A NEW APPROXIMATE ANALYTICAL METHOD FOR A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

by

Shao- Wen YAO , Kang- Le WANG*

School of Mathematics and information Science, Henan Polytechnic University, Jiaozuo, China

In this paper, a new approximate analytical method is established , and it is useful in constructing approximate analytical solution to a system of fractional differential equations . The results show that our method is reliable and efficient for solving the fractional system.

Key words: *caputo fractional derivative, analytical solution, fractional differential equations, DGJ method*

Introduction

In 2006, DGJ method (DGJM) was proposed by Daftardar-Gejji and Jafari in [1]. This method is now widely used by many researchers to study fractional partial differential equations. It was shown that the method is more powerful than existing techniques such as the Adomian method [2,3], travelling-wave method [4-8] and homotopy analysis method (HAM) [9]. The method gives rapidly convergent successive approximations of the exact solution, and it has no specific requirements [10,11], such as small parameters, linearization, Adomian polynomials for nonlinear terms, etc. Recently, Daftardar-Gejji and co-authors [12,13] have found the exact solution and approximate solution of fractional differential equations by using DGJ method.

In this paper, we have applied DGJM to study the following systems of fractional partial differential equations:

$$\begin{cases} D_t^{\alpha} u(x,t) = f(u,v,u_x,v_x), \\ D_t^{\beta} v(x,t) = g(u,v,u_{x,}v_x), \end{cases} \tag{1}$$

with the initial conditions

$$u(x,0) = \varphi(x),\ v(x,0) = \psi(x), \tag{2}$$

where the fractional derivative is understood in the Caputo sense [14], α and β are parameters describing the order of the Caputo fractional derivative($0 < \alpha \le 1,\ 0 < \beta \le 1$), and f, g, φ and ψ are given functions.

Such systems arise in various areas, especially in the study of chemical reactions, in population dynamics and in mathematical biology [15-21].

In [16], Jafari and Seifi studied a special case of the above systems:

* Corresponding author; e-mail: kangle83718@163.com

$$\begin{cases} D_t^{\alpha} u(x,t) = v_x - v - u, \\ D_t^{\beta} v(x,t) = u_x - v - u, \end{cases} \tag{3}$$

with the initial conditions as

$$u(x,0) = sh(x), v(x,0) = ch(x). \tag{4}$$

From [16], we can see that the structure of the solutions to the systems of fractional differential equations are very different from those of integer-order differential equations.

The aim of the present work is to construct approximate solution to the problem of Eqs. (1)-(2) by using DGJM. Our results show that the method introduces a reliable and efficient process for solving the systems .

The structure of this paper is as follows. In section 2, we introduce the basic definitions and properties of fractional calculus. In section 3, we illustrate the basic idea of DGJM. In section 4, we solve the systems of Eqs.(1)-(2) by using DGJ method. Finally, a brief conclusion is given in section 5.

Basic definitions of fractional calculus

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper [14].

Definition 1. A real function $f(x), x > 0$ is said to be in the space $C_{\mu}, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_{\mu}, n \in N$.

The Riemann-Liouville fractional integral operator is defined as follows [14]:

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x) \in C_{\mu}, \mu \ge -1$ is defined as:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)ds,$$

$$J^0 f(x) = f(x).$$

The properties of the operator J^{α} can be found in and we mention only the following [14]: For $\alpha, \beta \ge 0, x > 0$ and $\gamma > -1$:

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x),$$
$$J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x),$$

$$J^{\alpha}(x^{\gamma}) = \frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha+\gamma)} x^{\gamma+\alpha}.$$

Definition 4. The fractional derivative of $f(x)$ in Caputo sense is defined as[14]:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) ds,$$

for $m-1 < \alpha \le m, m \in N^+, x > 0$ and $f \in C_{-1}^m$.

We recall here two of its basic properties [14]:

$$D^{\alpha} J^{\alpha} f(x) = f(x),$$

$$J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \; x > 0$$

DGJ method

In this section, we illustrate the basic idea of DGJM [1]. We consider the following general function equation:

$$u = L(u) + N(u) + f, \tag{9}$$

where L is a linear operator, N is a nonlinear operator from a Banach space $B \to B$ and f a known function. We are looking for a solution u of equation (9) having the seriesform

$$u = \sum_{i=0}^{\infty} u_i. \tag{10}$$

The nonlinear operator N can be decomposed as :

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} u_j \right) - N\left(\sum_{j=0}^{i-1} u_j \right) \right\}. \tag{11}$$

From Eqs.(10) and (11), equation(9) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} u_j \right) - N\left(\sum_{j=0}^{i-1} u_j \right) \right\}. \tag{12}$$

Define the recurrence relation:

$$u_0 = f(x), \; u_1 = L(u_0) + H_0,$$

$$u_m = L(u_m) + H_m, \; m = 1,2,\cdots. \tag{13}$$

where

$$H_0 = N(u_0),$$

and

$$H_m = N\left(\sum_{i=0}^{m} u_i\right) - N\left(\sum_{i=0}^{m-1} u_i\right),\ m = 1,2,\cdots.$$

Then k-term approximate solution of equation (9) is given by

$$u = u_0 + u_1 + \cdots + u_{k-1}. \tag{14}$$

Solution for the system of Eqs. (1)-(2)

In this section we derive an algorithm of the DGJM for solving the systems of Eqs. (1)-(2).

To use DGJM , we rewrite the systems of Eqs. (1)-(2) as :

$$\begin{cases} D_t^{\alpha} u(x,t) = h_1(x) + L_1(X) + N_1(X), \\ D_t^{\beta} v(x,t) = h_2(x) + L_2(X) + N_2(X), \end{cases} \tag{15}$$

with the initial conditions

$$u(x,0) = \varphi(x),\ v(x,0) = \psi(x), \tag{16}$$

where h_1 and h_2 are known functions, L_1 and L_2 are linear operators, N_1 and N_2 are nonlinear operators, and $X = (u, v, u_x, v_x)$.

From the basic properties of the operators J^{α} and D^{α}, the systems of Eqs.(15)-(16) are equivalent to

$$\begin{cases} u(x,t) = J_t^{\alpha}(h_1(x) + L_1(X) + N_1(X)) + \varphi(x), \\ v(x,t) = J_t^{\beta}(h_2(x) + L_2(X) + N_2(X)) + \psi(x), \end{cases} \tag{17}$$

Suppose that the solution of equation (17) takes the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t),\ v(x,t) = \sum_{k=0}^{\infty} v_k(x,t).$$

Let

$$G_0 = N_1(X_0),\ D_0 = N_2(X_0),\ G_1 = N_1(X_0 + X_1) - N_1(X_0),\ D_1 = N_2(X_0 + X_1) - N_2(X_0),$$

$$G_m = N_1(X_0 + X_1 + \cdots X_m) - N_1(X_0 + X_1 + \cdots + X_{m-1}),$$

$$D_m = N_2(X_0 + X_1 + \cdots X_m) - N_2(X_0 + X_1 + \cdots + X_{m-1}),$$

where $m = 1, 2, \cdots,$ and $X_k = (u_k, v_k, u_{kx}, v_{kx})$, $k = 0,1,\cdots.$

Then we can construct the recurrence relation:

$$u_0(x,t) = \varphi(x),\ v_0(x,t) = \psi(x),\ u_1(x,t) = J_t^{\alpha}(h_1(x) + L_1(X_0) + G_0(X_0)),$$

$$v_1(x,t) = J_t^{\beta}(h_2(x) + L_2(X_0) + D_0(X_0)),$$

$$u_2(x,t) = J_t^{\alpha}(L_1(X_1) + G_1(X_0, X_1)),$$

$$v_2(x,t) = J_t^{\beta}(L_2(X_1) + D_1(X_0, X_1)),$$

$$u_3(x,t) = J_t^{\alpha}(L_1(X_2) + G_2(X_0, X_1, X_3)),$$

$$v_3(x,t) = J_t^{\beta}(L_2(X_2) + D_2(X_0, X_1, X_2)), \tag{18}$$

and so on.

Thus the solutions in the series form are given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots,$$

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \cdots.$$

To give a clear over of the algorithm introduced above, two illustrative system of partial differential equations, a linear and another nonlinear, have been selected to demonstrate the reliability and efficiency of the method.

Example 1. Consider the following linear system [16]:

$$\begin{cases} D_t^{\alpha} u(x,t) = v_x - v - u, \\ D_t^{\beta} v(x,t) = u_x - v - u, \end{cases}$$

with the initial conditions as

$$u(x,0) = sh(x), v(x,0) = ch(x).$$

The initial value problem are equivalent to the following fractional integral equations:

$$\begin{cases} u(x,t) = J_t^{\alpha}(v_x - v - u) + sh(x), \\ v(x,t) = J_t^{\beta}(u_x - v - u) + ch(x). \end{cases}$$

Following the algorithm given in equation (18):

$$u_0(x,t) = sh(x), \quad v_0(x,t) = ch(x),$$

$$u_1(x,t) = J_t^{\alpha}(v_{0x} - v_0 - u_0) = -\frac{ch(x)}{\Gamma(1+\alpha)} t^{\alpha},$$

$$v_1(x,t) = J_t^{\beta}(u_{0x} - v_0 - u_0) = -\frac{sh(x)}{\Gamma(1+\beta)} t^{\beta},$$

$$u_2(x,t) = J_t^{\alpha}(v_{1x} - v_1 - u_1) = \frac{ch(x)}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{sh(x) - ch(x)}{\Gamma(1+\alpha+\beta)} t^{\alpha+\beta},$$

$$v_2(x,t)=J_t^{\beta}(u_{1x}-v_1-u_1)=\frac{sh(x)}{\Gamma(1+2\beta)}t^{2\beta}+\frac{ch(x)-sh(x)}{\Gamma(1+\alpha+\beta)}t^{\alpha+\beta},$$

and so on.

Hence

$$u(x,t)=sh(x)-\frac{ch(x)}{\Gamma(1+\alpha)}t^{\alpha}+\frac{ch(x)}{\Gamma(1+2\alpha)}t^{2\alpha}+\frac{sh(x)-ch(x)}{\Gamma(1+\alpha+\beta)}t^{\alpha+\beta}+\cdots,$$

$$v(x,t)=ch(x)-\frac{sh(x)}{\Gamma(1+\beta)}t^{\beta}+\frac{sh(x)}{\Gamma(1+2\beta)}t^{2\beta}+\frac{ch(x)-sh(x)}{\Gamma(1+\alpha+\beta)}t^{\alpha+\beta}+\cdots.$$

If $\alpha=\beta=1$, then

$$u(x,t)=sh(x)(1+\frac{t^2}{2!}+\cdots)-ch(x)(t+\frac{t^3}{3!}+\cdots),$$

$$v(x,t)=ch(x)(1+\frac{t^2}{2!}+\cdots)-sh(x)(t+\frac{t^3}{3!}+\cdots),$$

which are exactly the same as the solutions obtained in [21] converging to the exact solutions, $u(x,t)=sh(x-t), v(x,t)=ch(x-t)$.

Example 2. Consider the following nonlinear system:

$$\begin{cases} D_t^{\alpha}u(x,t)=1-u-u_x v, \\ D_t^{\beta}v(x,t)=1+v+u_x v_x, \end{cases}$$

with the initial conditions as

$$u(x,0)=e^x,\ v(x,0)=e^{-x}.$$

Here $L_1=-u$, $L_2=v$, $N_1=-u_x v$ and $N_2=uv_x$.

Following equation (18), we obtain

$$u_0(x,t)=e^x,\ v_0(x,t)=e^{-x},\quad u_1(x,t)=\frac{-e^x t^{\alpha}}{\Gamma(1+\alpha)},\ v_1(x,t)=\frac{e^{-x}t^{\beta}}{\Gamma(1+\beta)},$$

$$u_2(x,t)=\frac{(e^x+1)t^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}+\frac{\Gamma(1+\alpha+\beta)t^{2\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+2\alpha+\beta)},$$

$$v_2(x,t)=\frac{(e^{-x}-1)t^{2\beta}}{\Gamma(1+2\beta)}+\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}+\frac{\Gamma(1+\alpha+\beta)t^{\alpha+2\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)},$$

Thus we find that the solutions are

$$u(x,t)=e^{x}-\frac{e^{x}t^{\alpha}}{\Gamma(1+\alpha)}+\frac{(e^{x}+1)t^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}$$
$$+\frac{\Gamma(1+\alpha+\beta)t^{2\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+2\alpha+\beta)}+\cdots,$$

$$v(x,t)=e^{-x}+\frac{e^{-x}t^{\beta}}{\Gamma(1+\beta)}+\frac{(e^{-x}-1)t^{2\beta}}{\Gamma(1+2\beta)}+\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}$$
$$+\frac{\Gamma(1+\alpha+\beta)t^{\alpha+2\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)}+\cdots.$$

If $\alpha=\beta=1,$ we obtain the close-form solutions[21]:

$$u(x,t)=e^{x-t},$$

and

$$v(x,t)=e^{-x+t}.$$

Conclusion

In this work, a new approximate analytical method was applied to handle the system of the fractional differential equations within the Caputo fractional derivative. The results show that it is more reliable , efficient and accurate than the other ones.

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Paper submitted: June 13, 2018
Paper revised: July 12, 2018
Paper accepted: Auguest 11, 2018