A LINEAR FINITE DIFFERENCE SCHMEME FOR THE GENERALIZED DISSIPATIVE SRLW EQUATION WITH DAMPING

by

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In this paper, we study and analyze a three-level linear finite difference scheme for the initial boundary value problem of the Symmetric Regularized Long Wave (SRLW) equation with damping. The proposed scheme has the second accuracy both for the spatial and temporal discretization. The convergence and stability of the numerical solutions are proved by the mathematical induction and the discrete functional analysis. Numerical results are given to verify the accuracy and the efficiency of proposed algorithm.

Key words: dissipative, damping, generalized SRLW equation, linear difference scheme, stability, convergence

Introduction

As a description of the symmetry for the Regularized Long Wave (RLW) equation, the Symmetric Regularized Long Wave (SRLW) equation is often used for investigating ion-acoustic waves in plasma, propagations of ion-acoustic waves, solitary waves with bidirectional propagation. This plays a very important role in many physical applications [1, 2]. In recent decades, SRLW equation has drawn more and more attention. For example, the hyperbolic secant squared solitary waves, four conservative laws and some numerical results are obtained [3]. Guo [4] studied the existence, uniqueness and regularity of numerical solutions for the periodic initial value problem of the generalized SRLW equation by spectral method. Duan et al. presented some Solitary Wave Solutions in [5] and some numerical simulations for the interaction of solitary waves are shown [6].

In fact, in many actual problems, some non-negligible issues such as the air resistance, the force of friction and the gravity are must be considered. Therefore, the dissipative SRLW equation is of great theoretical value and practical significance. In this paper, we consider the following initial boundary value problem of the dissipative Symmetric Regularized Long Wave (SRLW) equation with damping,

$$u_t + \rho_x - \upsilon u_{xx} + \frac{1}{p}(u^p)_x - u_{xxt} = 0,\quad (x,t) \in (x_L, x_R) \times (0,T] \tag{1}$$

$$\rho_t + u_x + \gamma\rho = 0,\quad (x,t) \in (x_L, x_R) \times (0,T] \tag{2}$$

$$u(x,0) = u_0(x),\ \rho(x,0) = \rho_0(x),\ x \in [x_L, x_R] \tag{3}$$

$$u(x_L,t) = u(x_R,t) = 0,\quad \rho(x_L,t) = \rho(x_R,t) = 0,\quad t \in [0,T] \tag{4}$$

where $\upsilon > 0$ is the dissipative coefficient, $\gamma > 0$ is the damping coefficient and, both $u_0(x)$ and $\rho_0(x)$ are smooth functions.

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If the dissipation is considered, equations (1)-(2) is a reasonable model to describe the nature of the phenomenon of the movement in nonlinear ion acoustic waves. Guo and Shang [8,9] discussed the well-posedness, the global existence and uniqueness and the long time behavior of the solutions for the periodic boundary value problem and the initial boundary value problem. Since it is highly difficult to find the analytic solution of the SRLW equation, various numerical methods were proposed and also applied in other physical models. Numerical solutions of the dissipative SRLW equation with damping have been discussed by employing various form of the finite difference method [10], finite element method [11], and the finite volume method [12,13]. As far as the finite difference schemes are concerned, a two-level fully implicit difference scheme [14] and a nonlinear conservative difference scheme [15] are shown in detail. Also, a three-level linear implicit scheme is proposed in [16, 17].

The remainder of this paper is arranged as follows. In Section 2, a three-level linear finite difference scheme for the dissipative SRLW equation with damping is presented. The convergence and the stability of the numerical solutions are proved in Section 3, and numerical experiments to verify the theoretical analysis are shown in Section 4.

The finite difference scheme

Firstly, for the domain $[x_L, x_R]\times[0,T]$, let $h=(x_R-x_L)/J$ be the step size for the spatial grid, and τ be the step size for the temporal direction such that $x_j = x_L + jh(0\le j\le J)$ $,t_n = n\tau\,(n=0,1,2,\cdots,N, N=[T/\tau])$.

Denote: $u_j^n = u(x_j,t_n)$, $U_j^n \approx u(x_j,t_n)$, $Z_h^0=\{U=(U_j)\,|\,U_0=U_J=0, j=0,1,\cdots,J-1,J\}$, and

$$(U_j^n)_x=\frac{U_{j+1}^n-U_j^n}{h},\quad (U_j^n)_{\bar{x}}=\frac{U_j^n-U_{j-1}^n}{h},\quad (U_j^n)_{\hat{x}}=\frac{U_{j+1}^n-U_{j-1}^n}{2h},\quad (U_j^n)_t=\frac{U_j^{n+1}-U_j^n}{\tau},$$

$$U_j^{n+\frac{1}{2}}=\frac{U_j^{n+1}+U_j^n}{2},\quad \langle U^n,V^n\rangle = h\sum_{j=1}^{J-1}U_j^nV_j^n,\quad \|U^n\|^2=\langle U^n,U^n\rangle,\quad \|U^n\|_\infty=\max_{1\le j\le J-1}|U_j^n|.$$

We propose a three-level linear finite difference scheme for the initial boundary value problem (1)–(4):

$$(U_j^n)_t-(U_j^n)_{x\bar{x}t}+(\phi_j^{n+\frac{1}{2}})_{\hat{x}}-\upsilon(U_j^{n+\frac{1}{2}})_{x\bar{x}}+\left(\frac{3}{2}(U_j^n)^{p-1}-\frac{1}{2}(U_j^{n-1})^{p-1}\right)(U_j^{n+\frac{1}{2}})_{\hat{x}}=0,$$
$$j=1,2,\cdots,J-1;\quad n=1,2,\cdots,N-1 \tag{5}$$

$$(\phi_j^n)_t+(U_j^{n+\frac{1}{2}})_{\hat{x}}+\gamma\phi_j^{n+\frac{1}{2}}=0,\quad j=1,2,\cdots,J-1;\, n=1,2,\cdots,N-1 \tag{6}$$

$$U_j^0=u_0(x_j),\,\phi_j^0=\rho_0(x_j),\quad j=0,1,2,\cdots,J \tag{7}$$

$$U^n\in Z_h^0,\,\phi^n\in Z_h^0,\quad n=0,1,2,\cdots,N \tag{8}$$

Let $u(x,t)$ and $\rho(x,t)$ be the solution of problem (1)-(4), $u_j^n=u(x_j,t_n)$, $\rho_j^n=\rho(x_j,t_n)$, then the truncation error of (1)–(4) is:

$$r_j^n=(u_j^n)_t-(u_j^n)_{x\bar{x}t}+(\rho_j^{n+\frac{1}{2}})_{\hat{x}}-\upsilon(u_j^{n+\frac{1}{2}})_{x\bar{x}}+[\frac{3}{2}(u_j^n)^{p-1}-\frac{1}{2}(u_j^{n-1})^{p-1}](u_j^{n+\frac{1}{2}})_{\hat{x}} \tag{9}$$

$$s_j^n=(\rho_j^n)_t+(u_j^{n+\frac{1}{2}})_{\hat{x}}+\gamma\rho_j^{n+\frac{1}{2}} \tag{10}$$

$$u_j^0=u_0(x_j),\quad \rho_j^0=\rho_0(x_j),\quad j=0,1,2,\cdots,J \tag{11}$$

$$u^n\in Z_h^0,\,\rho^n\in Z_h^0,\quad n=0,1,2,\cdots,N \tag{12}$$

Making use of Taylor expansion, we know that

$$|r_j^n| + |s_j^n| = O(\tau^2 + h^2) \tag{13}$$

Holds if $h, \tau \to 0$.

Convergence and stability

Lemma 1.（Discrete Sobolev inequality[18]）Suppose that $U = \{U_j \mid j = 0,1,2,\cdots, J\}$ are any mesh functions defined on the finite interval $[x_L, x_R]$. Then, the inequality

$$\|U\|_\infty \le C_0\sqrt{\|U\|}\sqrt{\|U_x\| + \|U\|},$$

holds,where C_0 is independent of $U = \{U_j \mid j = 0,1,2,\cdots, J\}$ and h.

Lemma 2.（Discrete Gronwall inequality[18]）Assume $\{w^n \mid n = 0,1,2,\cdots, N; N\tau \le T\}$ is non-negative sequence and satisfies

$$w^n \le A + \tau\sum_{l=1}^{n} B_l w^l,$$

where A and B_l ($l = 0,1,2,\cdots, N$) are non-negative constants. Then w satisfies

$$\max_{1\le n\le N} | w^n | \le A\exp(2\tau\sum_{l=1}^{N} B_l),$$

for $0 \le n \le N$, where τ is sufficiently small satisfying $\tau \cdot (\max_{1\le n\le N} B_l) \le 1/2$.

Lemma 3. [10] Suppose that $u_0 \in H^1$ and $\rho_0 \in L_2$, then the solutions of the initial boundary value problem (1)-(4) satisfy

$$\|u\|_{L_2} \le C,\ \ \|u_x\|_{L_2} \le C,\ \ \|\rho\|_{L_2} \le C,\ \ \|u\|_{L_\infty} \le C.$$

Theorem 1. Suppose $u_0 \in H^1$, $\rho_0 \in L_2$. For sufficiently small temporal step τ and spatial step h, the solutions of the scheme (5)-(8) converge to the solution of the initial boundary value problem (1)-(4) with the convergence order of $O(\tau^2 + h^2)$ by the norm $\|\cdot\|_\infty$ for U^n and by the norm $\|\cdot\|_{L^2}$ for ϕ^n.

Proof. Subtracting (9)-(12) from (5)-(8) and letting $e_j^n = u_j^n - U_j^n$, we have

$$r_j^n = (e_j^n)_t - (e_j^n)_{x\bar{x}t} + (\eta_j^{n+\frac{1}{2}})_{\hat{x}} - \upsilon(e_j^{n+\frac{1}{2}})_{x\bar{x}} + R_j \tag{14}$$

$$s_j^n = (\eta_j^n)_t + (e_j^{n+\frac{1}{2}})_{\hat{x}} + \gamma\eta_j^{n+\frac{1}{2}} \tag{15}$$

$$e_j^0 = 0,\ \ \eta_j^0 = 0\ \ j = 0,1,2,\cdots, J \tag{16}$$

$$e^n \in Z_h^0,\ \ \eta^n \in Z_h^0, n = 0,1,2,\cdots, N \tag{17}$$

where

$$R_j = [\frac{3}{2}(u_j^n)^{p-1} - \frac{1}{2}(u_j^{n-1})^{p-1}](u_j^{n+\frac{1}{2}})_{\hat{x}} - [\frac{3}{2}(U_j^n)^{p-1} - \frac{1}{2}(U_j^{n-1})^{p-1}](U_j^{n+\frac{1}{2}})_{\hat{x}}.$$

Next, we use the mathematical induction to prove the error estimates. From Lemma 3 and (13), there exist constants C_u, C_r and C_s, which are independent of τ and h,satisfy that

$$\|u^n\|_\infty \le C_u,\ \ \|r^n\|_\infty \le C_r(\tau^2 + h^2),\ \ \|s^n\|_\infty \le C_s(\tau^2 + h^2),\ \ n = 1,2,\cdots, N \tag{18}$$

It follows from the initial conditions that the following estimates

$$\|e^0\| = 0,\ \ \|\eta^0\| = 0,\ \ \|U^0\|_\infty \le C_u \tag{19}$$

On the other hand, we must compute U^1 and ϕ^1 using other methods, for instance, the finite difference scheme in [16] and have the estimate,

$$\left\|e^1\right\|+\left\|e_x^1\right\|+\left\|\eta^1\right\|\le C_1(\tau^2+h^2) \tag{20}$$

where C_1 is also independent of τ and h.

Suppose that

$$\left\|e^l\right\|+\left\|e_x^l\right\|+\left\|\eta^l\right\|\le C_l(\tau^2+h^2)\,,\quad l=2,3,\cdots,n\,,\,(n\le N-1) \tag{21}$$

where C_l ($l=2,3,\cdots,n$) are also independent of τ and h. By Lemma 1 and the Cauchy-Schwarz inequality,we get

$$\left\|e^l\right\|_\infty\le C_0\sqrt{\left\|e^l\right\|}\sqrt{\left\|e_x^l\right\|+\left\|e^l\right\|}\le\frac{1}{2}C_0\left(2\left\|e^l\right\|+\left\|e_x^l\right\|\right)\le\frac{3}{2}C_0C_l(\tau^2+h^2)\,,\quad l=1,2,\cdots,n \tag{22}$$

$$\left\|U^l\right\|_\infty\le\left\|u^l\right\|_\infty+\left\|e^l\right\|_\infty\le C_u+\frac{3}{2}C_0C_l(\tau^2+h^2)\,,\quad l=2,3,\cdots,n \tag{23}$$

Taking the inner product of (14) with $e^{n+\frac{1}{2}}$,and using the summation by part[18], we get

$$\begin{aligned}
&\frac{1}{2}\left\|e^n\right\|_t^2+\frac{1}{2}\left\|e_x^n\right\|_t^2+\langle\eta_{\hat{x}}^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\rangle\\
&=-\upsilon\left\|e_x^{n+\frac{1}{2}}\right\|^2+\langle r^n,e^{n+\frac{1}{2}}\rangle-\langle R,e^{n+\frac{1}{2}}\rangle\\
&=-\upsilon\left\|e_x^{n+\frac{1}{2}}\right\|^2+\langle r^n,e^{n+\frac{1}{2}}\rangle-h\sum_{j=1}^{J-1}[\frac{3}{2}(U_j^n)^{p-1}-\frac{1}{2}(U_j^{n-1})^{p-1}](e_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}\\
&\qquad-\frac{3}{2}h\sum_{j=1}^{J-1}e_j^n\sum_{k=0}^{p-2}(u_j^n)^{p-2-k}(U_j^n)^k(u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}\\
&\qquad+\frac{1}{2}h\sum_{j=1}^{J-1}e_j^n\sum_{k=0}^{p-2}(u_j^{n-1})^{p-2-k}(U_j^{n-1})^k(u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}
\end{aligned} \tag{24}$$

According to Lemma 3 and the mean value theorem, the following result

$$(u_j^{n+\frac{1}{2}})_{\hat{x}}=\frac{u_{j+1}^{n+\frac{1}{2}}-u_{j-1}^{n+\frac{1}{2}}}{2h}=\frac{u(x_{j+1},\frac{t_n+t_{n+1}}{2})-u(x_{j-1},\frac{t_n+t_{n+1}}{2})}{2h}=\frac{\partial}{\partial x}u(x_{\xi_j},\frac{t_n+t_{n+1}}{2})\,,(x_{j-1}\le x_{\xi_j}\le x_{j+1})$$

holds, that is,

$$\left\|u_{\hat{x}}^{n+\frac{1}{2}}\right\|_\infty\le C_u \tag{25}$$

If h and τ are sufficiently small and satisfy that,

$$\frac{3}{2}C_0\cdot(\max_{0\le l\le n}C_l)(\tau^2+h^2)\le 1 \tag{26}$$

then, it follows from (23), (25), (26) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
-h\sum_{j=1}^{J-1}[\frac{3}{2}(U_j^n)^{p-1}-\frac{1}{2}(U_j^{n-1})^{p-1}](e_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}&\le\max_{1\le j\le J-1}[\frac{3}{2}\left|U_j^n\right|^{p-1}+\frac{1}{2}\left|U_j^{n-1}\right|^{p-1}]\cdot h\sum_{j=1}^{J-1}(e_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}\\
&\le[\frac{3}{2}\left\|U_j^n\right\|_\infty^{p-1}+\frac{1}{2}\left\|U_j^{n-1}\right\|_\infty^{p-1}]\cdot h\sum_{j=1}^{J-1}(|(e_j^{n+\frac{1}{2}})_{\hat{x}}|\cdot|e_j^{n+\frac{1}{2}}|)
\end{aligned}$$

$$\le 2[C_u+\frac{3}{2}C_0\max(C_n,C_{n-1})(\tau^2+h^2)]^{p-1}h\sum_{j=1}^{J-1}|(e_j^{n+\frac{1}{2}})_{\hat{x}}|\cdot|e_j^{n+\frac{1}{2}}|$$

$$\le \frac{1}{2}(C_u+1)^{p-1}(\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|e^{n+1}\|^2+\|e^n\|^2) \tag{27}$$

$$-\frac{3}{2}h\sum_{j=1}^{J-1}e_j^n\sum_{k=0}^{p-2}(u_j^n)^{p-2-k}(U_j^n)^k(u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}$$

$$\le \frac{3}{2}\sum_{k=0}^{p-2}\left[(C_u)^{p-2-k}(C_u+\frac{3}{2}C_0C_n(\tau^2+h^2))^kC_u\right]\cdot h\sum_{j=1}^{J-1}e_j^n\cdot e_j^{n+\frac{1}{2}}$$

$$\le \frac{3}{2}\sum_{k=0}^{p-2}\left[(C_u)^{p-2-k}(C_u+1)^{k+1}\right]\cdot h\sum_{j=1}^{J-1}e_j^n\cdot e_j^{n+\frac{1}{2}}$$

$$\le \frac{3}{2}(p-1)(C_u+1)^{p-1}\cdot h\sum_{j=1}^{J-1}|e_j^n|\cdot\left|e_j^{n+\frac{1}{2}}\right|$$

$$\le \frac{3}{8}(p-1)(C_u+1)^{p-1}\left[\|e^{n+1}\|^2+3\|e^n\|^2\right] \tag{28}$$

$$\frac{1}{2}h\sum_{j=1}^{J-1}e_j^n\sum_{k=0}^{p-2}(u_j^{n-1})^{p-2-k}(U_j^{n-1})^k(u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}$$

$$\le \frac{1}{2}\sum_{k=0}^{p-2}\left[(C_u)^{p-2-k}\left(C_u+\frac{3}{2}C_0C_{n-1}(\tau^2+h^2)\right)^kC_u\right]\cdot h\sum_{j=1}^{J-1}e_j^n\cdot e_j^{n+\frac{1}{2}}$$

$$\le \frac{1}{2}\sum_{k=0}^{p-2}\left[(C_u)^{p-2-k}(C_u+1)^{k+1}\right]\cdot h\sum_{j=1}^{J-1}e_j^n\cdot e_j^{n+\frac{1}{2}}$$

$$\le \frac{1}{8}(p-1)(C_u+1)^{p-1}\left[\|e^{n+1}\|^2+3\|e^n\|^2\right] \tag{29}$$

and

$$\langle r^n,e^{n+\frac{1}{2}}\rangle=\frac{1}{2}\langle r^n,e^{n+1}+e^n\rangle\le\frac{1}{2}\|r^n\|^2+\frac{1}{4}[\|e^{n+1}\|^2+\|e^n\|^2] \tag{30}$$

Substituting (27)-(30) into (24), we get

$$(\|e^{n+1}\|^2-\|e^n\|^2)+(\|e_x^{n+1}\|-\|e_x^n\|^2)+2\tau\left\langle\eta_{\hat{x}}^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right\rangle$$

$$\le\tau\|r^n\|^2+5\tau(p-1)(C_u+1)^{p-1}\left[\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|e^{n+1}\|^2+\|e^n\|^2\right] \tag{31}$$

Similarly, taking the inner product of (15) with $\eta^{n+\frac{1}{2}}$, we derive that

$$(\|\eta^{n+1}\|^2-\|\eta^n\|^2)+2\tau\langle e_{\hat{x}}^{n+\frac{1}{2}},\eta^{n+\frac{1}{2}}\rangle=-2\tau\gamma\left\|\eta^{n+\frac{1}{2}}\right\|+2\tau\left\langle s^n,\eta^{n+\frac{1}{2}}\right\rangle$$

$$\le\tau\|s^n\|^2+\frac{\tau}{2}[\|\eta^{n+1}\|^2+\|\eta^n\|^2] \tag{32}$$

Add (31) to (32) and note that

$$\langle\eta_{\hat{x}}^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\rangle=-\langle e_{\hat{x}}^{n+\frac{1}{2}},\eta^{n+\frac{1}{2}}\rangle.$$

We have

$$(\|e^{n+1}\|^2-\|e^n\|^2)+(\|e_x^{n+1}\|^2-\|e_x^n\|^2)+(\|\eta^{n+1}\|^2-\|\eta^n\|^2)$$

$$\le\tau\|r^n\|^2+\tau\|s^n\|^2+\tau\left[5(p-1)(C_u+1)^{p-1}\right]\left[\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2+\|\eta^{n+1}\|^2+\|\eta^n\|^2\right] \tag{33}$$

Summing up (33) from 1 to n ,we get

$$\left\|e^{n+1}\right\|^2+\left\|e_x^{n+1}\right\|^2+\left\|\eta^{n+1}\right\|^2\le\left\|e^1\right\|^2+\left\|e_x^1\right\|^2+\left\|\eta^1\right\|^2+\tau\sum_{k=1}^{n}\left\|r^k\right\|^2+\tau\sum_{k=1}^{n}\left\|s^k\right\|^2$$
$$+\tau\sum_{k=0}^{n+1}[10(p-1)(C_u+1)^{p-1}]\left(\left\|e^k\right\|^2+\left\|e_x^k\right\|^2+\left\|\eta^k\right\|^2\right) \tag{34}$$

and

$$\tau\sum_{k=1}^{n}\left\|r^k\right\|^2\le n\tau\max_{1\le k\le n}\left\|r^k\right\|^2\le T(C_r)^2(\tau^2+h^2)^2 \tag{35}$$

$$\tau\sum_{k=1}^{n}\left\|s^k\right\|^2\le n\tau\max_{1\le k\le n}\left\|s^k\right\|^2\le T(C_s)^2(\tau^2+h^2)^2 \tag{36}$$

Then, the substitution of (20), (35) and (36) into (34) leads to

$$\left\|e^{n+1}\right\|^2+\left\|e_x^{n+1}\right\|^2+\left\|\eta^{n+1}\right\|^2$$
$$\le\left\{T[(C_r)^2+(C_s)^2]+C_1^2\right\}(\tau^2+h^2)^2+\tau\sum_{k=0}^{n+1}[10\left(p-1\right)\left(C_u+1\right)^{p-1}][\left\|e^k\right\|^2+\left\|e_x^k\right\|^2+\left\|\eta^k\right\|^2].$$

By Lemma 2, if τ is sufficiently small satisfy that

$$\tau<\frac{1}{20(p-1)(C_u+1)^{p-1}},$$

the following result

$$\left\|e^{n+1}\right\|^2+\left\|e_x^{n+1}\right\|^2+\left\|\eta^{n+1}\right\|^2\le\left\{T\left[(C_r)^2+(C_s)^2\right]+C_1^2\right\}(\tau^2+h^2)^2\,\mathrm{e}^{2T[10(p-1)(C_u+1)^{p-1}]}$$
$$\le(C_{n+1})^2(\tau^2+h^2)^2,\ (n=1,2,\cdots,N-1),$$

holds, where

$$C_{n+1}=[\sqrt{T}(C_r+C_s)+C_1]\mathrm{e}^{10T(p-1)(C_u+1)^{p-1}}.$$

Obviously, C_{n+1} is a constant which is independent of n. Therefore, by the mathematical induction, we get

$$\left\|e^n\right\|\le O(\tau^2+h^2),\ \left\|e_x^n\right\|\le O(\tau^2+h^2),\ \left\|\eta^n\right\|\le O(\tau^2+h^2),\ (n=1,2,\cdots,N).$$

Finally, It follows from Lemma 1 that

$$\left\|e^n\right\|_\infty\le O(\tau^2+h^2),\quad (n=1,2,\cdots,N).$$

This completes the proof.

By the same fashion, we can prove the following results and the detail is skipped for the brevity:

Theorem 2. Under the conditions defined in Theorem 1, the solutions of the scheme (5)-(8) are unconditionally stable by the norm $\|\cdot\|_\infty$ for U^n and by the norm $\|\cdot\|_{L^2}$ for ϕ^n.

Numerical experiments

When $t=0$, the damping does not effect and the dissipative term will not appear, so in the following experiments, the initial function of the problem (1)-(4) can be set in the following[18, 19]:

$$u_0(x)=\left[\frac{5p(p+1)}{12}\right]^{1/(p-1)}\mathrm{sech}^{2/(p-1)}\frac{p-1}{6}\sqrt{5}x,$$

$$\rho_0(x)=\frac{2}{3}\left[\frac{5p(p+1)}{12}\right]^{1/(p-1)}\mathrm{sech}^{2/(p-1)}\frac{p-1}{6}\sqrt{5}x,$$

which come from the generalized SRLW equation

$$u_t + \rho_x + \frac{1}{p}(u^p)_x - u_{xxt} = 0 , \quad \rho_t + u_x = 0 \qquad (37)$$

We choose three different group of parameters to compute the error estimate by the norm $\|\cdot\|_\infty$ and $\|\cdot\|$, respectively. We take $-x_L = x_R = 20$ and $T = 2.0$. Since the exact solutions of (1)-(2) are unknown, we set the numerical solutions on the mesh $\tau = h = 1/160$ as the reference solution. For some different τ and h, the error comparison between the numerical solution and the solitary wave solution at different parameters is shown in Tab. 1.

Tab. 1 Error estimates of numerical solution with $p = 3$ and $\upsilon = \gamma = 1.0$

		$\|\cdot\|_\infty$			$\|\cdot\|$		
		$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$
u	$t = 0.4$	6.6257e-4	1.6494e-4	3.9568e-5	8.7534e-4	2.1896e-4	5.1774e-5
	$t = 0.8$	1.0342e-3	2.5731e-4	6.0590e-5	1.3412e-3	3.4933e-4	7.9610e-5
	$t = 1.2$	1.1678e-3	2.8857e-4	6.8711e-5	1.5475e-3	3.8241e-4	9.1057e-5
	$t = 1.6$	1.1717e-3	2.8978e-4	6.9022e-5	1.5890e-3	3.9267e-4	9.3500e-5
	$t = 2.0$	1.1032e-3	2.7287e-4	6.4982e-5	1.5333e-3	3.7889e-4	9.0217e-5
ρ	$t = 0.4$	6.5138e-4	1.5532e-4	3.7829e-5	8.1300e-4	2.0953e-4	4.8414e-5
	$t = 0.8$	9.1754e-4	2.2675e-4	5.4504e-5	1.1055e-3	2.8065e-4	6.8709e-5
	$t = 1.2$	9.7255e-4	2.4053e-4	5.7285e-5	1.1550e-3	2.8561e-4	6.8016e-5
	$t = 1.6$	9.3766e-4	2.3200e-4	5.5281e-5	1.1394e-3	2.8178e-4	6.7107e-5
	$t = 2.0$	8.8385e-4	2.1894e-4	5.2164e-5	1.1036e-3	2.7296e-4	6.5008e-5

From these results, the stability and convergence of the scheme are verified. And it shows that the proposed algorithm is effective and reliable.

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