THE GENERALIZED GIACHETTI-JOHNSON HIERARCHY AND ALGEBRO-GEOMETRIC SOLUTIONS OF THE COUPLED KDV-MKDV EQUATION

by

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By using a Lie algebra $A_1$, an isospectral Lax pair is introduced from which a generalized Giachetti-Johnson (GGJ) hierarchy is generated, which reduce to the coupled KdV-MKdV equation. Furthermore, the algebro-geometric solutions of the coupled KdV-MKdV equation are constructed in terms of Riemann theta functions.

Key words: algebro-geometric solution, Riemann theta function, coupled KdV-MKdV equation

Introduction

In recent years, a family of methods were developed to find the exact solutions for the linear and nonlinear partial differential equations. Among them, there are Adomian decomposition method\cite{1}, traveling wave transformation method\cite{2,3}, Riccati equation method\cite{4} and algebro-geometric method\cite{5-7}. The mathematical model of shallow water waves was rediscovered by Korteweg and de Vries\cite{8}, which is commonly known as KdV equation, many physical quantities of KdV equation and MKdV equation have been discussed later\cite{9-11}. In this paper, we first use a Lie algebra $A_1$ to obtain the generalized Giachetti-Johnson (GGJ) hierarchy, which reduce to the coupled KdV-MKdV equation. Then in terms of Riemann theta functions, the algebro-geometric solutions of the coupled KdV-MKdV equation are constructed.

The GGJ hierarchy and coupled KdV-MKdV equation

The Lie algebra $A_1$ has a basis \cite{12,13}:

$$e_1=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[e_1,e_2]=2e_3, [e_1,e_3]=-2e_2, [e_2,e_3]=e_1,$$

(1)

Consider the isospectral problem

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\[ \psi \varepsilon U, U = (\alpha \lambda + s) e_1(0) + u e_2(0) \]
\[ \psi \varepsilon V, V = \sum_{n=0}^{\infty} \left( a_n e_1(n-m) + b_n e_2(n-m) + c_n e_3(n-m) \right), \]

Note \( V^{(\alpha)} = \sum_{n=0}^{\infty} \left( a_n e_1(n-m) + b_n e_2(n-m) + c_n e_3(n-m) \right). \)

\[ -V^{(\alpha)} + \left[ U, V^{(\alpha)} \right] = -2\alpha b_{2\alpha} + 2\alpha a_{2\alpha} e_1(0). \]

Set \( V^{(\alpha)} = V^{(\alpha)} - a_0 e_1(0) \), then \( U - V^{(\alpha)} + \left[ U, V^{(\alpha)} \right] = 0 \) leads to the GGJ hierarchy

\[ u_i = \begin{pmatrix} u_1 \\ u_2 \\ s \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 2\alpha \\ -u_1 & (\alpha + u_2) \\ 0 & -\frac{1}{2} \end{pmatrix} J P_r. \]

When taking \( n = 3, s = 0 \) in (3), we have the coupled KdV-MKdV equation

\[ u_i = \frac{1}{4} \alpha^2 \left[ u_{xxx} - 4u_1 (\alpha + u_2) u_{x2} - 2u_1 u_{x2} \right], u_{x2} = \frac{1}{4} \alpha^2 \left[ u_{xxx} - 4u_1 (\alpha + u_2) u_{x2} - 2(\alpha + u_2)^2 u_{x2} \right]. \]

Algebro-geometric solutions of the coupled KdV-MKdV equation

We introduce the Lenard gradient sequence \( \{s_j\}_{j=0}^{\infty} \),

\[ KS_j = JS_{j+1}, s_0 = (0,0,1)^T, \]

\[ K = \begin{pmatrix} 0 & \partial & 2u_1 \\ -\partial & 0 & 2(\alpha + u_2) \\ -u_1 & (\alpha + u_2) & \partial \end{pmatrix}, J = \begin{pmatrix} 0 & 2\alpha & 0 \\ 2\alpha & 0 & 0 \\ -u_1 & (\alpha + u_2) & \partial \end{pmatrix}. \]

Let \( X = (X_1, X_2)^T \) and \( Y = (Y_1, Y_2)^T \) be two basic solutions of spectral problems

\[ \Phi_x = U \Phi_x U = \begin{pmatrix} \alpha \lambda & u_1 \\ \alpha_1 + u_2 & -\alpha \lambda \end{pmatrix}, \Phi_y = V^{(m)} \Phi_y V^{(m)} = \begin{pmatrix} A^{(m)} & B^{(m)} \\ C^{(m)} & -A^{(m)} \end{pmatrix}, \]

\[ A^{(m)} = -a_m + \sum_{j=0}^{m} a_j \lambda^{m-j}, B^{(m)} = \sum_{j=0}^{m} b_j \lambda^{m-j}, C^{(m)} = \sum_{j=0}^{m} c_j \lambda^{m-j}, \]

then \( W = \frac{1}{2} (XY^T + YX^T) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \hat{g} & \hat{f} \\ \hat{h} & -\hat{g} \end{pmatrix} \) satisfies the Lax equation

\[ \dot{W} = [U, W], \quad W_0 = [V^{(m)}, W]. \]

which implies that \( \det W \) is a constant independent of \( x \) and \( t_m \). From (8), we get

\[ \dot{g}_x = u_1 \dot{h} - (\alpha_1 + u_2) \dot{f}, \quad \dot{f} = 2\alpha \lambda \dot{f} - 2u_1 \dot{g}, \quad \dot{h} = -2\alpha \lambda \dot{h} + 2(\alpha_1 + u_2) \dot{g}. \]

\[ \hat{g}_x = B^{(m)} \dot{h} - C^{(m)} \dot{f}, \quad \hat{f} = 2A^{(m)} \dot{f} - 2B^{(m)} \dot{g}, \quad \hat{h} = 2C^{(m)} \dot{g} - 2A^{(m)} \dot{h}. \]

\[ \hat{g} = \sum_{j=0}^{N} \hat{g}_x \lambda^{N+1-j}, \quad \hat{f} = \sum_{j=0}^{N} \hat{f}_x \lambda^{N+1-j}, \quad \hat{h} = \sum_{j=0}^{N} \hat{h}_x \lambda^{N+1-j}, \quad KQ_{j+1} = JQ_{j}, JQ_0 = 0, KQ_{N+1} = 1. \]
\[ Q_j = (\hat{h}, \hat{j}, \hat{g})^T. \quad Q_0 = \beta_0 S_0 = \beta_0(0,0,1)^T, \quad Q_i = \sum_{j=0}^{k-1} \beta_j S_{i-j}, \quad k = 0,1, \ldots, \quad \beta_0 \tilde{S}_N + \ldots + \beta_N \tilde{S}_N = 0. \quad (12) \]

Set \( \beta_0 = 1 \) in Eq. (12), from Eqs. (5) and (12), we have

\[ Q = \begin{pmatrix} \alpha - (a_i + u_j) \\ \alpha^2 u_j \\ \beta \end{pmatrix}, \quad Q_1 = \begin{pmatrix} -1/2 \alpha^2 u_i + \alpha^2 \beta (a_i + u_j) \\ \alpha^2 u_i + \alpha^2 \beta u_i \\ -1/2 \alpha^2 (a_i + u_j) + \beta_i \end{pmatrix}, \quad \hat{f} = \alpha^{-1} u_j \sum_{j=0}^{k} (\lambda - \mu), \quad \hat{h} = \alpha^{-1} (a_i + u_j) \sum_{j=0}^{k} (\lambda - v_j). \quad (13) \]

By comparing the coefficients of \( \lambda^{N-1}, \lambda^{N-2} \) and combining Eqs. (11) and (13), we have

\[ \frac{1}{2} \alpha^{-1} u_i + \beta_i = -\sum_{j=0}^{k} \mu_j - \frac{1}{2} \alpha^{-1} (a_i + u_j), \quad \beta_i = -\sum_{j=0}^{k} \beta_j, \quad \frac{1}{2} \alpha^{-1} u_i = -\frac{1}{2} \sum_{j=0}^{k} \mu_j, \quad \frac{1}{2} \alpha^{-1} (a_i + u_j) = \sum_{j=0}^{k} \beta_j, \quad \frac{1}{2} \alpha^{-1} \beta_i = \sum_{j=0}^{k} \beta_j. \quad (14) \]

\[ \frac{1}{4} \alpha^2 \beta_i u_i - 2u_i (a_i + u_j), \quad \frac{1}{2} \alpha^{-1} \beta_j u_i + \beta_j = \sum_{j=0}^{k} \mu_j \mu_j, \quad (15) \]

\[ \frac{1}{4} \alpha^2 \beta_i (a_i + u_j) - 2u_i (a_i + u_j), \quad \frac{1}{2} \alpha^{-1} \beta_j (a_i + u_j) + \beta_j = \sum_{j=0}^{k} \mu_j v_j, \quad (16) \]

\[ \det W = \hat{g} \hat{h} + \hat{h} = \prod_{j=0}^{k} (\lambda - \lambda_j) = R(\lambda). \quad (17) \]

\[ 2\hat{g}_i \hat{g}_j = -\sum_{j=0}^{k} \lambda_j \hat{g}_j + 2\hat{g}_i \hat{g}_j + \hat{h} = \sum_{j=0}^{k} \lambda_j \lambda_j, \quad \beta = \frac{1}{2} \sum_{j=0}^{k} \lambda_j, \quad \beta_i = \frac{1}{2} \left[ \sum_{j=0}^{k} \lambda_j \lambda_j - \frac{1}{4} \left( \sum_{j=0}^{k} \lambda_j \right)^2 \right]. \quad (18) \]

Thus, we get

\[ \hat{g} \mid_{\lambda=\mu} = \sqrt{R(\mu_i)}, \quad \hat{j} \mid_{\lambda=\mu} = -\alpha^{-1} u_i u_i \prod_{j=0}^{k} (\mu_j - \mu_j) = -2u_i \hat{g} \mid_{\lambda=\mu}, \quad \mu_i = \frac{2\alpha \sqrt{R(\mu_i)}}{\prod_{j=0}^{k} (\mu_j - \mu_j)}, \quad (19) \]

\[ \hat{g} \mid_{\lambda=v_i} = \sqrt{R(v_i)}, \quad \hat{h} \mid_{\lambda=v_i} = -\alpha^{-1} (a_i + u_j) v_j \prod_{j=0}^{k} (v_j - v_j) = 2(a_i + u_j) \hat{g} \mid_{\lambda=v_i}, \quad v_i = \frac{2\alpha \sqrt{R(v_i)}}{\prod_{j=0}^{k} (v_j - v_j)} \]

which gives rise to

\[ \mu_i = \frac{2\alpha \sqrt{R(\mu_i)}}{\prod_{j=0}^{k} (\mu_j - \mu_j)}, \quad v_i = \frac{2\alpha \sqrt{R(v_i)}}{\prod_{j=0}^{k} (v_j - v_j)}. \quad (20) \]

\[ \mu_i = \frac{2(\sum_{j=0}^{k} \mu_j + \beta_j) \mu_j + \sum_{j=0}^{k} u_i \mu_i + \sum_{j=0}^{k} \mu_j + \beta_j \beta_j \sqrt{R(\mu_j)}}{\prod_{j=0}^{k} (\mu_j - \mu_j)}, \quad (21) \]

\[ v_i = \frac{2(\sum_{j=0}^{k} v_j + \beta_j) v_j + \sum_{j=0}^{k} v_j v_j + \sum_{j=0}^{k} v_j + \beta_j \beta_j \sqrt{R(v_j)}}{\prod_{j=0}^{k} (v_j - v_j)}, \]

then \( (u_i, u_j) \) determined by Eq. (14) is a solution of equation (4).

We consider the hyper-elliptic Riemann Surface \( \Gamma \): \( \hat{g}^2 = R(\lambda), \quad R(\lambda) = \prod_{j=0}^{k} (\lambda - \lambda_j), \quad \lambda_{2k+2} = 0, \) for a fixed point \( \lambda_0 \), we introduce the Abel-Jacobi coordinate as \( \rho_m = (\rho_m^1, \rho_m^2, \ldots, \rho_m^n)^T, m = 1,2 \) with

\[ \rho_m^1(x_i, t_i) = \sum_{k=0}^{n} \int_{\lambda_{m+1}}^{\lambda_{m+1}} C_{\mu} \frac{2^{-1} d\sqrt{R(\lambda)}}{\sqrt{R(\lambda)}} \]

\[ \rho \rho_m^2(x_i, t_i) = \sum_{k=0}^{n} \int_{\lambda_{m+1}}^{\lambda_{m+1}} C_{\mu} \frac{2^{-1} d\sqrt{R(\lambda)}}{\sqrt{R(\lambda)}}. \quad (22) \]
\[ \partial_i \rho^{(i)} = \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ij} \frac{\mu_j \mu_k}{\sqrt{R(\lambda)}} + \sum_{j=1}^{N} \sum_{k=1}^{N} 2\alpha \mu_j \mu_k C_{ij}, \quad \text{and} \quad \sum_{j=1}^{N} \frac{\mu_j}{\prod_{j=1, j \neq l}^{N} (\mu_j - \mu_l)} = \delta_{\eta, l}, l = 1, 2, \ldots, N. \]

In a similar way, we obtain from (20)-(22)
\[ \partial_i \rho^{(j)} = -\Omega^{(j)} \omega_i (\tau_j, \zeta_j), \quad \partial_i \rho^{(j)} = -\Omega^{(j)}, \quad j = 1, 2, \ldots, N, \quad \rho_j = \Omega \omega_j + \Omega \tau_j + \gamma_j, \]
\[ \rho_j = -\Omega \omega_j + \Omega \tau_j + \gamma_j, \quad \gamma_j = \sum_{k=1}^{N} n_k \omega_k. \quad \eta_j = (\eta^{(0)}, \eta^{(2)}, \eta^{(4)}), m = 1, 2. \]

We define an Abel map on \( \Gamma \) as \( A(p) = \int_{p_0}^{p} \omega = \omega_1, \omega_2, \ldots, \omega_N \), \( A(\sum \eta_j p_j) = \sum \eta_j A(p_j) \).

Consider two special divisors \( \sum_{m=1}^{N} \rho^{(m)} \) (m = 1, 2), then we have
\[ A(\sum_{m=1}^{N} \rho^{(m)}) = A(\sum_{m=1}^{N} \rho^{(m)}) = \sum_{j=1}^{N} \omega_j = \rho_j, \quad A(\sum_{m=1}^{N} \rho^{(m)}) = \sum_{j=1}^{N} \omega_j = \rho_j. \]

with \( p^{(1)} = (\mu_j, \xi, \mu_k) \) and \( p^{(2)} = (\nu_j, \xi, \nu_k) \). The Riemann theta function of \( \Gamma \) is defined as
\[ \theta(\xi) = \sum_{x \in \mathbb{Z}^N} \exp(\pi i (\tau_j, \zeta_j) + 2\pi i (\xi, \zeta_j)) = \exp((\tau_j, \zeta_j) + (\xi, \zeta_j)) = \sum_{x \in \mathbb{Z}^N} \exp((\xi, \zeta_j)), j \in \mathbb{Z}^N, \]
\[ \theta(\xi, \zeta_j) = \sum_{x \in \mathbb{Z}^N} \exp((\tau_j, \zeta_j) + 2\pi i (\xi, \zeta_j)), j \in \mathbb{Z}^N. \]

Then we have
\[ \sum_{j=1}^{N} \mu_j = I + \sum_{j=1, j \neq l}^{N} \frac{\lambda_j}{\partial_{\lambda_j}} \ln \frac{\theta^{(m)}}{\partial_{\theta^{(m)}}} + \frac{\lambda_j}{\theta^{(m)}}, m = 1, 2, \]
\[ \theta^{(m)} = \theta(\Omega, \omega_j + \Omega \tau_j + \gamma_j), m = 1, 2, \]
\[ \theta^{(m)} = \theta(-\Omega, \omega_j + \Omega \tau_j + \gamma_j), m = 1, 2. \]

Substituting Eq. (24) into (23), we have
\[ \frac{d}{dz} \ln F_n (z^{-1}) = \frac{1}{2} \alpha (-1)^{s-1} \partial_z \ln \theta^{(s-1)} + o(z), \quad F_n (z^{-1}) = \theta^{(s-1)} + \frac{z}{2} \alpha (-1)^{s-1} \partial_z \theta^{(s-1)} + o(z), \]
\[ \text{and} \]
\[ \text{Res}_{\lambda_j} \lambda_j \ln F_n (\lambda) = \frac{1}{2} \alpha (-1)^{s-1} \partial_{\lambda_j} \ln \theta^{(s-1)}, s = 1, 2, m = 1, 2, \]
\[ \theta^{(1)} = \theta(\Omega, \omega_j + \Omega \tau_j + \gamma_j), \theta^{(2)} = \theta(-\Omega, \omega_j + \Omega \tau_j + \gamma_j). \]

Substituting Eq. (26) into Eq. (14), we obtain the algebro-geometric solutions of the coupled KdV-MKdV equation (4)
\[ u_1 = \frac{\theta^{(1)}}{\theta^{(1)}} \exp \left[ \alpha \left( 2I - \sum_{j=1}^{N} \lambda_j \right) x + u_1^0 (t) \right], \quad u_2 = \frac{\theta^{(2)}}{\theta^{(2)}} \exp \left[ -\alpha \left( 2I - \sum_{j=1}^{N} \lambda_j \right) x + u_2^0 (t) \right] - \alpha. \]

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Nomenclature

\[ t \text{-time, [s]} \quad x \text{-space,[m]} \]

References


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