APPROXIMATE ANALYTIC SOLUTIONS OF MULTI-DIMENSIONAL FRACTIONAL HEAT-LIKE MODELS WITH VARIABLE COEFFICIENTS

by

Jianshe SUN [a, b *]

[a] School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China
[b] School of Mathematics, Jiaozuo Teacher's College, Jiaozuo,454150, China

In this work, the fractional power series method (FPSM) is applied to solve the two-dimensional and three-dimensional fractional heat-like models with variable coefficients. The fractional derivatives are described in the Liouville-Caputo sense. The analytical approximate solutions and exact solutions for the two-dimensional and three-dimensional fractional heat-like models with variable coefficients are obtained. It is shown that the proposed method provides a very effective, convenient and powerful mathematical tool for solving fractional differential equations in mathematical physics.

Key words: *heat-like models， fractional power series, fractional differential equation with variable coefficients, Liouville-Caputo fractional derivative*

Introduction

In this paper, we consider the three-dimensional fractional order heat-like model

$$D_t^{\alpha} u = f(x,y,z) D_x^{\beta_1} u + g(x,y,z) D_y^{\beta_2} u + h(x,y,z) D_z^{\beta_3} u, \tag{1}$$

with the initial condition

$$u(x,y,z,0) = \mu_1(x,y,z), \tag{2}$$

and the two-dimensional order heat-like model

$$D_t^{\alpha} u = f(x,y) D_x^{\beta_1} u + g(x,y) D_y^{\beta_2} u, \tag{3}$$

with the initial condition

$$u(x,y,0) = \mu_2(x,y), \tag{4}$$

where $u = u(x,y,z,t)$, $0 < \alpha \le 1$, $D_t^{\alpha} u(x,y,z,t)$ is the Liouville-Caputo fractional derivative [1], $f(x,y,z), g(x,y,z)$ and $h(x,y,z)$ are any functions with respect to the variables x, y and z.

* Corresponding author; e-mail: sunjianshe@126.com

In the case of $\beta_j = 2(j=1,2,3)$, then (1) reduces to a fractional heat-like equation with variable coefficients [2].

The fractional power series method (FPSM) have played an important role in solving the fractional differential equations in applied and engineering sciences [3-6]. The FPSM was proposed to solve the fractional diffusion equation within Caputo fractional derivative (see[7]) and was used to solve the fractional one dimensional heat-like equations with variable coefficients (see[8]). The target of the paper is to solve the two-dimensional and three-dimensional fractional heat-like models with variable coefficients.

The structure of the paper is given as follows. In Section 2, we present the idea of the FPSM. In Section 3, we suggest the applications of the FPSM to find the solutions for the fractional heat-like model with variable coefficients. Finally, the conclusion is drawn in Section 4.

The FPS and FPSM

The basic idea of the FPS

A power series of the form

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \cdots, \tag{5}$$

is called a fractional power series (FPS) about t_0, where $0 \le m-1 < \alpha \le m$, $m \in N^+$, t $(t \ge t_0)$ is a variable and c_n are the coefficients of the series.

Let the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ be the radius of convergence, denoted as $r > 0$. If $f(t)$ is a function defined by $f(t) = \sum_{n=0}^{\infty} c_n t^{na}$ by on $0 \le t < r$, then for $m-1 < \alpha \le m$ and $0 \le t < r$, we have

$$D^{\alpha} f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha}. \tag{6}$$

For more information for the FPS, see[3-8].

Applications

The FPSM for solving the three-dimensional fractional heat-like model with variable coefficients

Suppose that the solution of (1) and (2) takes the form

$$u(x,y,z,t) = \sum_{k=0}^{\infty} a_k (x,y,z) t^{\alpha k}. \tag{7}$$

where $a_k(x,y,z)$ $(k=1,2,\cdots)$, is denoted as the components of the function $u(x,y,z,t)$, which will be determined recursively.

Making use of (2), one obtains

$$a_0(x,y,z) = \mu_1(x,y,z). \tag{8}$$

From Eq. (6), one gets

$$D_t^{\alpha}u(x,y,z,t)=\sum_{k=1}^{\infty}\frac{a_k(x,y,z)\Gamma(\alpha k+1)}{\Gamma(\alpha(k-1)+1)}t^{\alpha(k-1)}. \tag{9}$$

From (7) , it is easy to see that

$$D_x^{\beta_1}u=D_x^{\beta_1}a_0(x,y,z)+t^{\alpha}D_x^{\beta_1}a_1(x,y,z)+t^{2\alpha}D_x^{\beta_1}a_2(x,y,z)+\cdots, \tag{10}$$

$$D_y^{\beta_2}u=D_y^{\beta_2}a_0(x,y,z)+t^{\alpha}D_y^{\beta_2}a_1(x,y,z)+t^{2\alpha}D_y^{\beta_2}a_2(x,y,z)+\cdots, \tag{11}$$

and

$$D_z^{\beta_3}u=D_z^{\beta_3}a_0(x,y,z)+t^{\alpha}D_z^{\beta_3}a_1(x,y,z)+t^{2\alpha}D_z^{\beta_3}a_2(x,y,z)+\cdots. \tag{12}$$

Substituting (9-12) into (1), we have

$$\begin{aligned}\sum_{k=1}^{\infty}a_k(x,y,z)\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}t^{(k-1)\alpha} &= f(x,y,z)\sum_{k=0}^{\infty}t^{k\alpha}D_x^{\beta_1}a_k(x,y,z)\\ +g(x,y,z)\sum_{k=0}^{\infty}t^{k\alpha}D_y^{\beta_2}a_k(x,y,z)&+h(x,y,z)\sum_{k=0}^{\infty}t^{k\alpha}D_z^{\beta_3}a_k(x,y,z).\end{aligned} \tag{13}$$

Comparing the coefficients of $t^{k\alpha}$ in (13) , we have

$$a_k(x,y,z)=\frac{\Gamma(\alpha(k-1)+1)}{\Gamma(\alpha k+1)}(fD_x^{\beta_1}a_{k-1}+gD_y^{\beta_2}a_{k-1}+hD_z^{\beta_3}a_{k-1}) \tag{14}$$

such that

$$u(x,y,z,t)=\sum_{k=0}^{\infty}t^{k\alpha}\frac{\Gamma(\alpha(k-1)+1)}{\Gamma(\alpha k+1)}(fD_x^{\beta_1}a_{k-1}+gD_y^{\beta_2}a_{k-1}+hD_z^{\beta_3}a_{k-1}) \tag{15}$$

where $k=1,2,\cdots$.

The FPSM for solving the two-dimensional fractional heat-like model with variable coefficients

Let us consider the two-dimensional fractional heat-like model with variable coefficients

$$D_t^{\alpha}u=\frac{1}{2}\left(y^2D_x{}^2u+x^2D_y^2u\right),0<x,y<1,t>0, \tag{16}$$

subject to the initial condition

$$u\left(x,y,0\right)=y^2. \tag{17}$$

In this case, we can write the solutions of (16) and (17) as follows:

$$u(x,y,t)=\sum_{k=0}^{\infty}a_k(x,y)t^{k\alpha}. \tag{18}$$

Obviously,

$$a_0(x,y)=y^2. \tag{19}$$

From (6) we present

$$D_t^{\alpha}u=\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}a_k(x,y)t^{(k-1)\alpha}$$

$$=\Gamma(\alpha+1)a_1(x,y)+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}a_2(x,y)t^{\alpha}+\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}a_3(x,y)t^{2\alpha}+\cdots. \quad (20)$$

With the aid of (18) , it is easy to see that

$$\frac{\partial^2 u}{\partial x^2}=\sum_{k=0}^{\infty}\frac{\partial^2 a_k}{\partial x^2}t^{k\alpha}=\frac{\partial^2 a_0}{\partial x^2}+\frac{\partial^2 a_1}{\partial x^2}t^{\alpha}+\frac{\partial^2 a_2}{\partial x^2}t^{2\alpha}+\cdots, \quad (21)$$

$$\frac{\partial^2 u}{\partial y^2}=\sum_{k=0}^{\infty}\frac{\partial^2 a_k}{\partial y^2}t^{k\alpha}=\frac{\partial^2 a_0}{\partial y^2}+\frac{\partial^2 a_1}{\partial y^2}t^{\alpha}+\frac{\partial^2 a_2}{\partial y^2}t^{2\alpha}+\cdots. \quad (22)$$

Substituting the expansion of (20)-(22) into (16), it follows that

$$\Gamma(\alpha+1)a_1(x,y)+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}a_2(x,y)t^{\alpha}+\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}a_3(x,y)t^{2\alpha}+\cdots$$

$$=\frac{1}{2}(y^2\frac{\partial^2 a_0}{\partial x^2}+x^2\frac{\partial^2 a_0}{\partial y^2})+\frac{1}{2}(y^2\frac{\partial^2 a_1}{\partial x^2}+x^2\frac{\partial^2 a_1}{\partial y^2})t^{\alpha}+\frac{1}{2}(y^2\frac{\partial^2 a_2}{\partial x^2}+x^2\frac{\partial^2 a_2}{\partial y^2})t^{2\alpha}+\cdots. \quad (23)$$

Comparing the coefficients of (20) and (23), we have

$$a_k(x,y)=\frac{\Gamma(\alpha(k-1)+1)}{\Gamma(\alpha k+1)}(\frac{1}{2}y^2\frac{\partial^2 a_{k-1}}{\partial x^2}+\frac{1}{2}x^2\frac{\partial^2 a_{k-1}}{\partial y^2}), \quad (k=1,2,\cdots). \quad (24)$$

Substituting (19) into (24), we present

$$a_1(x,y)=\frac{x^2}{\Gamma(\alpha+1)},\ a_2(x,y)=\frac{y^2}{\Gamma(2\alpha+1)},\ a_3(x,y)=\frac{x^2}{\Gamma(3\alpha+1)}, \quad (25)$$

$$a_4(x,y)=\frac{y^2}{\Gamma(4\alpha+1)}, \quad (26)$$

and so on.

Therefore, we obtain

$$u(x,y,t)=y^2+\frac{x^2t^{\alpha}}{\Gamma(\alpha+1)}+\frac{y^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^2t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{y^2t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots. \quad (27)$$

If $\alpha=1,$ then we have the exact solution as follows:

$$u(x,y,t)=y^2+x^2t+\frac{y^2t^2}{2!}+\frac{x^2t^3}{3!}+\frac{y^2t^4}{4!}+\cdots=x^2\sinh t+y^2\cosh t. \quad (28)$$

Conclusion

In the present task, the FPSM has been successfully applied to solve two-dimensional and three-dimensional fractional heat-like models with variable coefficients. It is shown that the FPSM is a simple and effective method for solving exact approximate solutions of fractional partial differential Equations with variable coefficients.

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Nomenclature

x, y, z - space co-ordinates, [m]
α – fractional order, [–]
k – natural number, [–]
t – time, [s]
β_i – fractional order, [–]
N – positive integer, [–]

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