A FORCED 3-D TIME FRACTIONAL ZK-BURGERS MODEL FOR ROSSBY SOLITARY WAVES WITH DISSIPATION AND THERMAL FORCING

by

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In the paper, beginning from the quasi-geostrophic potential vorticity equation with the dissipation and thermal forcing in stratified fluid, by employing multi-scale analysis and perturbation method, we derive a forced 3-D Zakharov Kuznetsor (ZK)-Burgers equation describe the propagation of the Rossby solitary waves within the fractional derivative. The exact solutions are given by virtue of the (G'/G)-expansion method to analyze the excitation effect of thermal forcing on the Rossby waves.

Key words*: 3-D time fractional forced ZK-Burgers equation, Rossby solitary waves, thermal forcing*

Introduction

In the atmosphere and ocean, due to the earth rotation and the spherical effect, a long life-history large-scale permanent wave is generated. This wave has the characteristics of organized consistency in structure, and this wave has the isolated characteristic of stable large amplitude, which is called Rossby solitary waves [1, 2]. The Rossby solitary waves are a special branch in the field of fluid solitary waves, which has important theoretical and practical significance especially in Marine atmospheric science [3]. The KdV and the modified KdV (mKdV) [4, 5] as well as Boussinesq equation [6, 7] were derived to describe the amplitude of the solitary waves. According to the different dimensions of propagation space, the equations which used to describe Rossby solitary waves are divided into one 1-D [8], 2-D [9], and 3-D. For describing general theoretical and practical problems in the study of Rossby solitary waves, compared with low-dimensional models (mKdV and KP) [10, 11] and 3-D models (ZK and S-KdV) [12] are more suitable, because the density of the fluids is depth or height dependent.

In recent years, the fractional PDE have attracted more and more attention [13, 14]. The solitary waves propagation of the fractional PDE have more research value compared with the integer order differential equation, especially high-dimensional fractional PDE [15]. For example, the fractional PDE to describe solitary waves were investigated [16]. However, the effects of dissipation and thermal forcing on the Rossby solitary waves on the basis of high-dimensional fractional PDE have not been considered. The aim of the paper is to derive a forced 3-D ZK-Burgers equation wthin the fractional derivatives, and analyze the excitation effect of the thermal forcing on the Rossby waves.

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Derivation of forced 3-D ZK-Burgers equation

The forced 3-D ZK-Burgers equation from the quasi-geostrophic vortex equation including the thermal forcing and dissipation can be written:

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi+f+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi}{\partial z}\right)\right]=\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}q\right) \tag{1}$$

where ψ the stream function, N – the Brunt-Vaisala frequency, s and f – the constants, ρ_s – the density, q – the thermal function, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the 3-D Laplace operator.

Suppose that the lower boundary exogenous and dissipation exists, we have:

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial z}+\mu_0 s\nabla^2\psi=q \tag{2}$$

where $\mu_0 s\nabla^2\psi$ denotes the dissipation effect, and $\mu_0 = |K/2f|^{1/2}$ – the dissipation coefficient.

Assuming that the stream function is composed of two parts. The basic stream function and the perturbation stream function can be given:

$$\psi=-\int^y (u(y,z)-c+\varepsilon\alpha)\mathrm{d}y+\varepsilon\psi' \tag{3}$$

where ε is a small parameter which is much less than 1, α – called the detuning parameter that is used to compare the degree of proximity of the weakness to resonate state, and c – the Rossby waves phase speed and can be taken as a constant.

Substituting eq. (3) into eq. (1) and introducing the non-linear β plane for approximation, we get the disturbance stream function. If the apostrophe is omitted here for the convenience of writing, then we have:

$$\left[\frac{\partial}{\partial t}+(u-c+\varepsilon\alpha)\frac{\partial}{\partial x}\right]\left[\nabla^2\psi+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi}{\partial z}\right)\right]+\left\{[\beta(y)y]'-\frac{\partial^2 u}{\partial y^2}-\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\right)\frac{\partial u}{\partial z}\right\}\cdot$$

$$\cdot\frac{\partial\psi}{\partial x}+\varepsilon\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi+\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}\frac{\partial\psi}{\partial z}\right)\right]=\frac{f}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{s}q\right) \tag{4}$$

The lower boundary condition, lateral boundary condition and upper boundary are taken:

$$\varepsilon\left[\frac{\partial}{\partial t}+\left(u-c+\varepsilon\alpha\right)\frac{\partial}{\partial x}\right]\frac{\partial\psi}{\partial z}-\varepsilon\frac{\partial u}{\partial z}\frac{\partial\psi}{\partial x}+\varepsilon^2\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\psi}{\partial z}+\varepsilon\mu_0\frac{N^2}{f}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=\varepsilon q \tag{5}$$

In order to make the non-linearity, dispersion, dissipation and thermal forcing balance, we introduce the time and space stretching transform:

$$T=\varepsilon^{3/2}t,\ X=\varepsilon^{1/2}x,\ Y=\varepsilon^{1/2}y,\ Z=\varepsilon^{1/2}z,\ q=\varepsilon^{3/2}q',\ \mu_0=\varepsilon^{3/2}\mu \tag{6}$$

Assuming that the ψ has the following:

$$\psi=\psi_0+\varepsilon\psi_1+\cdots \tag{7}$$

Substituting (6) and (7) into (4) and (5), and letting ψ_0 become:

$$\psi_0=A(X,T)\varphi_0(y,z) \tag{8}$$

For the determined basic stream function $u(y, z)$, the proper function $\varphi_0(y, z)$ and eigenvalue c can be obtained. In order to obtain the control model of Rossby solitary waves amplitude, the higher order equation of ε needs to be solved.

After a series of calculations with boundary conditions, we may have:

$$A_t + \alpha A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} + a_4 A = a_5 Q \tag{9}$$

where AA_x represents the non-linear effect, A represents the dissipation effect, and Q is forcing term which caused by thermal source. It is called the 3-D forced ZK-Burgers equation. We have the following cases:

(I) When the thermal forcing is absent *i. e.,* $a_5 = 0$*,* it degenerates to the 3-D ZK-Burgers equation:

$$A_t + \alpha A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} + a_4 A = 0 \tag{10}$$

(II) When dissipation is absent *i. e.,* $a_4 = 0$*,* it degenerates to 3-D forced ZK equation, given:

$$A_t + \quad A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} = a_5 Q \tag{11}$$

(III) When a_4 and a_5 are both absent, it is the standard 3-D ZK equation, *e. g.*:

$$A_t + \alpha A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} = 0 \tag{12}$$

Derivation of forced 3-D time fractional ZK-Burgers equation

In this section, to learn more the propagation feature of Rossby solitary waves, the semi-inverse method and the fractional variational principle are applied to derive the 3-D time fractional ZK equation.

Suppose that $A(x, y, z, t) = B_x(x, y, z, t)$, a potential function $B_x(x, y, z, t)$ provides the potential equation of the eq. (12) in the form:

$$B_{xt} + \alpha B_{xx} + a_0 B_x B_{xx} + a_1 B_{xxxx} + a_2 B_{xxyy} + a_3 B_{xxzz} = 0 \tag{13}$$

The functional of eq. (13) can be represented:

$$J(B) = \int_R \mathrm{d}x \int_R \mathrm{d}y \int_R \mathrm{d}z \int_T \mathrm{d}t [B(x,y,z,t)(c_1 B_{xt} + c_2 \alpha B_{xx} + c_3 a_0 B_x B_{xx} + c_4 a_1 B_{xxxx} + c_5 a_2 B_{xxyy} + c_6 a_2 B_{xxzz})] \tag{14}$$

where $c_i (i = 1,...,6)$ are Lagrangian coefficient, which can be confirmed later.

Integrating eq. (14) by parts and taking $B_t|_T = B_x|_R = B_{xxx}|_R = B_{xyy}|_R = B_{xzz}|_R = 0$, we get:

$$J(B) = \int_R \mathrm{d}x \int_R \mathrm{d}y \int_R \mathrm{d}z \int_T \mathrm{d}t \, (-c_1 B_x B_t - c_2 \alpha B_x^2 - \frac{1}{2} c_3 a_0 B_x^3 + c_4 a_1 B_{xx}^2 + c_5 a_2 B_{xy}^2 + c_6 a_3 B_{xz}^2) \tag{15}$$

Applying the variation of this function, integrating each parts, optimizing of this variation, $\delta J(B) = 0$ and comparing with eq. (13), we can have the Lagrangian coefficiens $c_1 = c_2 = c_4 = c_5 = c_6 = 1/2$, and $c_3 = 1/3$.

The Lagrangian form of the regular ZK equation is given:

$$F(B_x B_t, B_x, B_{xx}, B_{xy}, B_{xz}) = -\frac{1}{2} B_x B_t - \frac{1}{2} \alpha B_x^2 - \frac{1}{6} a_0 B_x^3 + \frac{1}{2} a_1 B_{xx}^2 + \frac{1}{2} a_2 B_{xy}^2 + \frac{1}{2} a_3 B_{xz}^2 \tag{16}$$

In a same manner, the Lagrangian form of the time fractional ZK equation could be expressed in the following form:

$$L({}_0\mathrm{D}_t^\beta B, B_x, B_{xx}, B_{xy}, B_{xz}) = -\frac{1}{2}\mathrm{D}_t^\beta B B_x - \frac{1}{2}\alpha B_x^2 - \frac{1}{6}a_0 B_x^3 + \frac{1}{2}a_1 B_{xx}^2 + \frac{1}{2}a_2 B_{xy}^2 + \frac{1}{2}a_3 B_{xz}^2 \quad (17)$$

where ${}_0\mathrm{D}_t^\beta B$ is the left Riemann-Liouville fractional derivative [2].

So, the functional of the time fractional ZK equation with the dissipation effect will take the form:

$$J(B) = \int_R \mathrm{d}x \int_R \mathrm{d}y \int_R \mathrm{d}z \int_T \mathrm{d}t\, L({}_0\mathrm{D}_t^\beta B, B_x, B_{xx}, B_{xy}, B_{xz}) \quad (18)$$

On the basis of the Agrawal's method, the variation of eq. (18) can be written:

$$\delta J(B) = \int_R \mathrm{d}x \int_R \mathrm{d}y \int_R \mathrm{d}z \int_T \mathrm{d}t [\left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^\beta B}\right)\delta\, {}_0\mathrm{D}_t^\beta B + \left(\frac{\partial L}{\partial B_x}\right)\delta B_x + \\ + \left(\frac{\partial L}{\partial B_{xx}}\right)\delta B_{xx} + \left(\frac{\partial L}{\partial B_{xy}}\right)\delta B_{xy} + \left(\frac{\partial L}{\partial B_{xz}}\right)\delta B_{xz}] \quad (19)$$

where ${}_t\mathrm{D}_b^\alpha$ is the right Riemann-Liouville fractional derivative [2].

Integrating the right-hand side of the eq. (19) using the fractional integration by parts rule, optimizing the variation of the function and $\delta J(B) = 0$, we get the Euler-Lagrange equation of the 3-D ZK equation:

$${}_t\mathrm{D}_{t_0}^\beta \left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^\beta B}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial B_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial B_{xx}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial L}{\partial B_{xy}}\right) + \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial L}{\partial B_{xz}}\right) = 0 \quad (20)$$

Substituting (17) into the Euler-Lagrange equation given by (20) and substituting the potential function $B_x(x, y, z, t) = A(x, y, z, t)$, we get the 3-D time fractional ZK equation with the regular function $A(x, y, z, t)$ as:

$$\frac{1}{2}\left({}_0\mathrm{D}_t^\beta A - {}_t\mathrm{D}_{t_0}^\beta A\right) + \alpha A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} = 0 \quad (21)$$

In addition, $A(x, y, z, t)$ is the Riesz fractional derivative [2], *e. g.*,

$${}_0^R\mathrm{D}_t^\beta f(t) = \frac{1}{2}\left[{}_a\mathrm{D}_t^\beta f(t) - (-1)^k\, {}_t\mathrm{D}_b^\beta f(t)\right] = \frac{1}{2}\frac{1}{\Gamma(k-\beta)}\frac{\mathrm{d}k}{\mathrm{d}t^k}\int_a^b \mathrm{d}\tau\, |t-\tau|^{k-\beta-1} f(\tau), \\ k-1 \le \beta \le k,\ t \in [a,b] \quad (22)$$

The another form of the 3-D time fractional ZK equation can be written:

$$\mathrm{D}_t^\beta A + \alpha A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} = 0 \quad 0 \le \beta \le 1,\ \ t \in [0, T_0] \quad (23)$$

If the dissipation is presented, eq. (23) may reduce to the 3-D time fractional ZK-Burgers model:

$$\mathrm{D}_t^\beta A + \alpha A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} + a_4 A = 0 \quad 0 \le \beta \le 1,\ \ t \in [0, T_0] \quad (24)$$

If the dissipation and thermal forcing are presented, eq. (23) can lead to the forced 3D time fractional ZK-Burgers model:

$$\mathrm{D}_t^\beta A + \alpha A_x + a_0 A A_x + a_1 A_{xxx} + a_2 A_{xyy} + a_3 A_{xzz} + a_4 A = a_5 Q \quad 0 \le \beta \le 1,\ t \in [0, T_0] \quad (25)$$

Analytical solutions of 3-D time fractional ZK-Burgers equation and dissipation effect

In order to analyze the influence of dissipation on Rossby solitary waves, we seek the analytical solution of the 3-D time fractional ZK-Burgers equation by applying (G'/G)--expansion method.

We now introduce a fractional complex transformation:

$$A(x,y,z,t)=\Psi(\zeta),\ \zeta=k_1x+k_2y+k_3z+\frac{\omega t^\beta}{\Gamma(\beta+1)} \tag{26}$$

where k_1, k_2, k_3, and ω are constants to be confirmed later, and get:

$$(\omega+\alpha k_1)\psi'(\zeta)+a_0k_1\psi(\zeta)\psi'(\zeta)+(a_1k_1^3+a_2k_1k_2^2+a_3k_1k_3^2)\psi'''(\zeta)+a_4\psi(\zeta)=0 \tag{27}$$

In order to balance the highest order and non-linear term, we assume the solution form of eq. (27) by a polynomial $\psi(\zeta)$:

$$\psi(\zeta)=b_0+\sum_{i=1}^{n}b_i\left(\frac{G'}{G}\right)^i \tag{28}$$

In this case, we get:

$$\psi(\zeta)=b_0+b_1\left(\frac{G'}{G}\right)+b_2\left(\frac{G'}{G}\right)^2 \tag{29}$$

where $G=G(\zeta)$ satisfies:

$$G''+mG'+nG=0 \tag{30}$$

In this case, we obtain the set of coefficients for the solutions:

$$\begin{gathered}m=m,\ n=n,\ k_1=k_1,\ k_2=k_2,\ k_3=k_3,\ \omega=(a_4-a_0k_1)b_0-\alpha k_1-h(m^2-4n),\\ b_0=\frac{h(m^3-2m^2+4m-28n)}{a_0k_1+a_0k_1m+a_4},\ b_1=\frac{12hm}{a_0k_1},\ b_2=-\frac{12h}{a_0k_1}\end{gathered} \tag{31}$$

Finally, we have a series of exact solutions to 3D time fractional ZK-Burgers as:

(I) When $\Delta m^2-4n>0$, the hyperbolic solution takes the form:

$$\psi_1=\frac{h(m^3-2m^2+4m-28n)}{a_0k_1+a_0k_1m+a_4}+\frac{12hm}{a_0k_1}(H_1)-\frac{12h}{a_0k_1}(H_1)^2 \tag{32}$$

(II) When $\Delta m^2-4n<0$, the trigonometric solution can be written:

$$\psi_2=\frac{h(m^3-2m^2+4m-28n)}{a_0k_1+a_0k_1m-a_4}+\frac{12hm}{a_0k_1}(H_2)-\frac{12h}{a_0k_1}(H_2)^2 \tag{33}$$

Excitation effect of the thermal forcing for Rossby solitary waves

Let's take the Gauss thermal function:

$$Q=\frac{1}{2}\exp\left[\frac{-(x-50)^2}{4}\right] \tag{34}$$

From eq. (25) we have:

$$\mathrm{D}_t^\beta A+\alpha A_x+a_0AA_x+a_1A_{xxx}=0,\ 0\le\beta\le1,\ t\in[0,T_0]$$

If the spatial period is normalized to [0, 2π], and the time intervals is divided into points, then $\Delta x = \pi/N$. In this case, we have:

$$\hat{A}(v,t) = Fu = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} A(j\Delta x, t) \mathrm{e}^{-\pi ijv/N}, \ v = 0, \pm 1, \ldots, \pm N \tag{35}$$

$$A(j\Delta x, t) = F^{-1}\hat{A} = \frac{1}{\sqrt{2N}} \sum_{v} \hat{A}(A,t) \mathrm{e}^{\pi ijv/N} \tag{36}$$

By the properties of the Fourier transform, we get:

$$A_x = F^{-1}(ivFu), \ A_{xxx} = -iF^{-1}(v^3 Fu) \tag{37}$$

In this case, the spectral form of eq. (35) can be given:

$$A(x,t+\Delta t) - A(x,t-\Delta t) + 2\Delta t\alpha F^{-1}(ivFu) - 2\Delta t a_0 A F^{-1}(ivFu) + 2a_1 \Delta t i F^{-1}(v^3 Fu) = 0 \tag{38}$$

Conclusion

In our work, we had investigated a forced 3-D ZK-Burgers equation containing the fractional derivatives for the first time. The soutions for the 3-D time fractional ZK-Burgers equation and dissipation effect had analyzed and the excitation effect the thermal forcing for Rossby solitary waves was discussed. The reuslts are efficient for the description of the quasi-geostrophic potential vorticity equation with the dissipation and thermal forcing in stratified fluid.

Nomenclature

q – thermal function, [K]
N – Brunt-Vaisala frequency, [s^{-1}]
t – space co-ordinate, [s]
x, y, z – space co-ordinates, [m]

Greek symbols

α – fractional order, [–]
∇^2 – 3-D Laplace operator, [–]
ρ_s – density, [gm^{-3}]
ψ – stream function, [–]

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