{(4)}(0)+f_1^{(6)}(0)\right]\frac{t^{3\lambda}}{\Gamma(3\lambda+1)}+\cdots \tag{17}$$

and

$$d_1(t)=f_1'(0)+\left[f_2''(0)+g'(0)+f_1'''(0)\right]\frac{t^{\lambda}}{\Gamma(\lambda+1)}+$$

$$+\left[f_2^{(4)}(0)+g'''(0)+f_1^{(5)}(0)\right]\frac{t^{2\lambda}}{\Gamma(2\lambda+1)}+\left[f_2^{(6)}(0)+g^{(5)}(0)+f_1^{(7)}(0)\right]\frac{t^{3\lambda}}{\Gamma(3\lambda+1)}+\cdots \tag{18}$$

From eq. (15) and eq. (17), we get:

$$h_1(0)=f_1(0),\quad h_1'(0)=f_2'(0)+g(0)+f_1''(0)$$

$$h_1''(0) = f_2''''(0) + g''(0) + f_1^{(4)}(0)$$

$$h_1'''(0) = f_2^{(5)}(0) + g^{(4)}(0) + f_1^{(6)}(0), \cdots \tag{19}$$

From eqs. (16) and (18), we have:

$$d_1(0) = f_1'(0),\ d_1'(0) = f_2'(0) + g'(0) + f_1''(0)$$

$$d_1'''(0) = f_2^{(4)}(0) + g'''(0) + f_1^{(5)}(0),\ d_1^{(4)}(0) = f_2^{(6)}(0) + g^{(5)}(0) + f_1^{(7)}(0), \cdots \tag{20}$$

Recalling the Taylor series expansion of the unknown function $g(x)$, and considering the previous data, we have the following:

$$g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2!} + g'''(0)\frac{x^3}{3!} + \cdots + g^{(n)}(0)\frac{x^n}{n!} + \cdots \tag{21}$$

Consequently, we obtain:

$$g(x) = \left[h_1'(0) - f_2'(0) - f_1''(0)\right] + \left[d_1'(0) - f_2''(0) - f_1''''(0)\right]x +$$

$$+\left[h_1''(0) - f_2''''(0) - f_1^{(4)}(0)\right]\frac{x^2}{2!} + \left[d_1'''(0) - f_2^{(4)}(0) - f_1^{(5)}(0)\right]\frac{x^3}{3!} + \cdots \tag{22}$$

A typical example

Consider the linear fractional heat systems with variable coefficient:

$$\mathrm{D}_t^\lambda u(x,t) = v_x(x,t) + g(x),\ \mathrm{D}_t^\lambda v(x,t) = u_x(x,t), x > 0,\ t > 0,\ 0 < \lambda \le 1 \tag{23}$$

with the initial and boundary conditions as:

$$u(x,0) = \mathrm{e}^x + \sin x,\ v(x,0) = \mathrm{e}^x + \cos x, \text{ and } u(0,t) = \mathrm{e}^{2t},\ u_x(0,t) = \mathrm{e}^{2t} + 1$$

By eq. (14), the solution $u(x, t)$ of eq. (23):

$$u(x,t) = \mathrm{e}^x + \sin x + \left[2\mathrm{e}^x - 2\sin x + g(x)\right]\frac{t^\lambda}{\Gamma(\lambda+1)} + \left[2\mathrm{e}^x + 2\sin x + g''(x)\right]\frac{t^{2\lambda}}{\Gamma(2\lambda+1)} \tag{24}$$

Then, we have:

$$u(0,t) = 1 + \left[2 + g(0)\right]\frac{t^\lambda}{\Gamma(\lambda+1)} + \left[2 + g''(0)\right]\frac{t^{2\lambda}}{\Gamma(2\lambda+1)} + \left[2 + g^{(4)}(0)\right]\frac{t^{3\lambda}}{\Gamma(3\lambda+1)} + \cdots \tag{25}$$

From eq. (15) and the boundary conditions in eq. (23), it may beequal to the Taylor series of e^{2t} in the space whose bases are $\{t^{n\lambda}/\Gamma(n\lambda + 1)\}_{n=0}^{\infty}$, $0 < \lambda \le 1$:

$$\mathrm{e}^{2t} = 1 + 2\frac{t^\lambda}{\Gamma(\lambda+1)} + 4\frac{t^{2\lambda}}{\Gamma(2\lambda+1)} + 8\frac{t^{3\lambda}}{\Gamma(3\lambda+1)} + \cdots \tag{26}$$

Thus, we get:

$$g(0) = 0,\ g''(0) = 2,\ g^{(4)}(0) = 6,\ g^{(6)}(0) = 14 \tag{27}$$

From eq. (24) we have:

$$u_x(x,t) = \mathrm{e}^x + \cos x + \left[2\mathrm{e}^x - 2\cos x + g'(x)\right]\frac{t^\lambda}{\Gamma(\lambda+1)} + \cdots \tag{28}$$

So, we have:

$$u_x(0,t) = 2 + g'(0)\frac{t^\lambda}{\Gamma(\lambda+1)} + \left[4 + g'''(0)\right]\frac{t^{2\lambda}}{\Gamma(2\lambda+1)} + g^{(5)}(0)\frac{t^{3\lambda}}{\Gamma(3\lambda+1)} + \cdots \tag{29}$$

From eq. (16), the boundary conditions in eqs. (23) and (26), it must be equal to the following series:

$$e^{2t} + 1 = 2 + 2\frac{t^\lambda}{\Gamma(\lambda+1)} + 4\frac{t^{2\lambda}}{\Gamma(2\lambda+1)} + 8\frac{t^{3\lambda}}{\Gamma(3\lambda+1)} + \cdots \tag{30}$$

Thus, we get:

$$g'(0) = 2,\ g'''(0) = 0,\ g^{(5)}(0) = 8,\ g^{(7)}(0) = 12, \cdots \tag{31}$$

Using eqs. (27) and (31), we have the Taylor series expansion of $g(x)$ as follows:

$$g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2!} + g'''(0)\frac{x^3}{3!} + g^{(4)}(0)\frac{x^4}{4!} + g^{(5)}(0)\frac{x^5}{5!} + \cdots =$$
$$= 2\left(x + x^2 + 3\frac{x^4}{4!} + 4\frac{x^5}{5!} + 5\frac{x^6}{6!} + 6\frac{x^7}{7!} + \cdots\right) \tag{32}$$

Conclusion

In the present task, by choosing the Adomian decomposition method, it has been found that it is very effective to the fractional heat systems. Moreover, we gave an example to illustrate our results. In the following works, we shall do some these problems by our results on fractional differential equation [21, 22] and our results on fixed point theorem [23].

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Nomenclature

t – space coordinate, [s]
$u(x, t)$ – temperature distribution, [K]
x – space coordinate, [m]

Greek symbol

λ – fractional order, [–]

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