APPLICATIONS OF THE FOURIER-LIKE INTEGRAL TRANSFORM IN THE WAVE AND HEAT-TRANSFER PROBLEMS

by

***Shanjie SU* [a b]*, Feng GAO* [a b*]*, and Zekai WANG* [a b]*, Menglin DU* [a b]**

[a] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou221116, China

[b] School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou 221116, China

In this article, some new properties of a novel integral transform termed the Fourier-Yang are explored. The Fourier-Yang integral transforms of several basic functions are given firstly. With the aid of the new integral transform, a 1-D wave equation and 2-D heat-transfer equation are solved. The results show that the Fourier-Yang integral transform is efficient in solving partial differential equations.

Keywords: *Fourier-like integral transform, analytical solution, heat-transfer equation, wave equation*

Introduction

Integral transforms have been applied to solving the key issues involving mechanics, chemistry, physics, thermal science and interdisciplinary areas [1,2]. For example, the Laplace integral transform plays the important role in transient thermal stresses [3], fluid mechanics [4] and viscoelastic fluids[5]. The Fourier integral transform has become the powerful tool in solving the volume integral equations and the Cauchy integral equation [6-8], and the Sumudu integral transform was utilized to solve partial differential equations in [9-11].With the development of the integral transform, some new integral transforms proposed, such as the Elzaki transform[12,13], the Fourier-Yang transform[14,15], and the Laplace-Carson transform[16], were also suggested to solve more differential equations. Recently, a new Fourier-like integral transform adopted to deal with a steady heat transfer problem is given in [17]. However, the properties of the new integral transform are incomplete, and it has not been employed to solve the wave and the two-dimension heat-transfer equations.

This paper aims to extend some new properties of the Fourier-Yang integral transform and give the integral transform of some basic functions. Moreover, the partial differential equations proposed in the one-dimensional wave and the two-dimensional heat-transfer problems are solved by the technique of the integral transform for the first time.

The Fourier-Yang integral transform

* Corresponding author; e-mail:13814430108@163.com

In this section, the definitions of the Fourier integral transform and The Fourier-Yang integral transform are recalled, some new properties of The Fourier-Yang integral transform are given firstly. In addition, the integral transforms of some functions are defined.

The Fourier integral transform of the function $\varphi(t)$ is denoted as [18]:

$$\Phi(\omega)=\mathrm{F}\left[\varphi(t)\right]=\int_{-\infty}^{\infty}\varphi(t)e^{-j\omega t}dt, \tag{1}$$

where ω is a constant, and F represents the Fourier integral transform operator.

The inverse Fourier integral transform is given as [18]:

$$\Phi(t)=\mathrm{F}^{-1}\left[\varphi(\omega)\right]=\frac{1}{2\pi j}\int_{-\infty}^{\infty}\varphi(\omega)e^{j\omega t}d\omega, \tag{2}$$

where F^{-1} represents the inverse Fourier integral transform operator.

The Fourier-Yang integral transform of the function $\varphi(t)$ is shown as [19]:

$$\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}\Sigma(t)\mathrm{e}^{-j\eta t}dt, \tag{3}$$

where η is a real-valued constant, $\mathbb{Z}$ represents the Fourier-Yang integral transform operator.

The inverse Fourier-Yang integral transform is given by [19]:

$$\varphi(t)=\mathbb{Z}^{-1}\left[\Phi(\eta)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\eta)\eta\mathrm{e}^{j\eta t}d\eta, \tag{4}$$

where $\mathbb{Z}^{-1}$ represents the inverse Fourier-Yang integral transform operator.

Substituting eq. (3) into eq. (4), we have the integral criterion as:

$$\varphi(t)=\mathbb{Z}^{-1}\left[\Phi(\eta)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta\left[\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}dt\right]\mathrm{e}^{j\eta t}d\eta. \tag{5}$$

The properties of the Fourier-Yang integral transform are as follows.

(T1) If $\varphi(\omega)=F\left[\varphi(t)\right]$ and $\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]$, then we have:

$$\varphi(\omega)=\eta\Phi(\eta), \tag{6}$$

$$\Phi(\eta)=\frac{1}{\omega}\varphi(\omega). \tag{7}$$

(T2) If $\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]$ and $\Psi(\eta)=\mathbb{Z}\left[\Psi(t)\right]$, then we have:

$$\mathbb{Z}\left[a\varphi(t)+b\Psi(t)\right]=a\Phi(\eta)+b\Psi(\eta), \tag{8}$$

where a and b are the constants.

(T3) If $\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]$, then we have:

$$\mathbb{Z}\left[\varphi(t-a)\right]=e^{j\eta a}\Phi(\eta), \tag{9}$$

where a is a constant.

(T4) If $\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]$, then we have:

$$\mathbb{Z}\left(\frac{\mathrm{d}\varphi(t)}{dt}\right)=j\eta\Phi(\eta). \tag{10}$$

(T5) If $\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]$, then we have:

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\varphi(t)dt\right]=\frac{1}{j\eta}\Phi(\eta). \tag{11}$$

(T6) If $\Phi'(\eta)=\dfrac{d\Phi(\eta)}{d\eta}$, then we have:

$$\Phi'(\eta)=-\frac{1}{\eta}\Phi(\eta)-j\eta\Phi(\eta). \tag{12}$$

(T7) If $\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]$ and $\Psi(\eta)=\mathbb{Z}\left[\Psi(t)\right]$, then we have the Fourier-Yang integral transform of convolution as follows:

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\varphi(t-\tau)\Psi(\tau)dt\right]=\eta\Psi(\eta)\Phi(\eta). \tag{13}$$

(T8) Let $\varphi(t)=e^{-at}\nu(t)$, where $\nu(t)$ is the Heaviside unit step function [18]. Then:

$$\Phi(\eta)=\frac{1}{\eta(a+j\eta)}, \tag{14}$$

where a is a constant.
(T9) Let $\varphi(t)=\delta(t)$, where $\delta(t)$ is the Dirac function. Then:

$$\Phi(\eta)=\frac{1}{\eta}. \tag{15}$$

(T10) Let $\varphi(t)=\begin{cases}C, |t|<T\\ 0 \ \ else\end{cases}$, where C and T are constants. Then:

$$\mathbb{Z}\left[\varphi(t)\right]=\frac{2\sin(\eta T)}{\eta^2}. \tag{16}$$

(T11)Let $\varphi(t)=\upsilon e^{-kt^2}, k>0,$ where υ is a constant. Then:

$$\mathbb{Z}\left[\varphi(t)\right]=\frac{\upsilon\pi}{\eta\sqrt{k\pi}}e^{-\frac{\eta^2}{4k}}. \tag{17}$$

Proof.
(T1) Taking $\omega=\eta$ in eqs. (1) and (3), we obtain:

$$\varphi(\omega)=F\left[\varphi(t)\right]=\int_{-\infty}^{\infty}\varphi(t)e^{-j\omega t}dt=\eta.\left[\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}dt\right]=\eta\Phi(\eta) \tag{18}$$

$$\Phi(\eta)=\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}dt=\frac{1}{\omega}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\omega t}dt=\varphi(\omega). \tag{19}$$

(T2)

$$\mathbb{Z}\left[a\varphi(t)+b\Psi(t)\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}a\varphi(t)e^{-j\eta t}dt+\frac{1}{\eta}\int_{-\infty}^{\infty}b\Psi(t)e^{-j\eta t}dt=a\Phi(\eta)+b\Psi(\eta). \tag{20}$$

(T3)

$$\mathbb{Z}\left[\varphi(t-a)\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t-a)\mathrm{e}^{-j\eta t}d\ , \tag{21}$$

If $\mathrm{t}=t-a$, then

$$\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}e^{j\eta a}dt=e^{j\eta a}\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}dt=e^{j\eta a}\Phi(\eta). \tag{22}$$

(T4)

$$\mathbb{Z}\left(\frac{\mathrm{d}\varphi(t)}{dt}\right)=\frac{1}{\eta}\int_{-\infty}^{\infty}\frac{\mathrm{d}\varphi(t)}{dt}\mathrm{e}^{-j\eta t}dt=\frac{1}{\eta}\int_{-\infty}^{\infty}\mathrm{e}^{-j\eta t}d\varphi(t)=j\eta\Phi(\eta). \tag{23}$$

Similarly, we have:

$$\mathbb{Z}\left(\frac{\mathrm{d}^n\varphi(t)}{dt^n}\right)=j^n\eta^n\Phi(\eta), \tag{24}$$

where n is the positive integer.

(T5)

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\varphi(t)dt\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\varphi(t)dt\right]\mathrm{e}^{-j\eta t}dt=\frac{1}{j\eta}\Phi(\eta). \tag{25}$$

Similarly, we have:

$$\mathbb{Z}\left(\int_1\cdots\int_n\varphi(t)dt\right)=\frac{1}{j^n\eta^n}\Phi(\eta), \tag{26}$$

(T6)

$$\Phi'(\eta)=-\frac{1}{\eta^2}\int_{-\infty}^{\infty}\varphi(t)e^{-j\eta t}dt-j\int_{-\infty}^{\infty}\varphi(t)e^{-j\eta t}dt=-\frac{1}{\eta}\Phi(\eta)-j\eta\Phi(\eta). \tag{27}$$

(T7)

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\varphi(t-\tau)\Psi(\tau)dt\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}\Psi(\tau)\left[\int_{-\infty}^{\infty}\varphi(t-\tau)\mathrm{e}^{-j\eta t}dt\right]d\tau, \tag{28}$$

If λ=t-τ, then

$$\frac{1}{\eta}\int_{-\infty}^{\infty}\Psi(\tau)\left[\int_{-\infty}^{\infty}\varphi(\lambda)\mathrm{e}^{-j(\lambda+\tau)\eta}d\lambda\right]d\tau=\frac{1}{\eta}\int_{-\infty}^{\infty}\Psi(\tau)e^{-j\tau\eta}d\tau\int_{-\infty}^{\infty}\varphi(\lambda)\mathrm{e}^{-j\lambda\eta}d\lambda=\eta\Psi(\eta)\varphi(\eta). \tag{29}$$

Similarly, we have:

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t_1-\tau_1,t_2-\tau_2)\Psi(\tau_1,\tau_2)dt_1dt_2\right]=\eta_1\eta_2\Psi(\eta_1,\eta_2)\varphi(\eta_1,\eta_2). \tag{30}$$

(T8) If the Heaviside unit step function in [18] is

$$\int_{-\infty}^{\infty}\phi(\mathrm{t})\nu(t)dt=\int_{0}^{\infty}\phi(\mathrm{t})dt, \tag{31}$$

then

$$\Phi(\eta)=\mathbb{Z}[\varphi(t)]=\frac{1}{\eta}\int_{-\infty}^{\infty}e^{-at}\nu(t)\mathrm{e}^{-j\eta t}dt=\frac{1}{\eta}\int_{0}^{\infty}\mathrm{e}^{-(a+j\eta)t}dt=\frac{-1}{\eta(a+j\eta)}\mathrm{e}^{-(a+j\eta)t}\Big|_0^{\infty}=\frac{1}{\eta(a+j\eta)}. \tag{32}$$

(T9) If the Dirac function is

$$\delta(\tau-c)=\begin{cases}\infty, & \tau=c\\ 0, & \tau\neq c\end{cases},\ \int_{-\infty}^{+\infty}\delta(\tau-c)f(\tau)d\tau=f(c), \tag{33}$$

then

$$\Phi(\eta)=\mathbb{Z}[\varphi(t)]=\frac{1}{\eta}\int_{-\infty}^{\infty}\delta(t)\mathrm{e}^{-j\eta t}dt=\frac{1}{\eta}. \tag{34}$$

(T10)

$$\mathbb{Z}[\varphi(t)]=\frac{2}{\eta}\int_{0}^{\mathrm{T}}C\cos(\eta t)dt=\frac{2\sin(\eta T)}{\eta^2}, \tag{35}$$

(T11)

$$\Sigma(\eta)=\frac{1}{\eta}\int_{-\infty}^{\infty}\upsilon e^{-kt^2}e^{-j\eta t}dt=\frac{\upsilon}{\eta}\int_{-\infty}^{\infty}e^{\left[-k\left(t+\frac{j\eta}{2k}\right)^2-\frac{\eta^2}{4k}\right]}dt. \tag{36}$$

If $t=t+\frac{j\eta}{2k}$, then we get

$$\mathbb{Z}\left[\upsilon e^{-kt^2}\right]=\frac{\upsilon}{\eta}e^{-\frac{\eta^2}{4k}}\int_{-\infty}^{\infty}e^{-kt^2}dt=\frac{\upsilon\pi}{\eta\sqrt{k\pi}}e^{-\frac{\eta^2}{4k}}, \tag{37}$$

where

$$\int_{-\infty}^{\infty}e^{-kt^2}dt=\sqrt{\frac{\pi}{k}}. \tag{38}$$

Applications

Solving the one-dimensional wave equation

In this section, with the help of the Fourier-Yang integral transform, the analytical solution of the one-dimensional wave equation is shown as follows.

The mathematical model of one-dimensional wave equation is defined as [20]:

$$\frac{\partial^2 \varphi(x,t)}{\partial t^2} - \lambda^2 \frac{\partial^2 \varphi(x,t)}{\partial x^2} = 0, \left(-\infty < x < \infty, t > 0\right), \tag{39}$$

where λ is a constant.

The initial conditions are given by:

$$\varphi(x,0) = \vartheta(x) \ , \ \frac{\partial \varphi(x,0)}{\partial t} = 0. \tag{40}$$

Using the eqs. (3) and (24), the Fourier-Yang integral transforms of eqs. (39) with respect to x are given as:

$$\frac{\partial^2 \Phi(\eta,t)}{\partial t^2} + \lambda^2 \eta^2 \Phi(\eta,t) = 0. \tag{41}$$

Similarly, the initial conditions becomes：

$$\Phi(\eta,0) = \vartheta(\eta) \ , \ \frac{\partial \Phi(\eta,0)}{\partial t} = 0. \tag{42}$$

Making use of eqs. (41) and (42), we have:

$$\Sigma(\eta,t) = \vartheta(\eta)\cos(\lambda\eta t). \tag{43}$$

Substituting eq. (43) into eq. (4), and with the help of eq. (9), we obtain:

$$\varphi(x,t) = \mathbb{Z}^{-1}\left(\Phi(\eta,t)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta\vartheta(\eta)\cos(\lambda\eta t)\, e^{j\eta t} d\eta = \frac{1}{2}\left[\vartheta(x-\lambda t) + \vartheta(x+\lambda t)\right], \tag{44}$$

the solution of eq. (44) is identical to the result in [27].

Solving the two-dimensional heat-transfer equation

The partial differential equation in the two-dimensional heat-transfer problem is:

$$\frac{\partial \varphi(x,y,t)}{\partial t} - \kappa^2 \left(\frac{\partial^2 \varphi(x,y,t)}{\partial x^2} + \frac{\partial^2 \varphi(x,y,t)}{\partial y^2} \right) = 0, \left(-\infty < x, y < \infty, t > 0\right), \tag{45}$$

where κ is the thermal conductivity.

The initial condition is:

$$\varphi(x,y,0) = \vartheta(x,y). \tag{46}$$

Combining eqs. (3) and (24) yields the Fourier-Yang integral transforms of eq. (45) with respect to x *and* y expressed as:

$$\frac{\partial \Phi(\eta_1,\eta_2,t)}{\partial t} + \kappa^2\left(\eta_1^{\ 2} + \eta_2^{\ 2}\right)\Phi(\eta_1,\eta_2,t) = 0. \tag{47}$$

Similarly, the initial conditions is：

$$\Phi(\eta_1,\eta_2,0) = \vartheta(\eta_1,\eta_2). \tag{48}$$

From eqs. (47) and (48), we have:

$$\Phi\left(\eta_1,\eta_2,t\right)=\eta_1\eta_2\vartheta\left(\eta_1,\eta_2\right)\Psi\left(\eta_1,\eta_2\right), \tag{49}$$

where

$$\Psi\left(\eta_1,\eta_2\right)=\frac{1}{\eta_1}\frac{1}{\eta_2}e^{-\kappa^2\left(\eta_1{}^2+\eta_2{}^2\right)t}. \tag{50}$$

Utilizing eqs. (4) and (37), the inverse Fourier-Yang integral transform of eq. (50) is written as:

$$\mathbb{Z}^{-1}\left[\frac{1}{\eta_1}\frac{1}{\eta_2}e^{-\kappa^2\left(\eta_1{}^2+\eta_2{}^2\right)t}\right]=\left[\frac{1}{2\pi\kappa}\sqrt{\frac{\pi}{\mathrm{t}}}\mathrm{e}^{-\frac{x^2}{4\kappa^2t}}\right]\left[\frac{1}{2\pi\kappa}\sqrt{\frac{\pi}{\mathrm{t}}}\mathrm{e}^{-\frac{y^2}{4\kappa^2t}}\right]=\frac{1}{4\pi\kappa^2\mathrm{t}}e^{\frac{-\left(x^2+y^2\right)}{4\kappa^2t}}. \tag{51}$$

Substution of eqs. (50) and (51) into eq. (30), we have the analytical solution of the two-dimension heat-transfer problem as:

$$\Phi\left(x,y,t\right)=\frac{1}{4\pi\kappa^2\mathrm{t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\vartheta\left(\tau_1,\tau_2\right)e^{\frac{-\left(\left(x-\tau_1\right)^2+\left(y-\tau_2\right)^2\right)}{4\kappa^2t}}d\tau_1 d\tau_2. \tag{52}$$

Conclusion

In this work, some new properties of the Fourier-Yang integral transform are extended firstly, and the integral transforms of some functions are given. Applying those, we obtain the analytical solutions of the differential equations in the one-dimensional wave and the two-dimensional heat-transfer problems. The results indicate that the Fourier-Yang integral transform is effective and precise in solving the partial differential equations.

Nomenclature

	Greek symbols
t *-time, [s]*	λ *- wave propagation rate, [m/s]*
x,y *-space co-ordinate, [m]*	κ *- thermal conductivity,* $[Wm^{-2}K^{-1}]$

Acknowledgments

The authors sincerely thanks the financial support from the State Key Research Development Program of China (No. 2016YFC0600705), the National Natural Science Foundation of China (No. 51604263) and the Natural Science Foundation of Jiangsu Province (No. BK20160252).

References

[1] Debnath, L., Bhatta, D., *Integral Transforms and Their Applications*, CRC Press, Boca Raton, Fla., USA, 2014

[2] Yang, X. J., A New Integral Transform Operator for Solving the Heat-Diffusion Problem, *Applied Mathematics Letters, 64* (2017), pp. 193-197

[3] Jin, Z. H., Paulino, G. H., Transient Thermal Stress Analysis of an Edge Crack in a Functionally Graded Material, *International Journal of Fracture, 107* (2001), 1, pp. 73-98

[4] Kulish, V. V., Lage, J. L., Application of Fractional Calculus to Fluid Mechanics, *Journal of Fluids Engineering, 124* (2002), 3, pp. 803-806

[5] Zhao, C., Yang, C., Exact Solutions for Electro-Osmotic Flow of Viscoelastic Fluids in Rectangular Micro-Channels, *Applied Mathematics & Computation*, *211* (2009), 2, pp. 502-509

[6] Greengard, L., Lee, J. Y., Accelerating the Nonuniform Fast Fourier Transform, *Siam Review*, *46* (2004), 3, pp. 443-454

[7] Fan, Z., *et al.*, Weak Form Nonuniform Fast Fourier Transform Method for Solving Volume Integral Equations, *Progress in Electromagnetics Research*, *89* (2009), 4, pp. 275-289

[8] Yokogawa, D., *et al*., An Integral Equation Theory for 3D Solvation Structure: A New Procedure Free from 3D Fourier Transform, *Chemical Physics Letters*, *432* (2006), 4, pp. 595-599

[9] Eltayeb, H., *et al*., A New Integral Transform and Associated Distributions, *Integral Transforms & Special Functions*, *21* (2010), 5, pp. 367-379

[10] Kumar, D., *et al*., An Efficient Approach for Fractional Harry Dym Equation by Using Sumudu Transform, *Abstract and Applied Analysis*, (2013), pp. 1-16

[11] Boyadjiev, L., Luchko, Y., Mellin Integral Transform Approach to Analyze the Multidimensional Diffusion-Wave Equations, *Chaos Solitons & Fractals*, 102 (2017), pp. 127-134

[12] Elzaki, T. M., The new Integral Transform "Elzaki Transform", *Global Journal of Pure & Applied Mathematics*, *7* (2011), 1, pp. 57-64

[13] Tarig, E., *et al*., *On the New Integral Transform "ELzaki Transform" Fundamental Properties Investigations and Applications*, Global Journal of Mathematical Sciences, 2012

[14] Celeghini, E., Del Olmo, M.A., Quantum Physics and Signal Processing in Rigged Hilbert Spaces by Means of Special Functions, Lie Algebras and Fourier and Fourier-Like Transforms, *Journal of Physics: Conference Series*, 597 (2015), pp. 012022

[15] Liang, X., *et al*., Applications of a Novel Integral Transform to Partial Differential Equations, *Journal of Nonlinear Science & Applications*, 10 (2017), 2, pp. 528-534

[16] Friba, L., *Vibration of Solids and Structures under Moving Loads*, Springer Science & Business Media ,1(1972), pp. 39-99

[17] Yang, X. J., New Integral Transforms for Solving a Steady Heat Transfer Problem, *Thermal Science*, 21 (2017), Suppl.1, pp. S79-S87

[18] Beerends, R. J., *et al*., *Fourier and Laplace Transforms*, Cambridge University Press, Cambridge, UK, 2003

[19] Dyke, P. P. G., *An Introduction to Laplace Transforms and Fourier Series// an Introduction to Laplace transforms and Fourier series*, Springer, 2001

[20] Usman, M., *et al*., Differential Transform Method for Unsteady Nanofluid Flow and Heat Transfer, *Alexandria Engineering Journal*, 57(2018), 3, PP.1867-1875

Paper submitted: May 20, 2018
Paper revised: June 28, 2018
Paper accepted: August 11, 2018