NUMERICAL SOLUTION OF A CLASS OF ADVECTION-REACTION-DIFFUSION SYSTEM

by

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In this article, the barycentric interpolation collocation methods is proposed for solving a class of nonlinear advection-reaction-diffusion system. Compared with other methods, the numerical experiment shows the barycentric interpolation collocation method is a high precision method to solve the advection-reaction-diffusion system.

Key words: nonlinear advection-reaction-diffusion problems; barycentric interpolation collocation method; numerical experiment

Introduction

The treatise is devoted to the numerical solution of a class of nonlinear advection-reaction-diffusion system. In this paper, the general expression of such system as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial u}{\partial x} + h_1(u,v), \quad a \leq x \leq b, \quad 0 \leq t \leq T, \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + a_2 \frac{\partial v}{\partial x} + h_2(u,v),
\end{align*}
\]

(1)

with the following initial boundary conditions:

\[
\begin{align*}
u(x,0) &= f_0(x), & v(x,0) &= g_0(x), \quad a \leq x \leq b, \quad 0 \leq t \leq T, \\
u(a,t) &= f_1(t), & u(b,t) &= f_2(t), \quad 0 \leq t \leq T, \\
v(a,t) &= g_1(t), & v(b,t) &= g_2(t).
\end{align*}
\]

(2)

where \(a_1\) and \(a_2\) represent the of the transport medium, such as water or air, and both \(d_1 > 0\) and \(d_2 > 0\) are diffusion coefficients, which include the parametrizations of the turbulence.

The advection-reaction-diffusion system has wide applications in thermal science, chemical and mechanics. There are some valuable efforts that focus on finding the analytical and numerical methods for solving the advection-reaction-diffusion system. These methods include B-spline method [1,2], the variational iteration method [3], homotopy perturbation method [4], integral transform [5] and so on.

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The barycentric interpolation collocation method [8-13] is a high precision method. In this paper, we mainly employ the barycentric Lagrange barycentric interpolation collocation method to solve the systems (1).

**Description of the numerical method**

We give two initial functions \( u_0, v_0 \), and construct following linear iterative format:

\[
\begin{cases}
\frac{\partial u_n}{\partial t} = d_1 \frac{\partial^2 u_n}{\partial x^2} + a_1 \frac{\partial u_n}{\partial x} + h_1(u_{n-1}, v_{n-1}), \\
\frac{\partial v_n}{\partial t} = d_2 \frac{\partial^2 v_n}{\partial x^2} + a_2 \frac{\partial v_n}{\partial x} + h_2(u_{n-1}, v_{n-1}),
\end{cases}
\]

\( n = 1, 2, 3, \ldots \) \hspace{1cm} (3)

Next, we use barycentric interpolation collocation method to solve Eq.(3).

Let \( a \leq x_1 < x_2 < \cdots < x_M \leq b, \quad 0 \leq t_1 < t_2 < \cdots < t_N \leq T \), respectively, These nodes can generate two dimensional nodes on the rectangular area \( \Omega = [a,b] \times [0,T] \), as follows:

\( \{(x_i, t_j), i = 1, 2, \cdots, M; j = 1, 2, \cdots, N; \} \)

The barycenter interpolation form of function \( u(x, t) \) can be expressed as

\[
u(x, t) = \sum_{i=1}^{M} \sum_{j=1}^{N} \xi_i(x) \eta_j(t) u(x_i, t_j),
\]

where

\[
\xi_i(x) = \prod_{k=1, k \neq i}^{M} \frac{x - x_k}{x_i - x_k}, \quad i = 1, 2, \cdots, M, \quad \eta_j(t) = \prod_{k=1, k \neq j}^{N} \frac{t - t_k}{t_j - t_k}, \quad j = 1, 2, \cdots, N.
\]

According to formula (4), the partial derivative of \( l + k \) order of function \( u(x, t) \) at the nodes \( (x_p, t_q) \) can be written as:

\[
\frac{\partial^{l+k} u(x_p, t_q)}{\partial x^l \partial t^k} = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{\partial^{l+k}}{\partial x^l \partial t^k} \xi_i(x_p) \eta_j(t_q) u(x_i, t_j), \quad p = 1, 2, \cdots M; q = 1, 2, \cdots N.
\]

The function values of the formula (4) and the formula (5) at the node form column vectors \( u, u^{(l,k)} \) and they are as follows:

\[
u = [u_1, u_2, \cdots, u_{M \times N}]^T, \quad u^{(l,k)} = [u^{(l,k)}_1, u^{(l,k)}_2, \cdots, u^{(l,k)}_{M \times N}]^T,
\]

\[
u_p = u(X_p, T_q), \quad u^{(l,k)}_p = u^{(l,k)}(X_p, T_q), \quad p = 1, 2, \cdots, M \times N.
\]

Therefore, the formula (5) can be expressed in the following matrix form:

\[
u^{(l,k)} = D^{(l,k)} u,
\]

in the formula (6), \( D^{(l,k)} = C^{(l)} \otimes D^{(k)} \) is the Kronecker product of matrix \( C^{(l)} \) and \( D^{(k)} \), and it is also called \( l + k \) order partial differential matrix at nodes \( \{(x_i, t_j), i = 1, 2, \cdots, M; j = 1, 2, \cdots, N; \} \)

\( C^{(l)} \) and \( D^{(k)} \) are \( l \) \( k \) order differential matrices formed by barycenter interpolation at node interval \([a, b] \) and interval \([0, T] \) respectively. Let

\[
D^{(0)} = I_M, \quad D^{(0)} = I_N,
\]

where \( I_M \) and \( I_N \) are \( M \) order unit matrix and \( N \) order unit matrix respectively.

So, the discrete form of the Eq. (3) can be written as:
\[
\begin{aligned}
D^{(0,1)} u_n - d_1 D^{(2,0)} u_n - a_1 D^{(1,0)} u_n &= \text{diag}(h_1(u_{n-1}, v_{n-1})), \\
D^{(0,1)} v_n - d_2 D^{(2,0)} v_n - a_2 D^{(1,0)} v_n &= \text{diag}(h_2(u_{n-1}, v_{n-1})),
\end{aligned}
\]  

(8)

So, Eq. (8) can be written in the following partitioned matrix form:

\[
\begin{bmatrix}
D^{(0,1)} - d_1 D^{(2,0)} & 0 \\
0 & D^{(0,1)} - d_2 D^{(2,0)}
\end{bmatrix}
\begin{bmatrix}
u_n \\
v_n
\end{bmatrix} =
\begin{bmatrix}
\text{diag}(h_1(u_{n-1}, v_{n-1})) \\
\text{diag}(h_2(u_{n-1}, v_{n-1}))
\end{bmatrix}
\]  

(9)

In this paper, we use the displacement method to impose the initial boundary conditions. The detailed procedure see Ref. [8, 9, 10]. In calculation, we choose the Chebyshev nodes. In the following numerical experiments, we set a calculation accuracy \( \varepsilon = 10^{-15} \), if \( |u_n(x,t) - u_{n+1}(x,t)| < \varepsilon \) then the iteration stops.

Numerical experiments

**Experiment 1.** We consider the following dvection-reaction-diffusion problem [6]

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial u}{\partial x} + (a_1 - d_1)t + d_1 u - a_1 v + (u - v)^2 + \sin(2x), \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + a_2 \frac{\partial v}{\partial x} + (a_2 - d_2)t + a_2 u + d_2 v + u^2 + v^2 - 2t(u + v) + 2t^2,
\end{aligned}
\]  

(10)

where the initial and boundary conditions are as follows:

\[
\begin{aligned}
u(x,0) &= \sin x, \\
u(0,t) &= u(4\pi, t) = t, \\
u(0,t) &= v(4\pi, t) = t+1, \\
u(0,t) &= v(4\pi, t) = t+1, \\
u(t) &= u(4\pi, t) = t+1,
\end{aligned}
\]  

(11)

The exact solution of the equations is given by \( u(x,t) = t + \sin(x) \), \( v(x,t) = t + \cos(x) \). We select \( d_1 = 0.5, d_2 = 1.0, a_1 = 0.5, a_2 = 0.1 \). Numerical results are showed in Figure 1-6 and Table 1-3. In Table 1, comparing the calculation time and error between the present method and other methods [6] under different nodes, we can find that the present method has the least calculation time and the least error. Table 2-3 show the absolute errors in different nodes. As can be seen from the Table 2-3, with the increase of nodes, the absolute error is also decreasing. Figure 1-6 are numerical solutions and absolute errors in different nodes, which we can see that the absolute error is very small. Obviously, our method is very suitable for solving such problems.

**Table 1: Comparison of \( L_\infty \) error norm for Experiment 1.**

<table>
<thead>
<tr>
<th>Method</th>
<th>M</th>
<th>CPU time</th>
<th>( L_\infty ) error.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method</td>
<td>20</td>
<td>6.9844</td>
<td>1.3107E-7</td>
</tr>
<tr>
<td>IMEX-TF [6]</td>
<td>20</td>
<td>114.551534</td>
<td>0.000006</td>
</tr>
<tr>
<td>IMEX-class[6]</td>
<td>20</td>
<td>7.254047</td>
<td>0.103574</td>
</tr>
<tr>
<td>Present method</td>
<td>40</td>
<td>34.1141</td>
<td>2.5390E-12</td>
</tr>
<tr>
<td>IMEX-class[6]</td>
<td>40</td>
<td>78.499703</td>
<td>0.025453</td>
</tr>
<tr>
<td>IMEX-class[6]</td>
<td>80</td>
<td>29.452989</td>
<td>0.006355</td>
</tr>
<tr>
<td>IMEX-class[6]</td>
<td>160</td>
<td>132.944052</td>
<td>0.001592</td>
</tr>
<tr>
<td>IMEX-class[6]</td>
<td>320</td>
<td>1627.52733</td>
<td>0.000398</td>
</tr>
<tr>
<td>IMEX-class[6]</td>
<td>640</td>
<td>14774.495908</td>
<td>0.000099</td>
</tr>
</tbody>
</table>

**Table 2: Absolute errors for Experiment 1 with \( M = 20, N = 20 \).**

Table 3: Absolute errors for Experiment 1 with \( M = 40, \ N = 40 \).

<table>
<thead>
<tr>
<th>( u(x,t) )</th>
<th>Numer. solution ( V )</th>
<th>Exact solution ( V )</th>
<th>Abs. error</th>
<th>Numer. solution ( U )</th>
<th>Exact solution ( U )</th>
<th>Abs. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3.18, 0.001)</td>
<td>0.0015</td>
<td>0.0015</td>
<td>0.0023E-12</td>
<td>0.1753</td>
<td>0.1753</td>
<td>0.4201E-12</td>
</tr>
<tr>
<td>(2.97, 0.014)</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0134E-12</td>
<td>0.2061</td>
<td>0.2061</td>
<td>0.5137E-12</td>
</tr>
<tr>
<td>(6.23, 0.028)</td>
<td>0.0138</td>
<td>0.0138</td>
<td>0.0223E-12</td>
<td>0.2388</td>
<td>0.2388</td>
<td>0.5328E-12</td>
</tr>
<tr>
<td>(7.59, 0.049)</td>
<td>0.0245</td>
<td>0.0245</td>
<td>0.1543E-12</td>
<td>0.3087</td>
<td>0.3087</td>
<td>0.6217E-12</td>
</tr>
<tr>
<td>(8.90, 0.301)</td>
<td>0.0381</td>
<td>0.0381</td>
<td>0.2321E-12</td>
<td>0.3455</td>
<td>0.3455</td>
<td>0.6357E-12</td>
</tr>
<tr>
<td>(8.95, 0.312)</td>
<td>0.0737</td>
<td>0.0737</td>
<td>0.3861E-12</td>
<td>0.3833</td>
<td>0.3833</td>
<td>0.7211E-12</td>
</tr>
</tbody>
</table>

Experiment 2. Experiment 1 is joined with the following initial conditions and Dirichlet boundary conditions:

\[
u(x,0) = \sin x, \quad \pi \leq x \\
u(0,t) = t, \quad u(\frac{5}{2} \pi, t) = t + 1, \quad v(0,t) = t + 1, \quad v(\frac{5}{2} \pi, t) = t, \quad t \in [0,1].
\]

The exact solution is given by \( u = t + \sin x, v = t + \cos x \). We take \( d_i = 0.5, a_i = 0.1, a_1 = 0.5, a_2 = 0.1 \).

From the Figs. 7-8 and Table 4, it can be seen that when the number of nodes is increased, the absolute error tends to be stable and convergence rapidly, which indicates that the present method has good stability and convergence. It can be seen from Experiment 1-2 that when spatial \( x \) selects different ranges, the influence of numerical solutions is small. With different spatial values, when the number of nodes is greater, the absolute error is smaller.

Fig. 1 Numerical solutions obtained by the present method for Experiment 1 with \( M = 40, \ N = 20 \).
Experiment 3 We consider the following system having time-dependent model parameters and a nonlinear reaction term [14]:

\[
\begin{align*}
\frac{\partial \hat{u}}{\partial t} &= \frac{t}{2} \frac{\partial^2 \hat{u}}{\partial x^2} - \frac{t}{2} \frac{\partial \hat{u}}{\partial x} + (u - \nu)^2 + u^2 + \nu^2 - 2t^2, \\
\frac{\partial \hat{\nu}}{\partial t} &= \frac{t}{2} \frac{\partial^2 \hat{\nu}}{\partial x^2} + \frac{t}{2} \frac{\partial \hat{\nu}}{\partial x} + uv - t^2 - \frac{\sin(4x)}{2} + 1,
\end{align*}
\]

(13)

with the following initial and periodic boundary condition

\[
\begin{align*}
\hat{u}(x,0) &= -\sin 2x, & \hat{\nu}(x,0) &= -\cos 2x, & x \in [0,4\pi] \\
\hat{u}(0,t) &= t = \hat{u}(4\pi,t), & \hat{\nu}(0,t) &= \hat{\nu}(4\pi,t) = t, & t \in [0,1].
\end{align*}
\]

The exact solution is \( \hat{u} = t - \sin 2x \), \( \hat{\nu} = t - \cos 2x \). The numerical solution and the absolute error diagram of Experiment 3 are given in Fig. 9, respectively. As can be seen from the Figure 9, the method can be used to obtain a smaller absolute error.
Fig. 4 Absolute errors for Experiment 1 with $M = 20$, $N = 20$

![Graphs showing absolute errors for Experiment 1 with $M = 20$, $N = 20$.]

Fig. 5 Numerical solutions and absolute errors for Experiment 1 with $M = 40$, $N = 40$

![Graphs showing numerical solutions and absolute errors for Experiment 1 with $M = 40$, $N = 40$.]

(a) ![Graph showing comparison of absolute errors for Experiment 1.](image)

(b) ![Graph showing comparison of absolute errors for Experiment 1.](image)

Figure 6: Comparison of absolute errors for Experiment 1

Table 4: Absolute errors for Experiment 2 with $M = 40$, $N = 40$.

<table>
<thead>
<tr>
<th>$u(x,t)$</th>
<th>Numer. solution $v$</th>
<th>Exact solution $v$</th>
<th>Abs. error $v$</th>
<th>Numer. solution $u$</th>
<th>Exact solution $u$</th>
<th>Abs. error $u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.21,0.001)</td>
<td>1.0222</td>
<td>1.0222</td>
<td>0.0007E-12</td>
<td>0.7256</td>
<td>0.7256</td>
<td>0.1034E-12</td>
</tr>
<tr>
<td>(1.34,0.018)</td>
<td>1.0711</td>
<td>1.0711</td>
<td>0.1015E-12</td>
<td>0.9756</td>
<td>0.9756</td>
<td>0.1124E-12</td>
</tr>
<tr>
<td>(2.18,0.003)</td>
<td>1.1216</td>
<td>1.1216</td>
<td>0.2037E-12</td>
<td>1.4756</td>
<td>1.4756</td>
<td>0.2157E-12</td>
</tr>
<tr>
<td>(0.17,0.508)</td>
<td>1.1733</td>
<td>1.1733</td>
<td>1.3009E-12</td>
<td>1.5971</td>
<td>1.5971</td>
<td>0.3210E-12</td>
</tr>
<tr>
<td>(2.10,0.019)</td>
<td>1.2256</td>
<td>1.2256</td>
<td>1.3472E-12</td>
<td>1.6301</td>
<td>1.6301</td>
<td>0.3789E-12</td>
</tr>
<tr>
<td>(2.55,0.149)</td>
<td>1.2778</td>
<td>1.2778</td>
<td>2.0097E-12</td>
<td>1.7146</td>
<td>1.7146</td>
<td>0.4321E-12</td>
</tr>
</tbody>
</table>
Fig. 7 Numerical solutions and absolute errors for Experiment 2 with $M = 20$, $N = 20$

Fig. 8 Numerical solutions and absolute errors for Experiment 2 with $M = 40$, $N = 40$

Fig. 9 Numerical solutions and absolute errors for Experiment 3 with $M = 50$, $N = 50$

Conclusions

In the present work, a class of advection-reaction-diffusion systems have solved by using barycentric interpolation collocation method (BICM). The numerical experiments show that the algorithm is high accuracy. We will apply this approach to more areas in the future.

Acknowledgements

The authors would like to express their thanks to the unknown referees for their careful reading and helpful comments. This paper is supported by Inner Mongolia Grassland Talent Project (No.12000-12102012), Project of Inner Mongolia Institute of Data Science and Big Data (No.BDY18007).
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Paper submitted: August 3, 2018
Paper revised: November 29, 2018
Paper accepted: February 21, 2019