AN ENERGY-STABLE PSEUDOSPECTRAL SCHEME FOR SWIFT-HOHENBERG EQUATION WITH ITS LYAPUNOV FUNCTIONAL

by

Jun ZHOU[a*] and Xiaomin DAI[b]

[a] School of Mathematics and Statistics, Yangtze Normal University, Chongqing, China
[b] Mathematics Teaching and Research Group, Fifth Middle School of Fuling, Chongqing, China

Original scientific paper
https://doi.org/10.2298/TSCI180612249Z

We analyze a first order in time Fourier pseudospectral scheme for Swift-Hohenberg equation. One major challenge for the higher order diffusion non-linear systems is how to ensure the unconditional energy stability and we propose an efficient scheme for the equation based on the convex splitting of the energy. Theoretically, the energy stability of the scheme is proved. Moreover, following the derived aliasing error estimate, the convergence analysis in the discrete l^2-norm for the proposed scheme is given.

Key words: *Swift-Hohenberg equation, convex splitting, energy stability, Fourier pseudospectral method*

Introduction

The 2-D Swift-Hohenberg (SH) equation was first proposed by Swift and Hohenberg to model the onset and evolution of roll patterns in Rayleigh-Benard convection in [1]. Since then, it has attracted considerable attention in several related fields, for instance as a model for the formation of hexagon patches in granular fluid systems and frictional fluids.

The SH equation is associated with the following Lyapunov energy functional:

$$E(u)=\int_\Omega\left\{\Psi(u)+\frac{\delta^2}{2}(\Delta u)^2-\frac{\varepsilon^2}{2}|\nabla u|^2\right\}\mathrm{d}x \tag{1}$$

where $\Omega=[0, L_x]\times[0, L_x]$, $\Psi(u)$ is a non-linear function of u, defined:

$$\Psi(u)=-\frac{\gamma}{2}u^2+\frac{1}{4}u^4 \tag{2}$$

and γ, δ^2, and ε^2 are positive real constants. The chemical potential of SH equation is defined as the variational derivative of the energy functional eq. (1), namely:

$$\frac{\delta E}{\delta u}=\delta_u E=\Psi^{'}(u)+\varepsilon^2\Delta u+\delta^2\Delta^2 u \tag{3}$$

and the SH equation [2-5] is given by the L^2 gradient flow associated with the energy eq. (1):

$$\frac{\partial u}{\partial t}=-\frac{\delta E}{\delta u}=\delta_u E=-u^3+\gamma u-\varepsilon^2\Delta u-\delta^2\Delta^2 u \tag{4}$$

* Corresponding author, e-mail: flzjzklm@126.com

In most cases, eq. (4) are performed using periodic boundary conditions in all directions. In the rest of the paper, periodic boundary conditions are imposed for u, Δu, and $\Delta^2 u$.

The SH equation is a fourth order non-linear PDE. One of the important features of SH equation is energy dissipation. As a matter of fact, the exact solutions of the SH model are hard to obtain, and numerical methods have played an important role in various simulations. If inadequate schemes are used, this important property of SH model will be lost after numerical discretization, leading to non-physical solutions [6]. There have been some related works to develop numerical schemes for the SH equation [7, 8].

Meanwhile, it is observed that, the long time energy stability could not be theoretically justified for these numerical works, due to the explicit treatment for the non-linear terms [9, 10]. In the existing literature, the only numerical algorithm for the SH equation with a long time energy stability could be found in [11-13]. In this paper, based on the convex-concave decomposition for the Lyapunov energy functional, a first-order convex splitting scheme, with a Fourier pseudospectral spatial discretization for the SH eq. (4) is proposed.

The numerical scheme and energy stability

Fourier pseudospectral discretization space

Assume that $L_x = N_x h_x$, $L_y = N_y h_y$, for some mesh sizes h_x, $h_y > 0$ and some positive integers N_x and N_y. For simplicity of presentation, we use a square domain, *i. e.*, $L_x = L_y = 1$, and a uniform mesh size $h_x = h_y = h$, $N_x = N_y = N$. We will always assume that $N = 2K + 1$ is always odd. All the variables are evaluated at the regular numerical grid (x_i, u_j) with $x_i = ih$, $y_j = jh$, $0 \le i, j \le N$.

For a periodic function f over the given 2-D numerical grid, its discrete Fourier expansion is given:

$$f_{i,j} = \sum_{l,m=-K}^{K} \hat{f}_{l,m} \exp[2\pi i(lx_i + my_j)] \tag{5}$$

its collocation Fourier spectral approximations to first and second order partial derivatives are given:

$$(\mathrm{D}_x f)_{i,j} = \sum_{l,m=-K}^{K} (2\pi il)\hat{f}_{l,m} \exp[2\pi i(lx_i + my_j)]$$

$$(\mathrm{D}_x^2 f)_{i,j} = \sum_{l,m=-K}^{K} (-4\pi^2 l^2)\hat{f}_{l,m} \exp[2\pi i(lx_i + my_j)] \tag{6}$$

and the corresponding collocation spectral differentiations in the y-direction can be defined in the same way. In turn, the discrete Laplacian, gradient and divergence become:

$$\Delta_N f = (\mathrm{D}_x^2 + \mathrm{D}_y^2) f, \quad \nabla_N f = \begin{pmatrix} \mathrm{D}_x f \\ \mathrm{D}_y f \end{pmatrix} \tag{7}$$

at the point-wise level.

Moreover, given any periodic grid functions f and g (over the 2-D numerical grid), the spectral approximations to the l^2 inner product and norm are introduced:

$$\| f \|_2 = \sqrt{< f, f >}, \quad < f, g >= \sum_{i,j=0}^{N-1} f_{i,j} g_{i,j} \tag{8}$$

Furthermore, a detailed calculation shows that the following formulas of integration by parts are also valid at the discrete level:

$$< f, \Delta_N g >= -< \nabla_N f, \nabla_N g >, \; < f, \Delta_N^2 g >=< \Delta_N f, \Delta_N g > \tag{9}$$

In addition the l standard l^2 norm, we also introduce the l^∞ and l^p, $1 \le p \le \infty$ norms for a grid function:

$$\| f \|_\infty = \max_{0\le i,j\le N-1} | f_{i,j} |, \ \| f \|_p = \left(h^2 \sum_{i,j=0}^{N-1} | f_{i,j} |^p \right)^{1/p}$$

To obtain a pseudospectral approximation at a given set of points, an interpolation operator J_N should be introduced. Given a uniform numerical grid with $N = 2K + 1$ points in each dimension and a discrete vector function, f, where each point is denoted by (x_i, y_j) and the corresponding function value is given by $f_{i,j}$ the interpolation of the function:

$$(I_N f)(x,y) = \sum_{l,m=-K}^{K} (\hat{f}_c^N)_{l,m} \exp[2\pi i(lx + my)] \tag{10}$$

where the $(2K + 1)^2$ pseudospectral coefficients $(f_c^N)_{l,m}$ are given by $f_{i,j} = (I_N f)(x_i, y_j)$.

Lemma 1. [14] Suppose that m and K are non-negative integers, and $N = 2K + 1$. For $\varphi \in P_{mK}$ any (with trigonometric polynomial up to degree mK in R^d we have the estimate:

$$\| I_N \varphi \|_{H^r} \le \left(\sqrt{m}\right)^d \| \varphi \|_{H^r} \tag{11}$$

for any non-negative integer r.

Semi-implicit Fourier pesudospectral scheme and energy stability

The fundamental observation is that the energy E admits a splitting into purely convex and concave energies, that is, $E = E_c - E_e$, where E_c and E_e are convex, though not necessarily strictly convex and the canon-ical splitting:

$$E_c(u) = \int_\Omega \left\{ \frac{1}{4}u^4 + \frac{\delta^2}{2}(\Delta u)^2 \right\} \mathrm{d}x, \ E_e(u) = \int_\Omega \left\{ \frac{\gamma}{2}u^2 + \frac{\varepsilon^2}{2} | \nabla u |^2 \right\} \mathrm{d}x \tag{12}$$

Therefore, by the time discretization:

$$\frac{u^{n+1} - u^n}{\Delta t} = -\delta_u[u^{n+1}] + \delta_u[u^n] \tag{13}$$

we propose the following fully discrete first-order (in time) Fourier pseudospectral scheme for SH eq. (4):

$$\frac{u^{n+1} - u^n}{\Delta t} = -(u^{n+1})^3 + \gamma u^n - \varepsilon^2 \Delta_N u^n - \delta^2 \Delta_N^2 u^{n+1} \tag{14}$$

where Δt is the discrete time step and u^n denotes the time-discrete approximation of $u(.,n\Delta t)$ and $n = 0, 1,..., M = [\mathrm{T}/\Delta t]$ and T is a given final time.

With the same convex splitting idea, the energy according to $E = E_c^L - E_e^L$ with:

$$E_c^L(u) = \int_\Omega \left\{ \frac{C}{2} | u |^2 + \frac{\delta^2}{2}(\Delta u)^2 \right\} \mathrm{d}x, \ E_e^L(u) = \int_\Omega \left(\frac{C}{2} | u |^2 - \frac{1}{4}u^4 + \frac{\gamma}{2}u^2 + \frac{\varepsilon^2}{2} | \nabla u |^2 \right) \mathrm{d}x \tag{15}$$

hold for a sufficiently large constant $C > 0$, which turns out to yield a stable linear scheme [11].

Energy stability

For any periodic grid function, ϕ, we define the discrete energy for the eq. (14):

$$E_N(\phi) := -\frac{\gamma}{2} \| \phi \|_2^2 + \frac{1}{4} \| \phi \|_4^4 + \frac{\delta^2}{2} \| \Delta_N \phi \|_2^2 - \frac{\varepsilon^2}{2} \| \nabla_N \phi \|_2^2 \tag{16}$$

It is clear that the energy of the numerical solution of the fully discrete first order scheme (14) is decreasing at the discrete level.

Theorem 1. The scheme (14) is unconditionally strongly energy stable, with respect to the discrete energy E_N, *i. e.*, $E_N(u^{n+1}) \le E_N(u^n)$ for any $n \ge 1$ and any $\Delta t \ge 0$.

Proof. Taking the discrete inner product of (14) with $u^{n+1} - u^n$, we have:

$$<\frac{u^{n+1}-u^n}{\Delta t},\ u^{n+1}-u^n> = < -(u^{n+1})^3 + \gamma u^n - \varepsilon^2 \Delta_N u^n - \delta^2 \Delta_N^2 u^{n+1},\ u^{n+1}-u^n > \tag{17}$$

Considering each term on the right side of eq. (17), we can get:

$$-<(u^{n+1})^3,\ u^{n+1}-u^n> \le -\| u^{n+1} \|_4^4 + \frac{3}{4}\| u^{n+1} \|_4^4 + \frac{1}{4}\| u^n \|_4^4 = -\frac{1}{4}\| u^{n+1} \|_4^4 + \frac{1}{4}\| u^n \|_4^4 \tag{18}$$

in which the Young inequality is applied. In turn:

$$<\gamma u^n,\ u^{n+1}-u^n> = \gamma < u^n,\ u^{n+1} > -\gamma < u^n, u^n > \le \frac{\gamma}{2}\| u^{n+1} \|_2^2 - \frac{\gamma}{2}\| u^n \|_2^2 \tag{19}$$

holds since $< u^n, u^{n+1} > \le (\| u^n \|_2^2 + \| u^{n+1} \|_2^2)/2$. According to the eq. (9), the following estimates are valid:

$$-<\varepsilon^2 \Delta_N u^n,\ u^{n+1}-u^n> = \varepsilon^2 < \nabla_N u^n, \nabla_N (u^{n+1}-u^n) > \le \frac{\varepsilon^2}{2}\| \nabla_N u^{n+1} \|_2^2 - \frac{\varepsilon^2}{2}\| \nabla_N u^n \|_2^2 \tag{20}$$

and

$$-<\delta^2 \Delta_N^2 u^n,\ u^{n+1}-u^n> = -\delta^2 < \Delta_N u^n,\ \Delta_N (u^{n+1}-u^n) > \le -\frac{\delta^2}{2}\| \Delta_N u^{n+1} \|_2^2 + \frac{\delta^2}{2}\| \Delta_N u^n \|_2^2 \tag{21}$$

Hence, the combination of eqs. (17)-(21) indicates:

$$-E_N(u^{n+1}) + E_N(u^n) \ge \| u^{n+1} - u^n \|_2^2 / \Delta t \ge 0 \tag{22}$$

which shows an unconditional stability with respect to the discrete energy is established.

As a result of the energy stability, the uniform in time $\| u^n \|_4$, $\| u^n \|_2$, and $\| \Delta_N u^n \|_2$ bound of the numerical solution (14) is available at a discrete level.

Lemma 2 Assume that u^n is the numerical solution of (14). Then, the following estimates are valid:

$$\| u^n \|_4 \le C_1,\ | u^n \|_2 \le C_2,\ \| \Delta_N u^n \|_2 \le C_3,\ \| u^n \|_\infty \le C_4 \tag{23}$$

for $n = 1, 2, ..., M$.

Proof. From *Theorem 1*, the following energy bound is available:

$$E_N(u^n) \le E_N(u^{n-1}) \le ... \le E_N(u^0) := C_5 \tag{24}$$

For any f we have:

$$\frac{1}{4}\| f \|_4^4 - \frac{\gamma}{2}\| f \|_2^2 \ge \frac{1}{8}\| f \|_4^4 + C_6 \| f \|_2^2 - 2\left(C_6 - \frac{\gamma}{2}\right)| \Omega |,\ C_6 \ge 0 \tag{25}$$

A combination with the discrete energy eq. (16) shows:

$$\frac{1}{8}\| u^n \|_4^4 + C_6 \| u^n \|_2^2 - 2\left(C_6 - \frac{\gamma}{2}\right)^2 | \Omega | + \frac{\delta^2}{2}\| \Delta_N u^n \|_2^2 - \frac{\varepsilon^2}{2}\| \nabla_N u^n \|_2^2 \le$$

$$\le \frac{1}{4}\| u^n \|_4^4 - \frac{\gamma}{2}\| u^n \|_2^2 + \frac{\delta^2}{2}\| \Delta_N u^n \|_2^2 - \frac{\varepsilon^2}{2}\| \nabla_N u^n \|_2^2 \le E_N(u^n) \le C_5 \tag{26}$$

Then, we get:

$$\frac{1}{8}\| u^n \|_4^4 + C_6 \| u^n \|_2^2 + \frac{\delta^2}{2}\| \Delta_N u^n \|_2^2 \le 2(C_6 - \frac{\gamma}{2})^2 |\Omega| + \frac{\varepsilon^2}{2}\| \nabla_N u^n \|_2^2 \tag{27}$$

for all $n \ge 1$. Noting that the integration by parts and ε-inequality imply:

$$\| \nabla_N u^n \|_2^2 = < \nabla_N u^n,\ \nabla_N u^n > = -< u^n,\ \Delta_N u^n > \le \frac{\varepsilon^2}{2\delta^2}\| u^n \|_2^2 + \frac{\delta^2}{2\varepsilon^2}\| \Delta_N u^n \|_2^2 \tag{28}$$

Substituting eq. (28) into eq. (27) yields:

$$\frac{1}{8}\| u^n \|_4^4 + C_6 \| u^n \|_2^2 + \frac{\delta^2}{2}\| \Delta_N u^n \|_2^2 \le \frac{\varepsilon^4}{4\delta^2}\| u^n \|_2^2 + \frac{\delta^2}{4}\| \Delta_N u^n \|_2^2 + 2\left(C_6 - \frac{\gamma}{2}\right)^2 |\Omega| + C_5 \tag{29}$$

that is:

$$\frac{1}{8}\| u^n \|_4^4 + \left(C_6 - \frac{\varepsilon^4}{4\delta^2}\right)\| u^n \|_2^2 + \frac{\delta^2}{4}\| \Delta_N u^n \|_2^2 \le 2\left(C_6 - \frac{\gamma}{2}\right)^2 |\Omega| + C_5 := C_7 \tag{30}$$

provided that $C_6 \ge \max\{\varepsilon^4/(4\delta^2), \gamma/2\}$. Consequently, we have:

$$\begin{gathered}\| u^n \|_4 \le (8C_4)^{1/4} := C_1,\ \| u^n \|_2 \le \{[C_6 - \varepsilon^4/(4\delta^2)]^{-1} C_7\}^{1/2} := C_2 \\ \| \Delta_N u^n \|_2 \le [(\delta^2/4)^{-1} C_7]^{1/2} := C_3\end{gathered} \tag{31}$$

Moreover, employing the elliptic regularity and Sobolev inequality, we can obtain the l^∞ bound of the numerical solution:

$$\| u^n \|_\infty \le \hat{C}(\| u^n \|_2 + \| \Delta_N u^n \|_2) \le \hat{C}(C_1 + C_3) := C_4 \tag{32}$$

This completes the proof of *Lemma 2*.

Convergence analysis

The point-wise numerical error grid function is given:

$$v_{i,j}^n = (u_e)_{i,j}^n - u_{i,j}^n \tag{33}$$

Likewise, to facilitate the presentation, we also denote $v_N^n \in P_K$ as the continuous version of the numerical error function v^n.

Theorem 2. For any final time $T > 0$, assume the exact solution u_e to the SH eq. (4) has a sufficient regularity. Denote u^n as the numerical solution given by the proposed eq. (14). Then, as Δt, $h \to 0$, the following convergence result is valid:

$$\| u^n - u_e^n \|_2 \le C^* (\Delta t + h^m) \tag{34}$$

provided that Δt is sufficiently small, and the constant C^* – the independent of h and Δt, but may depend on ε, δ, γ, $\|u_e\|_{L^\infty(0,\ T,\ H^{m+4})}$, Ω and the final time T.

Proof. Because the exact solution u_e of eq. (4) has a sufficient regularity, detailed Taylor expansions imply the following truncation error:

$$\frac{u_e^{n+1} - u_e^n}{\Delta t} = -(u_e^{n+1})^3 + \gamma u_e^n - \varepsilon^2 \Delta_N u_e^n - \delta^2 \Delta_N^2 u_e^{n+1} + \tau^n \tag{35}$$

with $\| \tau^l \|_2 \le C(\Delta t + h^m)$ for $0 \le l \le M$, in which C is dependent on T, Ω and $\|u_e\|_{L^\infty(0,\ T,\ H^{m+4})}$.

Subtracting eq. (14) from eq. (35), we get the following error equation:

$$\frac{v^{n+1} - v^n}{\Delta t} = -[(u_e^{n+1})^3 - (u^{n+1})^3] + \gamma v^n - \varepsilon^2 \Delta_N v^n - \delta^2 \Delta_N^2 v^{n+1} + \tau^n \tag{36}$$

Taking an inner product of eq. (36) with v^{n+1} gives:

$$< v^{n+1} - v^n, v^{n+1} >= \Delta t < [(u^{n+1})^3 - (u_e^{n+1})^3] + \gamma v^n - \varepsilon^2 \Delta_N v^n - \delta^2 \Delta_N^2 v^{n+1} + \tau^n, v^{n+1} > \tag{37}$$

The time marching term and the truncation error term can be analyzed:

$$< v^{n+1} - v^n, v^{n+1} >= \frac{1}{2}(\| v^{n+1} \|_2^2 - \| v^n \|_2^2 + \| v^{n+1} - v^n \|_2^2) \ge \frac{1}{2}(\| v^{n+1} \|_2^2 - \| v^n \|_2^2) <$$

$$< \tau^n, v^{n+1} > \le \frac{1}{2}(\| v^{n+1} \|_2^2 + \| \tau^n \|_2^2) \tag{38}$$

and

$$< v^n, v^{n+1} > \le -(\| v^{n+1} \|_2^2 + \| v^n \|_2^2) \tag{39}$$

A discrete version of the integration by eq. (9) can be applied to analyze the diffusion terms:

$$-\varepsilon^2 < \Delta_N v^n, v^{n+1} >= \varepsilon^2 < \nabla_N v^n, \nabla_N v^{n+1} > \le \frac{\varepsilon^2}{2}(\| \nabla_N v^{n+1} \|_2^2 + \| \nabla_N v^n \|_2^2) -$$

$$-\delta^2 < \Delta_N^2 v^{n+1}, v^{n+1} >= -\delta^2 < \Delta_N v^{n+1}, \Delta_N v^{n+1} > \le -\delta^2 \| \Delta_N v^{n+1} \|_2^2 \tag{40}$$

Next, we focus on the analysis for non-linear inner product terms:

$$< [(u^{n+1})^3 - (u_e^{n+1})^3], v^{n+1} > \le \| (u^{n+1})^3 - (u_e^{n+1})^3 \|_2 \cdot \| v^{n+1} \|_2 \tag{41}$$

Noting:

$$\| (u^{n+1})^3 - (u_e^{n+1})^3 \|_2 \le \| (u_e^{n+1})^2 v^{n+1} \|_2 + \| u_e^{n+1} u^{n+1} v^{n+1} \|_2 + \| (u^{n+1})^2 v^{n+1} \|_2 \tag{42}$$

We begin with the first term on the right side of (42) and recall that $v_N^n \in P_K$ is the continuous version of the discrete grid error function v^{n+1}. It is obvious:

$$\| (u_e^{n+1})^2 v^{n+1} \|_2 = \| I_N [(u_e^{n+1})^2 v_N^{n+1}] \|_{L^2} \le 3C^2 \| v_N^{n+1} \|_{L^2} \le 3C^2 (\| v^{n+1} \|_2 + h^m) \tag{43}$$

The first step is based on the fact that v^{n+1} and v_N^{n+1} have the same interpolation values. At last step, we use the spectral accuracy in spatial space because of the collocation spectral approximation.

A similar analysis can be applied to the second and the third term on the right side of eq. (42):

$$\| u_e^{n+1} u^{n+1} v^{n+1} \|_2 \le 3C^2 (\| v^{n+1} \|_2 + h^m), \ \| \| (u^{n+1})^2 v^{n+1} \|_2 \le 3C^2 (\| v^{n+1} \|_2 + h^m) \tag{44}$$

in which the L^∞ bound of u^{n+1}, as the continuous version of the l^∞ bound of u^{n+1} also was used.

Therefore, it follows from eqs. (41)-(44):

$$< [(u^{n+1})^3 - (u_e^{n+1})^3], v^{n+1} > \le \frac{27}{2} C^2 \| v^{n+1} \|_2^2 + \frac{9}{2} C^2 h^{2m} \tag{45}$$

Additionally, from eq. (40), by ε-inequality and for $k = n$, $n + 1$, we get:

$$\frac{\varepsilon^2}{2} \| \nabla_N v^k \|_2^2 = -\frac{\varepsilon^2}{2} < \nabla_N v^k, \nabla_N v^k > \le \frac{\varepsilon^4}{8\delta^2} \| v^k \|_2^2 + \frac{\delta^2}{2} \| \Delta_N v^k \|_2^2 \tag{46}$$

Consequently, a substitution of eqs. (37)-(40) and eqs. (44) and (45) into eq. (36) results:

$$\frac{1}{2}(\| v^{n+1} \|_2^2 - \| v^n \|_2^2) + \frac{\delta^2}{2}\Delta t \| \Delta_N v^{n+1} \|_2^2 - \frac{\delta^2}{2}\Delta t \| \Delta_N v^n \|_2^2 \le$$

$$\le \left(\frac{\gamma}{2} + \frac{\varepsilon^4}{8\delta^2} + \frac{1}{2} + \frac{27}{2}C^2\right)\Delta t \| v^{n+1} \|_2^2 + \left(\frac{\gamma}{2} + \frac{\varepsilon^4}{8\delta^2}\right)\Delta t \| v^n \|_2^2 + \frac{1}{2}\Delta t \| \tau^n \|_2^2 + \frac{9}{2}C^2 \Delta t h^{2m} \tag{47}$$

We introduce a modified energy for the error function:

$$G^{n+1} = \frac{1}{2}\| v^{n+1} \|_2^2 + \frac{\delta^2}{2}\Delta t \| \Delta_N v^{n+1} \|_2^2 \tag{48}$$

and the previous inequality now takes the form:

$$G^{n+1} - G^n \le \tilde{C}\Delta t(G^{n+1} + G^n) + \frac{1}{2}\Delta t \| \tau^n \|_2^2 + \frac{9}{2}C^2 \Delta t h^{2m} \tag{49}$$

with $\tilde{C} = \varepsilon^4 / (4\delta^2) + \gamma + 27C^2 + 1$.

Summing over n from 0 to l, and using $G^0 = O(h^{2m})$ because of the spectral accuracy, we obtain:

$$G^{l+1} \le 2\tilde{C}\Delta t \sum_{k=0}^{l} G^k + \tilde{C}\Delta t G^{l+1} + \frac{1}{2}\Delta t \sum_{k=0}^{l} \| \tau^k \|_2^2 + \frac{9}{2}C^2 \Delta t \sum_{k=0}^{l} h^{2m} + h^{2m} \tag{50}$$

Noting that $l\Delta t \le T$, from previous inequality, we have:

$$G^{l+1} \le 2\tilde{C}\Delta t \sum_{k=0}^{l} G^k + \tilde{C}\Delta t G^{l+1} + C_8 T(\Delta t^2 + h^{2m}) \tag{51}$$

where $C = 9C/2+1$. Manipulating eq. (51), we get:

$$G^{l+1} \le \frac{2\tilde{C}\Delta t}{1-\tilde{C}\Delta t}\sum_{k=0}^{l} G^k + \frac{C_8 T}{1-\tilde{C}\Delta t}(\Delta t^2 + h^{2m}) \tag{52}$$

provided that $\tilde{C}\Delta t \le 1$. As a result, an application of the discrete Gronwall inequality gives:

$$G^l \le \frac{C_8 T}{1-\tilde{C}\Delta t}\left(1 + \frac{2\tilde{C}\Delta t}{1-\tilde{C}\Delta t}\right)^{l-1}(\Delta t^2 + h^{2m}) = \frac{C_8 T}{1-\tilde{C}\Delta t}\left(\frac{1+\tilde{C}\Delta t}{1-\tilde{C}\Delta t}\right)^{l-1}(\Delta t^2 + h^{2m}) \tag{53}$$

which is equivalent to the following convergence result:

$$\| v^{n+1} \|_2 + \sqrt{\Delta t}\delta \| \Delta_N v^{n+1} \|_2 \le C^*(\Delta t + h^m) \tag{54}$$

in which:

$$C^* = \left(\frac{C_8 T}{1-\tilde{C}\Delta t}\right)^{1/2}\left(\frac{1+\tilde{C}\Delta t}{1-\tilde{C}\Delta t}\right)^{(l-1)/2} \tag{55}$$

From the form of C^*, the coefficient C^* may be bounded by a positive constant that is dependent on T (exponentially), but otherwise independent of h and Δt. This completes the convergence analysis.

Conclusion

In this paper we presented an energy-stable first-order numerical scheme for the SH equation. The temporal discretization follows the convex splitting of the Lyapunov energy functional. With the help of the l^∞ estimate of control variable indicated by the energy stability and the aliasing error estimate, we obtained the convergence analysis in the discrete l^2-norm.

Acknowledgment

This work is supported by the Scientific Research Innovation Team Project of Yangtze Normal University (No. 2016XJTD01) and the Research Project of Chongqing Science and Technology Committee (No. CSTC2016jcyj-A0492).

References

[1] Swift, J., *et al.*, Hydrodynamic Fluctuations at the Convective Instability, *Physical Review A*, *15* (1977), 1, pp. 319-328
[2] Bordeu, I., *et al.*, From Localized Spots to the Formation of Invaginated Labyrinthine Structures in a Swift-Hohenberg Model, *Communications in Non-Linear Science and Numerical Simulation, 29* (2015), 1-3, pp. 482-4874
[3] Lloyd, D., *et al.*, Localized Radial Solutions of the Swift-Hohenberg Equation, *Non-Linearity, 22* (2009), 2, pp. 485-524
[4] Mccalla, S., *et al.*, Snaking of Radial Solutions of the Multi-Dimensional Swift-Hohenberg Equation: A Numerical Study, *Physica D*, *239* (2010), 16, pp. 1581-1592
[5] Mccalla, S., *et al.*, Spots in the Swift-Hohenberg Equation, *SIAM Journal on Applied Dynamical Systems, 12* (2013), 2, pp. 831-877
[6] Gomez, H., *et al.*, A New Space-Time Discretization for the Swift-Hohenberg Equation that Strictly Respects the Lyapunov Functional, *Communications in Non-linear Science and Numerical Simulation, 17* (2012), 12, pp. 4930-4946
[7] Christov, C. I., *et al.*, Implicit Time Splitting for Fourth order Parabolic Equations, *Computer Methods in Applied Mechanics and Engineering*, *148* (1997), 3-4, pp. 209-224
[8] Christov, C. I., *et al.*, Numerical Scheme for Swift-Hohenberg Equation with Strict Implementation of Lyapunov Functional, *Mathematical and Computer Modelling, 35* (2002), 1, pp. 87-99
[9] Cheng, K., *et al.*, A Second-Order, Weakly Energy-Stable Pseudo-Spectral Scheme for the Cahn-Hilliard Equation and its Solution by the Homogeneous Linear Iteration Method, *Journal of Scientific Computing*, *69* (2016), 3, pp. 1083-1114
[10] Cheng, K., *et al.*, An Energy Stable Fourth Order Finite Difference Scheme For The Cahn-Hilliard Equation, *Journal of Computational and Applied Mathematics*, On-line first, https://doi.org/10.1016/j.cam.2018.05.039, 2018
[11] Elsey, M., *et al.*, A Simple and Efficient Scheme for Phase Field Crystal Simulation, *ESAIM Mathematical Modelling and Numerical Analysis, 47* (2013), 5, pp. 1413-1432
[12] Wise, S. M., *et al.*, An Energy Stable and Convergent Finite-Difference Scheme for the Phase Field Crystal Equation, *SIAM Journal on Numerical Analysis*, *47* (2009), 3, pp. 2269-2288
[13] Eyre, D. J., Unconditionally Gradient Stable Time Marching the Cahn-Hilliard Equation, *MRS Online Proceeding Library Archive*, *529* (2011), 1, pp. 39-46
[14] Gottlieb, S., *et al.*, Stability and Convergence Analysis of Fully Discrete Fourier Collocation Spectral Method for 3-D Viscous Burgers' Equation, *Journal of Scientific Computing, 53* (2012), 1, pp.102-128

Paper submitted: June 12, 2018
Paper revised: September 20, 2018
Paper accepted: November 25, 2018