We analyze a first order in time Fourier pseudospectral scheme for Swift-Hohenberg equation. One major challenge for the higher order diffusion non-linear systems is how to ensure the unconditional energy stability and we propose an efficient scheme for the equation based on the convex splitting of the energy. Theoretically, the energy stability of the scheme is proved. Moreover, following the derived aliasing error estimate, the convergence analysis in the discrete $l^2$-norm for the proposed scheme is given.

Key words: Swift-Hohenberg equation, convex splitting, energy stability, Fourier pseudospectral method

Introduction

The 2-D Swift-Hohenberg (SH) equation was first proposed by Swift and Hohenberg to model the onset and evolution of roll patterns in Rayleigh-Benard convection in [1]. Since then, it has attracted considerable attention in several related fields, for instance as a model for the formation of hexagon patches in granular fluid systems and frictional fluids.

The SH equation is associated with the following Lyapunov energy functional:

\[
E(u) = \int_{\Omega} \left\{ \Psi(u) + \frac{\delta^2}{2} (\Delta u)^2 - \frac{\epsilon^2}{2} |\nabla u|^2 \right\} \, dx
\]

where $\Omega = [0, L_x] \times [0, L_y]$, $\Psi(u)$ is a non-linear function of $u$, defined:

\[
\Psi(u) = -\frac{\gamma u^2}{2} + \frac{1}{4} u^4
\]

and $\gamma$, $\delta^2$, and $\epsilon^2$ are positive real constants. The chemical potential of SH equation is defined as the variational derivative of the energy functional eq. (1), namely:

\[
\frac{\delta E}{\delta u} = \delta_x E = \Psi'(u) + \epsilon^2 \Delta u + \delta^2 \Delta^2 u
\]

and the SH equation [2-5] is given by the $L^2$ gradient flow associated with the energy eq. (1):

\[
\frac{\delta u}{\delta t} = -\frac{\delta E}{\delta u} = -u^3 + \gamma u - \epsilon^2 \Delta u - \delta^2 \Delta^2 u
\]
In most cases, eq. (4) are performed using periodic boundary conditions in all directions. In the rest of the paper, periodic boundary conditions are imposed for \( u, \Delta u, \) and \( \Delta^2 u \).

The SH equation is a fourth order non-linear PDE. One of the important features of SH equation is energy dissipation. As a matter of fact, the exact solutions of the SH model are hard to obtain, and numerical methods have played an important role in various simulations. If inadequate schemes are used, this important property of SH model will be lost after numerical discretization, leading to non-physical solutions [6]. There have been some related works to develop numerical schemes for the SH equation [7, 8].

Meanwhile, it is observed that, the long time energy stability could not be theoretically justified for these numerical works, due to the explicit treatment for the non-linear terms [9, 10]. In the existing literature, the only numerical algorithm for the SH equation with a long time energy stability could be found in [11-13]. In this paper, based on the convex-concave decomposition for the Lyapunov energy functional, a first-order convex splitting scheme, with a Fourier pseudospectral spatial discretization for the SH eq. (4) is proposed.

**The numerical scheme and energy stability**

**Fourier pseudospectral discretization space**

Assume that \( L_x = N_x h_x, L_y = N_y h_y \) for some mesh sizes \( h_x, h_y > 0 \) and some positive integers \( N_x \) and \( N_y \). For simplicity of presentation, we use a square domain, i.e., \( L_x = L_y = 1 \), and a uniform mesh size \( h_x = h_y, N_x = N_y = N \). We will always assume that \( N = 2K + 1 \) is always odd. All the variables are evaluated at the regular numerical grid \((x_i, y_j)\) with \( x_i = ih, y_j = jh, 0 \leq i, j \leq N \).

For a periodic function \( f \) over the given 2-D numerical grid, its discrete Fourier expansion is given:

\[
f_{ij} = \sum_{l,m=-K}^{K} \hat{f}_{lm} \exp[2\pi i(lx_i + my_j)]
\]

its collocation Fourier spectral approximations to first and second order partial derivatives are given:

\[
(D_x f)_{ij} = \sum_{l,m=-K}^{K} (2\pi i l) \hat{f}_{lm} \exp[2\pi i(lx_i + my_j)]
\]

\[
(D_x^2 f)_{ij} = \sum_{l,m=-K}^{K} (-4\pi^2 l^2) \hat{f}_{lm} \exp[2\pi i(lx_i + my_j)]
\]

and the corresponding collocation spectral differentiations in the \( y \)-direction can be defined in the same way. In turn, the discrete Laplacian, gradient and divergence become:

\[
\Delta f = (D^2_x + D^2_y) f, \quad \nabla f = \begin{bmatrix} D_x f \\ D_y f \end{bmatrix}
\]

at the point-wise level.

Moreover, given any periodic grid functions \( f \) and \( g \) (over the 2-D numerical grid), the spectral approximations to the \( \mathbb{F} \) inner product and norm are introduced:

\[
\| f \|_{\mathbb{F}} = \sqrt{< f, f >}, \quad < f, g >= \sum_{i,j=0}^{N-1} f_{ij} g_{ij}
\]

Furthermore, a detailed calculation shows that the following formulas of integration by parts are also valid at the discrete level:

\[
< f, \Delta g >= -< \nabla_x f, \nabla_g >, \quad < f, \Delta^2 g >= < \Delta_x f, \Delta_y g >
\]
In addition the $l^p$ standard $F$ norm, we also introduce the $l^p$ and $l^p$, $1 \leq p \leq \infty$ norms for a grid function:

$$
\| f \|_p = \max_{0 \leq i,j \leq N-1} | f_{i,j} |, \quad \| f \|_p = \left( h^d \sum_{i,j=0}^{N-1} | f_{i,j} |^p \right)^{1/p}
$$

To obtain a pseudospectral approximation at a given set of points, an interpolation operator $J_N$ should be introduced. Given a uniform numerical grid with $N = 2K + 1$ points in each dimension and a discrete vector function, $f$, where each point is denoted by $(x_i, y_j)$ and the corresponding function value is given by $f_{i,j}$, the interpolation of the function:

$$
(I_N f)(x, y) = \sum_{l=-K}^{K} \sum_{m=-K}^{K} (J_N^l)_{i,m} \exp[2\pi i(lx + my)]
$$

where the $(2K + 1)^2$ pseudospectral coefficients $(J_N^l)_{i,m}$ are given by $f_{i,j} = (J_N f)(x_i, y_j)$.

Lemma 1. [14] Suppose that $m$ and $K$ are non-negative integers, and $N = 2K + 1$. For $\phi \in P_m$ any (with trigonometric polynomial up to degree $mK$ in $R^d$ we have the estimate:

$$
\| I_N \phi \|_{H^r} \leq (\sqrt{m})^r \| \phi \|_{H^r}
$$

for any non-negative integer $r$.

**Semi-implicit Fourier pseudospectral scheme and energy stability**

The fundamental observation is that the energy $E$ admits a splitting into purely convex and concave energies, that is, $E = E_c - E_e$, where $E_c$ and $E_e$ are convex, though not necessarily strictly convex and the canonical splitting:

$$
\frac{1}{2}\int_\Omega \left( u^r \cdot \Delta u \right) + \frac{\gamma}{2} |u|^2 - \frac{\sigma_1}{2} |\nabla u|^2 \right) dx
$$

Therefore, by the time discretization:

$$
\frac{u^{n+1} - u^n}{\Delta t} = -\delta_t [u^{n+1}] + \delta_t [u^n]
$$

we propose the following fully discrete first-order (in time) Fourier pseudospectral scheme for SH eq. (4):

$$
\frac{u^{n+1} - u^n}{\Delta t} = -u^{n+1} + \gamma u^n - \sigma_1 \Delta_x u^n - \sigma_2 \Delta_y u^{n+1}
$$

where $\Delta t$ is the discrete time step and $u^n$ denotes the time-discrete approximation of $u(\cdot, n\Delta t)$ and $n = 0, 1, ..., M = [T/\Delta t]$ and $T$ is a given final time.

With the same convex splitting idea, the energy according to $E = E_c - E_e$ with:

$$
E_c(u) = \int_\Omega \frac{1}{2} |u|^2 + \frac{\sigma_1}{2} |\nabla u|^2 \right) dx, \quad E_e(u) = \int_\Omega \frac{1}{2} |u|^2 - \frac{\gamma}{2} |u|^2 + \frac{\sigma_1}{2} |\nabla u|^2 \right) dx
$$

hold for a sufficiently large constant $C > 0$, which turns out to yield a stable linear scheme [11].

**Energy stability**

For any periodic grid function, $\phi$, we define the discrete energy for the eq. (14):

$$
E_n(\phi) := -\frac{\gamma}{2} ||\phi||^2 + \frac{1}{4} ||\phi||^2 + \frac{\sigma_1}{2} || \Delta_x \phi ||^2 - \frac{\sigma_2}{2} || \nabla \phi ||^2
$$
It is clear that the energy of the numerical solution of the fully discrete first order scheme (14) is decreasing at the discrete level.

Theorem 1. The scheme (14) is unconditionally strongly energy stable, with respect to the discrete energy \( E_N \), i.e., \( E_N (u^{n+1}) \leq E_N (u^n) \) for any \( n \geq 1 \) and any \( \Delta t \geq 0 \).

Proof. Taking the discrete inner product of (14) with \( u^{n+1} - u^n \), we have:
\[
\frac{u^{n+1} - u^n}{\Delta t}, u^{n+1} - u^n = - (E_N (u^{n+1}) + \gamma u^n - \varepsilon^2 \Delta_n u^n - \delta^2 \Delta_n u^{n+1}, u^{n+1} - u^n ) \tag{17}
\]
Considering each term on the right side of eq. (17), we can get:
\[
-(u^{n+1})^2, u^{n+1} - u^n < - \| u^{n+1} \|_4^2 + \frac{3}{4} \| u^{n+1} \|_4^2 + \frac{1}{4} \| u^n \|_4^2 = - \frac{1}{4} \| u^{n+1} \|_4^2 + \frac{1}{4} \| u^n \|_4^2 \tag{18}
\]
in which the Young inequality is applied. In turn:
\[
-\gamma u^n, u^{n+1} - u^n = \gamma < u^n, u^{n+1} > - \gamma < u^n, u^n > \leq \frac{\gamma}{2} \| u^{n+1} \|_2^2 - \frac{\gamma}{2} \| u^n \|_2^2 \tag{19}
\]
holds since \( < u^n, u^{n+1} > (\| u^n \|_2^2 + \| u^{n+1} \|_2^2 )/2 \). According to the eq. (9), the following estimates are valid:
\[
- \varepsilon^2 \Delta_n u^n, u^{n+1} - u^n < \varepsilon^2 \nabla_n u^n, \nabla_n (u^{n+1} - u^n ) \leq \frac{\varepsilon^2}{2} \| \nabla_n u^{n+1} \|_2^2 - \frac{\varepsilon^2}{2} \| \nabla_n u^n \|_2^2 \tag{20}
\]
and
\[
- \delta^2 \Delta_n u^n, u^{n+1} - u^n = - \delta^2 \nabla_n u^n, \nabla_n (u^{n+1} - u^n ) \leq - \frac{\delta^2}{2} \| \Delta_n u^{n+1} \|_2^2 + \frac{\delta^2}{2} \| \Delta_n u^n \|_2^2 \tag{21}
\]
Hence, the combination of eqs. (17)-(21) indicates:
\[
-E_N(u^{n+1}) + E_N(u^n) \geq \| u^{n+1} - u^n \|_2^2 / \Delta t \geq 0 \tag{22}
\]
which shows an unconditional stability with respect to the discrete energy is established.

As a result of the energy stability, the uniform in time \( \| u^n \|_2, \| u^n \|_1 \), and \( \| \Delta_n u^n \|_2 \) bound of the numerical solution (14) is available at a discrete level.

Lemma 2. Assume that \( u^n \) is the numerical solution of (14). Then, the following estimates are valid:
\[
\| u^n \|_\infty \leq C_1, \| u^n \|_2 \leq C_2, \| \Delta_n u^n \|_1 \leq C_3, \| u^n \|_2 \leq C_4 \tag{23}
\]
for \( n = 1, 2, ..., M \).

Proof. From Theorem 1, the following energy bound is available:
\[
E_N (u^n) \leq E_N (u^{n+1}) \leq ... \leq E_N (u^0) = C_5 \tag{24}
\]
For any \( f \) we have:
\[
\frac{1}{4} \| f \|_4^2 + \frac{\gamma}{2} \| f \|_2^2 \leq \frac{1}{8} \| f \|_4^2 + C_6 \| f \|_2^2 - 2 \left( C_6 - \frac{\gamma}{2} \right) \| \nabla_n u^n \|_2^2, C_6 \geq 0 \tag{25}
\]
A combination with the discrete energy eq. (16) shows:
\[
\frac{1}{8} \| u^n \|_4^2 + C_6 \| u^n \|_2^2 - 2 \left( C_6 - \frac{\gamma}{2} \right) \| \nabla_n u^n \|_2^2 \leq \frac{3}{4} \| u^n \|_4^2 + \frac{\delta^2}{2} \| \Delta_n u^n \|_2^2 - \frac{\delta^2}{2} \| \nabla_n u^n \|_2^2 \leq E_N (u^n) \leq C_5 \tag{26}
\]
Then, we get:

$$\frac{1}{8} \left\| a^+ \right\|_2^2 + C_k \left\| a^- \right\|_2^2 + \frac{\delta^2}{2} \left\| \Delta X a^+ \right\|_2^2 \leq 2 \left( C_k - \frac{\gamma}{2} \right) \left\| \mathcal{L} a^+ \right\|_2^2 \quad (27)$$

for all $n \geq 1$. Noting that the integration by parts and $\varepsilon$-inequality imply:

$$\left\| \nabla \mathcal{L} a^+ \right\|_2 = \varepsilon \left\| \Delta X a^+ \right\|_2 \leq \frac{\varepsilon^2}{2\delta^2} \left\| a^+ \right\|_2^2 + \frac{\delta^2}{2\varepsilon} \left\| \Delta X a^+ \right\|_2^2 \quad (28)$$

Substituting eq. (28) into eq. (27) yields:

$$\frac{1}{8} \left\| a^+ \right\|_2^2 + C_k \left\| a^- \right\|_2^2 + \frac{\delta^2}{2} \left\| \Delta X a^+ \right\|_2^2 \leq 2 \left( C_k - \frac{\gamma}{2} \right) \left\| \mathcal{L} a^+ \right\|_2^2 + 2 \left( C_k - \frac{\gamma}{2} \right) \left\| \mathcal{L} a^+ \right\|_2^2 \quad (29)$$

that is:

$$\frac{1}{8} \left\| a^+ \right\|_2^2 + \left( C_k - \frac{\varepsilon^2}{2\delta^2} \right) \left\| a^- \right\|_2^2 + \frac{\delta^2}{4} \left\| \Delta X a^+ \right\|_2^2 \leq 2 \left( C_k - \frac{\gamma}{2} \right) \left\| \mathcal{L} a^+ \right\|_2^2 + C_k := C_7 \quad (30)$$

provided that $C_k \geq \max \{ \varepsilon^4 / (4\delta^2), \gamma / 2 \}$. Consequently, we have:

$$\left\| a^+ \right\|_2^2 \leq (8C_k)^{1/4} \left( 8C_k \right)^{1/4} = C_1, \quad \left\| a^- \right\|_2 \leq \left\{ \left[ C_k - \frac{\varepsilon^4}{(4\delta^2)} \right] \right\}^{1/4} \left( 8C_k \right)^{1/4} = C_2$$

$$\left\| \Delta X a^+ \right\|_2 \leq \left\{ (\delta^2 / 4) \right\}^{1/4} C_1 = C_3 \quad (31)$$

Moreover, employing the elliptic regularity and Sobolev inequality, we can obtain the $L^\infty$ bound of the numerical solution:

$$\left\| a^+ \right\|_\infty \leq \tilde{C} \left( \left\| a^- \right\|_2 + \left\| \Delta X a^+ \right\|_2 \right) \leq \tilde{C} (C_1 + C_2) := C_4 \quad (32)$$

This completes the proof of Lemma 2.

**Convergence analysis**

The point-wise numerical error grid function is given:

$$\nu_{i,j}^n = (u^n)^i_{i,j} - u^n_{i,j} \quad (33)$$

Likewise, to facilitate the presentation, we also denote $\nu_{i,j}^n \in P_k$ as the continuous version of the numerical error function $\nu^n$.

**Theorem 2.** For any final time $T > 0$, assume the exact solution $u^e$ to the SH eq. (4) has a sufficient regularity. Denote $u^n$ as the numerical solution given by the proposed eq. (14). Then, as $\Delta t, h \to 0$, the following convergence result is valid:

$$\left\| u^n - u^e \right\|_2 \leq \tilde{C}(\Delta t + h^n) \quad (34)$$

provided that $\Delta t$ is sufficiently small, and the constant $C^*$ — the independent of $h$ and $\Delta t$, but may depend on $\varepsilon, \delta, \gamma, \left\| u_e \right\|_{L^\infty(0,T,H^{4+m})}, \Omega$ and the final time $T$.

Proof. Because the exact solution $u^e$ of eq. (4) has a sufficient regularity, detailed Taylor expansions imply the following truncation error:

$$\frac{u^{e+1} - u^n_e}{\Delta t} = -u^{e+1} + \gamma u^n e - \varepsilon^2 \Delta X u^n e - \delta^2 \Delta X u^{e+1} + \tau^n \quad (35)$$

with $\left\| \tau^l \right\| \leq C(\Delta t + h^n)$ for $0 \leq l \leq M$, in which $C$ is dependent on $T, \Omega$ and $\left\| u_e \right\|_{L^\infty(0,T,H^{4+m})}$.

Subtracting eq. (14) from eq. (35), we get the following error equation:

$$\frac{v^{e+1} - v^n}{\Delta t} = -[(u^{e+1} - u^n e)] + \gamma v^n - \varepsilon^2 \Delta X v^n - \delta^2 \Delta X v^{e+1} + \tau^n \quad (36)$$
Taking an inner product of eq. (36) with \( v^{n+1} \) gives:

\[
< v^{n+1} - v^n, v^{n+1} > = \Delta t \left< \left( (u^{n+1})^{3} - (u_v^{n+1})^{3} \right), v^{n+1} \right> + \gamma v^n - \varepsilon^2 \Delta_x v^n - \delta^2 \Delta_x^2 v^{n+1} + r^n, v^{n+1} >
\]

The time marching term and the truncation error term can be analyzed:

\[
< v^{n+1} - v^n, v^{n+1} > = \frac{1}{2} (\| v^{n+1} \|_2^2 - \| v^n \|_2^2 + \| v^{n+1} - v^n \|_2^2) \geq \frac{1}{2} (\| v^{n+1} \|_2^2 - \| v^n \|_2^2)
\]

(37)

and

\[
< r^n, v^{n+1} > \leq \frac{1}{2} (\| v^{n+1} \|_2^2 + \| r^n \|_2^2)
\]

(38)

A discrete version of the integration by eq. (9) can be applied to analyze the diffusion terms:

\[
- \varepsilon^2 < \Delta_x v^n, v^{n+1} > = - \varepsilon^2 \left< \nabla_x v^n, \nabla_x v^{n+1} \right> \leq \frac{\varepsilon^2}{2} (\| \nabla_x v^{n+1} \|_2^2 + \| \nabla_x v^n \|_2^2) - \Delta_x v^n, v^{n+1} > \geq - \frac{\delta^2}{2} \| \Delta_x v^{n+1} \|_2^2
\]

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Next, we focus on the analysis for non-linear inner product terms:

\[
< (u^{n+1})^{3} - (u_v^{n+1})^{3}, v^{n+1} > \leq \| (u^{n+1})^{3} - (u_v^{n+1})^{3} \|_2 \| v^{n+1} \|_2
\]

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Noting:

\[
\| (u^{n+1})^{3} - (u_v^{n+1})^{3} \|_2 \leq \| (u^{n+1})^{3} \|_2 \| v^{n+1} \|_2 + \| u_v^{n+1} v^{n+1} \|_2 + \| (u_v^{n+1})^{3} v^{n+1} \|_2
\]

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We begin with the first term on the right side of (42) and recall that \( v_v^n \in P_K \) is the continuous version of the discrete grid error function \( v_v^{n+1} \). It is obvious:

\[
\| u^{n+1} ||_2 = \| I_N(\{u_v^{n+1}\}^2) \|_2 \leq 3C^2 \| v^{n+1} \|_2 \leq 3C^2 (\| v^{n+1} \|_2 + h^n)
\]

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The first step is based on the fact that \( v^{n+1} \) and \( v_v^{n+1} \) have the same interpolation values. At last step, we use the spectral accuracy in spatial space because of the collocation spectral approximation.

A similar analysis can be applied to the second and the third term on the right side of eq. (42):

\[
\| u_v^{n+1} v^{n+1} \|_2 \leq 3C^2 (\| v^{n+1} \|_2 + h^n), \| u^{n+1} \|_2 \leq 3C^2 (\| v^{n+1} \|_2 + h^n)
\]

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in which the \( L^\infty \) bound of \( u^{n+1} \), as the continuous version of the \( L^\infty \) bound of \( u_v^{n+1} \) also was used. Therefore, it follows from eqs. (41)-(44):

\[
< (u^{n+1})^{3} - (u_v^{n+1})^{3}, v^{n+1} > \leq \frac{27}{2} C^2 \| v^{n+1} \|_2^2 + \frac{9}{2} C^2 h^{2n}
\]

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Additionally, from eq. (40), by \( \varepsilon \)-inequality and for \( k = n, n + 1 \), we get:

\[
\frac{\varepsilon^2}{2} \| \nabla_x v^n \|_2^2 \leq \frac{\varepsilon^2}{2} < \nabla_x v^n, \nabla_x v^n > \leq \frac{\varepsilon^4}{8\delta} \| v^n \|_2^2 + \frac{\delta^2}{2} \| \Delta_x v^n \|_2^2
\]

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Consequently, a substitution of eqs. (37)-(40) and eqs. (44) and (45) into eq. (36) results:
We introduce a modified energy for the error function:

\[ G^{n+1} = \frac{1}{2} \| v^{n+1} \|_2^2 + \frac{\delta^2}{2} \Delta t \| \Delta_x v^{n+1} \|_2^2 \]  

and the previous inequality now takes the form:

\[ G^{n+1} - G^n \leq \tilde{C} \Delta t (G^{n+1} + G^n) + \frac{1}{2} \Delta t \| \tau^n \|_2^2 + \frac{9}{2} C^2 \Delta t \Delta h^2 n \]  

with \( \tilde{C} = \epsilon/\gamma + 4 \). Summing over \( n \) from 0 to \( l \), and using \( C^0 = O(h^{2n}) \) because of the spectral accuracy, we obtain:

\[ G^{n+1} \leq 2 \tilde{C} \Delta t \sum_{k=0}^{l} G^k + \tilde{C} \Delta t G^{n+1} + \frac{1}{2} \Delta t \| \tau^n \|_2^2 + \frac{9}{2} C^2 \Delta t \sum_{k=0}^{l} h^{2n} + h^{2n} \]  

Noting that \( l \Delta t \leq T \), from previous inequality, we have:

\[ G^{n+1} \leq 2 \tilde{C} \Delta t \sum_{k=0}^{l} G^k + \tilde{C} \Delta t G^{n+1} + C_\eta \tau (\Delta t^2 + h^{2n}) \]  

where \( C = 9C/2+1 \). Manipulating eq. (51), we get:

\[ G^{n+1} \leq \frac{2 \tilde{C} \Delta t}{1-C\Delta t} \sum_{k=0}^{l} G^k + \frac{C_\eta \tau}{1-C\Delta t} (\Delta t^2 + h^{2n}) \]  

provided that \( \tilde{C} \Delta t \leq 1 \). As a result, an application of the discrete Gronwall inequality gives:

\[ G^l \leq \frac{C_\eta \tau}{1-C\Delta t} \left( 1 + \frac{2 \tilde{C} \Delta t}{1-C\Delta t} \right)^{l-1} (\Delta t^2 + h^{2n}) = \frac{C_\eta \tau}{1-C\Delta t} \left( 1 + \frac{2 \tilde{C} \Delta t}{1-C\Delta t} \right)^{l-1} (\Delta t^2 + h^{2n}) \]  

which is equivalent to the following convergence result:

\[ \| v^{n+1} \|_2^2 + \sqrt{\Delta t} \| \Delta_x v^{n+1} \|_2^2 \leq C^* (\Delta t + h^n) \]  

in which:

\[ C^* = \left( \frac{C_\eta \tau}{1-C\Delta t} \right)^{1/2} \left( 1 + \frac{2 \tilde{C} \Delta t}{1-C\Delta t} \right)^{(l-1)/2} \]  

From the form of \( C^* \), the coefficient \( C^* \) may be bounded by a positive constant that is dependent on \( T \) (exponentially), but otherwise independent of \( h \) and \( \Delta t \). This completes the convergence analysis.

**Conclusion**

In this paper we presented an energy-stable first-order numerical scheme for the SH equation. The temporal discretization follows the convex splitting of the Lyapunov energy functional. With the help of the \( L^1 \) estimate of control variable indicated by the energy stability and the aliasing error estimate, we obtained the convergence analysis in the discrete \( L^1 \)-norm.
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