THE GENERALIZED GIACHETTI-JOHNSON HIERARCHY AND ALGEBRO-GEOMETRIC SOLUTIONS OF THE COUPLED KdV-MKdV EQUATION

by

Chao YUE [a*], ***Tiecheng XIA*** [b], ***Guijuan LIU*** [a], ***Qiang LU*** [a], **and *Ning ZHANG*** [c]

[a] College of Medical Information Engineering,
Shandong First Medical University & Shandong Academy of Medical Sciences, Taian, China
[b] Department of Mathematics, Shanghai University, Shanghai, China
[c] Department of Basical Courses, Shandong University of Science and Technology, Taian, China

Original scientific paper
https://doi.org/10.2298/TSCI180719257Y

By using a Lie algebra A_1, an isospectral Lax pair is introduced from which a generalized Giachetti-Johnson hierarchy is generated, which reduce to the coupled KdV-MKdV equation, furthermore, the algebro-geometric solutions of the coupled KdV-MKdV equation are constructed in terms of Riemann theta functions.

Key words: *algebro-geometric solution, Riemann theta function, coupled KdV-MKdV equation*

Introduction

In recent years, a family of methods were developed to find the exact solutions for the linear and non-linear PDE. Among them, there are adomian decomposition method [1], traveling wave transformation method [2, 3], Riccati equation method [4] and algebro-geometric method [5-7]. The mathematical model of shallow water waves was rediscovered by Korteweg and de Vries [8], which is commonly known as KdV equation, many physical quantities of KdV equation and MKdV equation have been discussed later [9-11]. In this paper, we first use a Lie algebra A_1 to obtain the generalized Giachetti-Johnson (GGJ) hierarchy, which reduce to the coupled KdV-MKdV equation, Then in terms of Riemann theta functions, the algebro-geometric solutions of the coupled KdV-MKdV equation are constructed.

The GGJ hierarchy and coupled KdV-MKdV equation

The Lie algebra A_1 has a basis [12, 13]:

$$e_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ e_2=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ e_3=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\ [e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_2,e_3]=e_1 \tag{1}$$

Consider the isospectral problem:

$$\begin{aligned}&\psi_x=U_\psi, U=(\alpha\lambda+s)e_1(0)+u_1e_2(0)+(\alpha_1+u_2)e_3(0),\\ &\psi_t=V_\psi, V=\sum_{m\geq 0}\left[a_me_1(-m)+b_me_2(-m)+c_me_3(-m)\right]\end{aligned} \tag{2}$$

---
[*] Corresponding author, e-mail: yuechao_71@163.com

Note:

$$V_+^{(n)}=\sum_{m=0}^{n}\left[a_m e_1(n-m)+b_m e_2(n-m)+c_m e_3(n-m)\right],$$
$$-V_{+x}^{(n)}+\left[U,V_+^{(n)}\right]=-2\alpha b_{n+1}e_2(0)+2\alpha c_{n+1}e_3(0)$$

Set $V^{(n)}=V_+^{(n)}-a_n e_1(0)$, then $U_t-V_+^{(n)}+[U,V^{(n)}]$ leads to the GGJ hierarchy:

$$u_{t_n}=\begin{pmatrix}u_1\\u_2\\s\end{pmatrix}_{t_n}=\begin{pmatrix}0&\partial-2s&0\\\partial+2s&0&0\\0&0&-\frac{1}{2}\partial\end{pmatrix}\begin{pmatrix}c_n\\b_n\\2a_n\end{pmatrix}=J_1P_n \tag{3}$$

When taking $n=3$, $s=0$ in eq. (3), we have the coupled KdV-MKdV equation:

$$u_{1t_3}=\frac{1}{4}\alpha^{-3}[u_{1xxx}-4u_1(\alpha_1+u_2)u_{1x}-2u_1^2u_{2x})],u_{2t_3}=$$
$$=\frac{1}{4}\alpha^{-3}[u_{2xxx}-4u_1(\alpha_1+u_2)u_{2x}-2(\alpha_1+u_2)^2u_{1x})] \tag{4}$$

Algebro-geometric solutions of the coupled KdV-MKdV equation

We introduce the Lenard gradient sequence $\{S_j\}_{j=0}^{\infty}$.

$$KS_j=JS_{j+1},S_0=(0,0,1)^T \tag{5}$$

$$K=\begin{pmatrix}0&\partial&2u_1\\-\partial&0&2(\alpha_1+u_2)\\-u_1&\alpha_1+u_2&\partial\end{pmatrix},\ J=\begin{pmatrix}0&2\alpha&0\\2\alpha&0&0\\-u_1&\alpha_1+u_2&\partial\end{pmatrix}$$

Let $X=(X_1,X_2)^T$ and $Y=(Y_1,Y_2)^T$ be two basic solutions of spectral problems:

$$\Phi_x=U\Phi,U=\begin{pmatrix}\alpha\lambda&u_1\\\alpha_1+u_2&-\alpha\lambda\end{pmatrix},\ \Phi_{t_m}=V^{(m)}\Phi,\ V^{(m)}=\begin{pmatrix}A^{(m)}&B^{(m)}\\C^{(m)}&-A^{(m)}\end{pmatrix} \tag{6}$$

$$A^{(m)}=-a_m+\sum_{j=0}^{m}a_j\lambda^{m-j},\ B^{(m)}=\sum_{j=0}^{m}b_j\lambda^{m-j},\ C^{(m)}=\sum_{j=0}^{m}c_j\lambda^{m-j} \tag{7}$$

then:

$$W=\frac{1}{2}(XY^T+YX^T)\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}\hat g&\hat f\\\hat h&-\hat g\end{pmatrix}$$

satisfies the Lax equation:

$$W_x=[U,W],\ W_{t_m}=\left[V^{(m)},W\right] \tag{8}$$

which implies that det W is a constant independent of x and t_m. From eq. (8), we get:

$$\hat g_x=u_1\hat h-(\alpha_1+u_2)\hat f,\ \hat f_x=2\alpha\lambda\hat f-2u_1\hat g,\quad \hat h_x=-2\alpha\lambda\hat h+2(\alpha_1+u_2)\hat g \tag{9}$$

$$\hat g_{t_m}=B^{(m)}\hat h-C^{(m)}\hat f,\ \hat f_{t_m}=2A^{(m)}\hat f-2B^{(m)}\hat g,\ \hat h_{t_m}=2C^{(m)}\hat g-2A^{(m)}\hat h \tag{10}$$

$$\hat{g}=\sum_{j=0}^{N+1}\hat{g}_j\lambda^{N+1-j},\ \hat{f}=\sum_{j=0}^{N+1}\hat{f}_j\lambda^{N+1-j},\ \hat{h}=\sum_{j=0}^{N+1}\hat{h}_j\lambda^{N+1-j},$$
$$KQ_{j-1}=JQ_j,\ JQ_0=0, KQ_{N+1}=1 \tag{11}$$

$$Q_j=(\hat{h}_j,\hat{f}_j,\hat{g}_j)^T,\ Q_0=\beta_0S_0=\beta_0(0,0,1)^T,\ Q_k=\sum_{j=0}^{k}\beta_jS_{k-j},\ k=0,1,...,\ \ \beta_0\tilde{S}_N+...+\beta_N\tilde{S}_N=0 \tag{12}$$

Set $\beta_0 = 1$ in eq. (12), from eqs. (5) and (12), we have:

$$Q_1=\begin{pmatrix}\alpha^{-1}(\alpha_1+u_2)\\ \alpha^{-1}u_1\\ \beta_1\end{pmatrix},\ Q_2=\begin{pmatrix}-\frac{1}{2}\alpha^{-2}u_{2x}+\alpha^{-1}\beta_1(\alpha_1+u_2)\\ \frac{1}{2}\alpha^{-2}u_{1x}+\alpha^{-1}\beta_1u_1\\ -\frac{1}{2}\alpha^{-2}u_1(\alpha_1+u_2)+\beta_2\end{pmatrix}$$

$$\hat{f}=\alpha^{-1}u_1\prod_{j=1}^{N}\left(\lambda-\mu_j\right),\ \hat{h}=\alpha^{-1}\left(\alpha_1+u_2\right)\prod_{j=1}^{N}\left(\lambda-\nu_j\right) \tag{13}$$

By comparing the coefficients of λ^{N-1}, λ^{N-2} and combining eqs. (11) and (13), we have:

$$\frac{1}{2}\alpha^{-1}\frac{u_{1x}}{u_1}+\beta_1=-\sum_{j=1}^{N}\mu_j,\ -\frac{1}{2}\alpha^{-1}\frac{(\alpha_1+u_2)_x}{\alpha_1+u_2}+\beta_1=-\sum_{j=1}^{N}\nu_j \tag{14}$$

$$\frac{1}{4}\alpha^{-2}\left[\frac{u_{1xx}}{u_1}-2u_1(\alpha_1+u_2)\right]+\frac{1}{2}\alpha^{-1}\beta_1\frac{u_{1x}}{u_1}+\beta_2=\sum_{j<k}^{N}\mu_j\mu_k \tag{15}$$

$$-\frac{1}{4}\alpha^{-2}\left[\frac{(\alpha_1+u_2)_{xx}}{\alpha_1+u_2}-2u_1(\alpha_1+u_2)\right]-\frac{1}{2}\alpha^{-1}\beta_1\frac{(\alpha_1+u_2)_x}{\alpha_1+u_2}+\beta_2=\sum_{j<k}^{N}\nu_j\nu_k \tag{16}$$

$$-\det W=\hat{g}^2+\hat{f}\hat{h}=\prod_{j=1}^{2N+2}\left(\lambda-\lambda_j\right)=R\left(\lambda\right) \tag{17}$$

$$2\hat{g}_0\hat{g}_1=-\sum_{j=1}^{2N+2}\lambda_j, \hat{g}_1^2+2\hat{g}_0\hat{g}_2+\hat{f}_1\hat{h}_1=\sum_{j<k}\lambda_j\lambda_k$$

$$\beta_1=-\frac{1}{2}\sum_{j=1}^{2N+2}\lambda_j,\ \beta_2=\frac{1}{2}\left[\sum_{j<k}\lambda_j\lambda_k-\frac{1}{4}\left(\sum_{j=1}^{2N+2}\lambda_j\right)^2\right] \tag{18}$$

Thus, we get:

$$\hat{g}\,|_{\lambda=\mu_k}=\sqrt{R\left(\mu_k\right)},\ \hat{f}_x\,|_{\lambda=\mu_k}=-\alpha^{-1}u_1u_{kx}\prod_{j=1,j\neq k}^{N}\left(\mu_k-\mu_j\right)=-2u_1\hat{g}\,|_{\lambda=\mu_k},\quad \mu_{kx}=\frac{2\alpha\sqrt{R\left(\mu_k\right)}}{\prod_{j=1,j\neq k}^{N}\left(\mu_k-\mu_j\right)}$$

$$\hat{g}\,|_{\lambda=\nu_k}=\sqrt{R\left(\nu_k\right)},\ \hat{h}_x\,|_{\lambda=\nu_k}=-\alpha^{-1}(\alpha_1+u_2)\nu_{kx}\prod_{j=1,j\neq k}^{N}\left(\nu_k-\nu_j\right)=2(\alpha_1+u_2)\hat{g}\,|_{\lambda=\nu_k} \tag{19}$$

$$\nu_{kx}\frac{2\alpha\sqrt{R\left(\nu_k\right)}}{\prod_{j=1,j\neq k}^{N}\left(\nu_k-\nu_j\right)}$$

which gives rise to:

$$\mu_{kx} = \frac{2\alpha\sqrt{R(\mu_k)}}{\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)},\ \nu_{kx}\ \frac{2\alpha\sqrt{R(\nu_k)}}{\prod_{j=1,j\neq k}^{N}(\nu_k-\nu_j)} \tag{20}$$

$$\begin{aligned}\mu_{kt} &= \frac{2\left[\mu_k^2-\left(\sum_{j=1}^{N}\mu_j+\beta_1\right)\mu_k+\sum_{j<k}u_ju_k+\left(\sum_{j=1}^{N}\mu_j+\beta_1\right)\beta_1-\beta_2\right]\sqrt{R(\mu_k)}}{\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)}\\ \nu_{kt} &= \frac{-2\left[\nu_k^2-\left(\sum_{j=1}^{N}\nu_j+\beta_1\right)\nu_k+\sum_{j<k}\nu_j\nu_k+\left(\sum_{j=1}^{N}\nu_j+\beta_1)\right)\beta_1-\beta_2\right]\sqrt{R(\nu_k)}}{\prod_{j=1,j\neq k}^{N}(\nu_k-\nu_j)}\end{aligned} \tag{21}$$

then (u_1, u_2) determined by eq. (14) is a solution of eq. (4).

We consider the hyper-elliptic Riemann surface:

$$\Gamma:\xi^2{=}R(\lambda),\ R(\lambda){=}\prod_{j=1}^{2N+2}(\lambda-\lambda_j),\ \lambda_{2N+2}=0$$

for a fixed point p_0, we introduce the Abel-Jacobi co-ordinate:

$$\rho_m=(\rho_m^{(1)},\rho_m^{(2)},\cdots,\rho_m^{(N)},)^T,\ m=1,2$$

with

$$\rho_1^{(j)}(x,t)=\sum_{k=1}^{N}\int_{p_0}^{\mu_k(x,t)}\omega_j=\sum_{k=1}^{N}\sum_{l=1}^{N}\int_{p_0}^{\mu_k}C_{jl}\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}},\ \rho_2^{(j)}(x,t)=\sum_{k=1}^{N}\int_{p_0}^{\upsilon_k(x,t)}\omega_j=\sum_{k=1}^{N}\sum_{l=1}^{N}\int_{p_0}^{\upsilon_k}C_{jl}\frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}}$$

$$\partial_x\rho_1^{(j)}=\sum_{k=1}^{N}\sum_{l=1}^{N}C_{jl}\frac{\mu_k^{l-1}\mu_{kx}}{\sqrt{R(\lambda)}}=\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{2\alpha\mu_k^{l-1}C_{jl}}{\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)},\ \text{and} \tag{22}$$

$$\sum_{l=1}^{N}\frac{\mu_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)}=\delta_{jN},l=1,2,\cdots,N$$

In a similar way, we obtain from (20)- (22):

$$\partial_t\rho_1^{(j)}=\Omega_1^{(j)}{=}2(C_{jN-2}-\beta_1C_{jN-1}+\beta_1^2C_{jN}-\beta_2C_{jN}),\partial_x\rho_2^{(j)}=-\Omega_0^{(j)},\ \partial_t\rho_2^{(j)}=-\Omega_1^{(j)}$$

$$j=1,2,\cdots,N.\ \rho_1=\Omega_0x+\Omega_1t+\gamma_1$$

$$\rho_2=-\Omega_0x-\Omega_1t+\gamma_2,\ \gamma_1^{(j)}=\sum_{k=1}^{N}\int_{p_0}^{\mu_k(0,0)}\omega_j,\gamma_2^{(j)}=\sum_{k=1}^{N}\int_{p_0}^{\upsilon_k(0,0)}\omega_j\Omega_m=(\Omega_m^{(1)},\Omega_m^{(2)},\cdots,\Omega_m^{(N)},)^T$$

$$\gamma_m=(\gamma_m^{(1)},\gamma_m^{(2)},\cdots,\gamma_m^{(N)},)^T,\ m=1,2$$

We define an Abel map on Γ:

$$A(p)=\int_{p_0}^{p}\omega,\omega=(\omega_1,\cdots,\omega_N)^T,\ A\left(\sum n_kp_k\right)=\sum n_kA(p_k)$$

Consider two special divisors $\sum_{k=1}^{N}P_m^{(k)}$ $(m = 1, 2)$, then we have:

$$A\left[\sum_{k=1}^{N} p_1^{(k)}\right]=\sum_{k=1}^{N} A[p_1^{(k)}]=\sum_{k=1}^{N}\int_{p_0}^{\mu_k}\omega=\rho_1,\ \ A\left[\sum_{k=1}^{N} p_2^{(k)}\right]=\sum_{k=1}^{N} A[p_2^{(k)}]=\sum_{k=1}^{N}\int_{p_0}^{\upsilon_k}\omega=\rho_2$$

with

$$p_1^{(k)}=[\mu_k,\xi(\mu_k)]\ \text{and}\ p_2^{(k)}=[\upsilon_k,\xi(\upsilon_k)]$$

The Riemann theta function of Γ is defined:

$$\theta(\zeta)=\sum_{z\in Z^N}\exp(\pi i\langle\tau z,z\rangle+2\pi i\langle\zeta,z\rangle),\ \ \zeta\in\mathbb{C}^N$$

where

$$\zeta=\left(\zeta_1,\cdots,\zeta_N\right)^T,\langle\zeta,z\rangle=\sum\nolimits_{k=1}^{N}\zeta_j z_j$$

Then we have:

$$\sum_{j=1}^{N}\mu_j=I-\sum_{s=1}^{2}\operatorname{Re}s_{\lambda=\infty_s}\lambda\mathrm{d}\ln F_1(\lambda),\ \ \sum_{j=1}^{N}\upsilon_j=I-\sum_{s=1}^{2}\operatorname{Re}s_{\lambda=\infty_s}\lambda\mathrm{d}\ln F_2(\lambda)$$

$$F_m\left(z^{-1}\right)=\theta_s^{(m)}+z(-1)^{s-1}\sum_{j=1}^{N}C_{jN}D_j\theta_s^{(m)}+o\left(z^2\right) \tag{23}$$

where

$$\theta_s^{(m)}=\theta(\rho_m+M_m+\eta_s)=\theta(\cdots,\rho_m^{(j)}+M_m^{(j)}+\eta_s^{(j)},\cdots)$$

It is easy to calculate that:

$$\partial_x\theta_s^{(m)}=\sum_{j=1}^{N}2\alpha C_{jN}D_j\theta_s^{(m)} \tag{24}$$

Substituting eq. (24) into eq. (23), we have:

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln F_m\left(z^{-1}\right)=\frac{1}{2}\alpha^{-1}(-1)^{s-1}\partial_x\ln\theta_s^{(m)}+o\left(z\right)\quad F_m\left(z^{-1}\right)=\theta_s^{(m)}+\frac{z}{2}\alpha^{-1}(-1)^{s-1}\partial_x\theta_s^{(m)}+o\left(z^2\right)$$

and

$$\operatorname{Re}s_{\lambda=\infty_s}\lambda\mathrm{d}\ln F_m\left(\lambda\right)=\frac{1}{2}\alpha^{-1}(-1)^{s-1}\partial_x\ln\theta_s^{(m)},\ \ s=1,2,\ m=1,2 \tag{25}$$

where

$$\theta_s^{(1)}=\theta(\Omega_0x+\Omega_1t+\pi_s),\ \ \theta_s^{(2)}=\theta(-\Omega_0x-\Omega_1t+\eta_s)$$

From eq. (23), we have:

$$\sum_{j=1}^{N}\mu_j=I+\frac{1}{2}\alpha^{-1}\partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}},\ \ \sum_{j=1}^{N}\upsilon_j=I+\frac{1}{2}\alpha^{-1}\partial_x\ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}} \tag{26}$$

Substituting eq. (26) into eq. (14), we obtain the algebro-geometric solutions of the coupled KdV-MKdV eq. (4):

$$u_1=\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\exp\left[\alpha\left(2I-\sum_{j=1}^{2N+2}\lambda_j\right)x+u_1^0\left(t\right)\right],\ u_2=\frac{\theta_1^{(2)}}{\theta_2^{(2)}}\exp\left[-\alpha\left(2I-\sum_{j=1}^{2N+2}\lambda_j\right)x+u_2^0\left(t\right)\right]-\alpha_1$$

Acknowledgment

This work was supported by the Natural Science Foundation of Shandong Province (Grant No. ZR2016AL04, ZR2016FL05, ZR2017MF039 and ZR2012AM021), the Natural Science Foundation of China (No.11805114), and the High-Level Trainning Project of Taishan Medical University (No.2015GCC07).

Nomenclature

t – time, [s]
x – space, [m]

References

[1] Adomian, G., A Review of the Decomposition Method and Sme Recent Results for Non-Linear Equations, *Computers and Mathematics with Applications*, *21* (1991), 5, pp. 101-127
[2] Yang, X. J., *et al.*, Exact Traveling-Wave Soluton for Local Fractional Boussinesq Equation in Fractal Domain, *Fractals, 25* (2017), 4, ID 1740006
[3] Yang, X. J., *et al.*, A New Computational Approach for Solving Non-Linear Local Fractional PDE, *Journal of Computational and Applied Mathematics*, *339* (2018), pp. 285-296
[4] Yang, X. J., *et al.*, Exact Travelling Wave Solutions for the Local Fractional 2-D Burgers-Type Equations, *Computers and Mathematics with Applications*, *73* (2017), 2, pp. 203-210
[5] Gesztesy, F., *et al.*, *Soliton Equations and their Algebro-Geometric Solutions*, Cambridge University Press, Cambridge, UK, 2003
[6] Hou, Y., *et al.*, Algebro-Geometric Solutions for the Gerdjikov-Ivanov Hierarchy, *Journal of Mathematical Physics*, *54* (2013), 073505-30
[7] Zhao, P., *et al.*, Algebro-Geometric Solutions for the Ruijsenaars-Toda Hierarch, *Chaos, Solitons and Fractals*, *54* (2012), Apr., pp. 8-25
[8] Korteweg, D. J., *et al.*, On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *39* (1895), 240, pp. 422-443
[9] Calogero, F., *et al.*, *Spectral Transform and Solitons: Tools to Solve and Investigate Non-Linear Evolution Equations*, North-Holland, New York, USA, 1982
[10] Yang, X. J., *et al.*, Modelling Fractal Waves on Shallow Water Surfaces via Local Fractional Korteweg-de Vries Equation, *Abstract and Applied Analysis*, *4* (2014), May, pp. 1-10
[11] Yang, X. J., *et al.*, *Local Fractional Integral Transforms and Their Applications*, Academic Press, New York, USA, 2015.
[12] Tu, G. Z., The Trace Identity, a Powerful Tool for Constructing the Hamiltonian Structure of Integrable Systems, *Journal of Mathematical Physics*, *30* (1989), 2, pp. 330-338
[13] Zhang, Y. F., *et al.*, Some Evolution Hierarchies Derived from Self-Dual Yang-Mills Equtions, *Communications In Theoretical Physics*, *56* (2011), 5, pp. 856-872

Paper submitted: July 19, 2018
Paper revised: October 20, 2018
Paper accepted: January 25, 2019