APPLICATION OF THE KUDRYASOV METHOD WITH CHARACTERISTIC SET ALGORITHM TO SOLVE SOME PARTIAL DIFFERENTIAL EQUATIONS IN FLUID MECHANICS

by

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In this paper, we pay attention to the analytical method named, the Kudryashov method combined with characteristic set algorithm for finding the exact travelling solutions of two nonlinear partial differential equations in fluid mechanics,which named surface wave equation and the generalized Kuramoto-Sivashinsky equation. The solution procedure of the Kudryashov method can be reduced to solve a large system of algebraic equations, which is hard to solve, then we use characteristic set algorithm to solve this problem. The obtained results show that the Kudryashov method combined with characteristic set algorithm is effective.

Key words: Kudryashov method, characteristic set algorithm, surface wave equation, generalized Kuramoto-Sivashinsky equation

Introduction

The partial differential equations (PDEs) arising in many physical fields like the condense matter physics, fluid mechanics, plasma physics and optics, etc. The investigation of the exact solutions plays an important role in the study of physical systems, and finding exact solutions of the PDEs is one of the central themes in mathematics and physics. In the past decades, a wealth of methods have been developed to obtain exact solutions of PDEs. Some of the most important methods are the homotopy perturbation method [1], variational iteration method[2], Riccati differential equation method [3], and other methods[4-9].

The objective of this article is to look for new study for relating to the Kudryashov method to explore exact travelling wave solution for the surface wave equation and the generalized Kuramoto- Sivashinsky equation. The solution procedure of the Kudryashov method can be reduced to solve a large system of algebraic equations, which is hard to solve, then we use the characteristic set algorithm to solve this problem. This application displays the simplicity, efficiency and effectiveness of the Kudryashov method with characteristic set algorithm [10]. To the best of our knowledge that the Kudryashov

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method has not been applied to the above mentioned equation in previous literature.

The Kudryashov method

Let us introduce the Kudryashov method as follows [10].

Consider the nonlinear partial differential equation in the following form:

$$Q(u, u_x, u_t, u_{xx}, u_{xt} \cdots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, Q is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved and the subscripts stands for the partial derivatives.

Step-1: We familiarize the travelling wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - ct, \tag{2}$$

where c is the speed of travelling wave, the travelling wave transformation eq.(2) transform eq.(1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$\Theta(u, u', -cu'', \cdots) = 0, \tag{3}$$

where Θ is a polynomial of u and its derivatives and the superscripts specify the ordinary derivatives with respect to ξ.

Step-2: We look for exact solution of eq. (3) in the form

$$u(\xi) = \sum_{i=0}^{N} a_i Q(\xi)^i,, \tag{4}$$

where $a_i (0 \le i \le N)$ are constants to be determined, such that $a_N \ne 0$, while $Q(\xi)$ has the form

$$Q(\xi) = \frac{1}{1 + \rho \exp(\xi)}, \tag{5}$$

a solution to the Riccati equation

$$Q'(\xi) = Q(\xi)^2 - Q(\xi), \tag{6}$$

where ρ is arbitrary constant.

Step-3: By balancing the highest order derivative terms with the nonlinear terms of the highest order come out in eq.(4), we can evaluated the value of the positive integer N.

Step-4: By substituting eq.(4) along with eq.(6) into eq.(3) and equating all the coefficients of same power of $Q(\xi)$ to zero, we obtained a system of algebraic equations. The obtaining system can be solved to find the value of c, $a_i (0 \le i \le N)$ substituting these terms into eq.(4) along with (5), the determination of solutions of eq.(1) will be completed.

Characteristic set algorithm

Let us give the characteristic set algorithm as follows [11].

A characteristic set CS of a polynomial system PS will be determined according to the following algorithm.

Input: A polynomial system PS.

Output: A characteristic set CS of PS.

Step 1: Set $PS_0 \leftarrow PS$;

Step 2: Take a basic set BS of PS_0;

Step 3: Form $RS = \mathrm{Re}\,md\mathrm{r}(PS_0 \setminus BS / BS) \setminus \{0\}$;

Step 4: If $RS = \phi$ then $CS \leftarrow BS$ and return. Otherwise $PS_0 = PS + BS + RS$ and go to **Step 2**.

The Step 2 will be achieved by the algorithm below:

Input: A polynomial system PS.

Output: A basic set BS of PS.

Step 1: Set $PS' = PS$ and $BS = \phi$;

Step 2: If $PS' = \phi$ then return BS. Otherwise take from PS' a polynomial B of least class and least degree and set $BS \leftarrow BS + \{B\}$;

Step 3: Set $PS' \leftarrow$ the set of polynomials in PS' which are reduced w.r.t. B;

Step 4: Go to **Step 2**.

(Well-Ordering Principle) Let CS be a characteristic set of a polynomials system PS. Then:

$$\mathrm{Zero}(PS / IP) = \mathrm{Zero}(CS / IP),$$
$$Zero(CS / IP) \subset Zero(PS) \subset Zero(CS),$$
$$Zero(PS) = Zero(CS / IP) \cup_i Zero(PS + \{I_i\}),$$

where I_i are initials in CS, and IP is the initial-product of CS.

Exact solutions of a surface wave equation in convecting fluid

Consider the following surface wave equation [12]:

$$u_t + a_0 u_x + a_1 u u_x + a_2 u_{xxx} + b_0 u_{xx} + b_1 (u u_x)_x + b_2 u_{xxxx} = 0, , \tag{7}$$

which describes oscillatory Rayleigh-Marangoni instability in a liquid layer with free boundary. Let's assume the traveling wave solution of eq.(7) in the form

$$u(x,t)=u(\xi), \qquad \xi = x - ct, \tag{8}$$

where c is a arbitrary constant. Using the wave variable (8), the eq. (7) is carried to

$$a_0 u' - cu' + a_1 u u' + b_0 u'' + b_1 (u'^2 + u u'') + a_2 u^{(3)} + b_2 u^{(4)} = 0, , \tag{9}$$

integrating eq.(9) once with respect to ξ and setting the integration constant as zero,we get:

$$(a_0 - c)u + \frac{a_1}{2} u^2 + (b_0 + b_1 u) u' + a_2 u'' + b_2 u^{(3)} = 0, \tag{10}$$

suppose that the solution of ODE (10) can be expressed:

$$u(\xi) = \sum_{i=0}^{N} c_i Q(\xi)^i, \tag{11}$$

where $c_i (0 \le i \le N)$ are constants to be determined, such that $c_N \ne 0$.

Consider the homogeneous balance between the highest order derivative $u^{(3)}$ and nonlinear term uu' appearing in (10), we have $N = 2$,we then suppose that eq.(10) has the following solutions:

$$u(\xi) = c_0 + c_1 Q(\xi) + c_2 Q(\xi)^2, \qquad c_2 \neq 0, \tag{12}$$

substituting eq.(12) along with eq.(6) into eq.(10) and collecting all the terms with the same power of $Q(\xi)$ together, equating each coefficient to zero, yields a set of algebraic equations, which is large and difficult to solve, with the aid of the characteristic set algorithm, we can distinguish the different cases namely:

Case (1)

$$c_2 = -\frac{12a_2}{a_1}, c_1 = \frac{12a_2}{a_1}, c_0 = 0, b_0 = b_1 = b_2 = 0, c = a_0 + a_2.$$

Case (2)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = \frac{24b_2}{b_1}, c_0 = 0, a_1 = -25b_1, a_2 = -30b_2, b_0 = 119b_2, c = a_0 - 150b_2.$$

Case (3)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = \frac{24b_2}{b_1}, c_0 = -\frac{12b_2}{b_1}, a_1 = -13b_1, a_2 = -18b_2, b_0 = 71b_2, c = a_0 + 78b_2.$$

Case (4)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = \frac{12b_2}{b_1}, c_0 = 0, a_1 = \frac{a_2 b_1}{b_2}, b_0 = -b_2, c = a_0 + a_2.$$

Case (5)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = \frac{12b_2}{b_1}, c_0 = 0, a_1 = -6b_1, a_2 = -6b_2, b_0 = -b_2, c = a_0 - 6b_2.$$

Case (6)

$$c_2 = \frac{12b_0}{5a_1}, c_1 = -\frac{24b_0}{5a_1}, c_0 = 0, a_2 = -\frac{b_0}{5}, b_1 = b_2 = 0, c = a_0 - \frac{6}{5}b_0.$$

Case (7)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = \frac{24b_2}{b_1}, c_0 = -\frac{12b_2}{b_1}, a_2 = -\frac{b_0 + 19b_2}{5}, a_1 = \frac{-b_0 b_1 + 6b_1 b_2}{5b_2}, c = a_0 + \frac{6b_0 - 36b_2}{5}.$$

Case (8)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = \frac{12b_2}{b_1}, c_0 = -\frac{b_0 + b_2}{b_1}, a_2 = a_1 = 0, c = a_0.$$

Case (9)

$$c_2 = \frac{12b_0}{5a_1}, c_1 = -\frac{24b_0}{5a_1}, c_0 = \frac{12b_0}{5a_1}, a_2 = -\frac{b_0}{5}, b_1 = b_2 = 0, c = a_0 + \frac{6}{5}b_0.$$

Case (10)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = c_0 = 0, a_2 = \frac{b_0 + 19b_2}{5}, a_1 = \frac{b_0 b_1 - 6b_1 b_2}{5b_2}, c = a_0 - \frac{6b_0 - 36b_2}{5}.$$

Case (11)

$$c_2=-\frac{12b_2}{b_1}, c_1=-\frac{2(a_2b_0-29a_2b_2+24b_0b_2-156b_2^2)}{b_1(5a_2+b_0+31b_2)}, c_0=0,$$

$$a_1=\frac{a_2b_0b_1+a_2b_1b_2+30b_0b_1b_2+30b_1b_2^2}{6b_2(5a_2+b_0+31b_2)}, b_0=\frac{a_2^2+6a_2b_2-6b_2^2}{6b_2}, c=a_0+a_2-b_0-b_2.$$

Case (12)

$$c_2=-\frac{12b_2}{b_1}, c_1=\frac{24b_2}{b_1}, c_0=0, a_2=-6b_2, a_1=-b_1, b_0=-b_2, c=a_0-6b_2.$$

Case (13)

$$c_2=-\frac{12b_2}{b_1}, c_1=\frac{22b_2}{b_1}, c_0=-\frac{10b_2}{b_1}, a_2=-5b_2, a_1=-\frac{5}{6}b_1, b_0=\frac{49}{6}b_2, c=a_0+\frac{25}{6}b_2.$$

Case (14)

$$c_2=-\frac{12b_2}{b_1}, c_1=\frac{24b_2}{b_1}, c_0=-\frac{b_0+6b_2}{b_1}, a_2=-5b_2, a_1=0, c=a_0.$$

Case (15)

$$c_2=-\frac{12b_0}{5a_1}, c_1=0, c_0=\frac{12b_0}{5a_1}, a_2=\frac{b_0}{5}, b_1=b_2=0, c=a_0+\frac{6}{5}b_0.$$

Case (16)

$$c_2=-\frac{12b_2}{b_1}, c_1=\frac{72b_2}{b_1}, c_0=0, a_2=-30b_2, a_1=-5b_1, b_0=119b_2, c=a_0-150b_2.$$

Case (17)

$$c_2=-\frac{12b_2}{b_1}, c_1=\frac{24b_2}{b_1}, c_0=0, a_2=-\frac{b_0+31b_2}{5}, a_1=\frac{-b_0b_1-6b_1b_2}{5b_2}, c=a_0-\frac{6b_0+36b_2}{5}.$$

Case (18)

$$c_2=-\frac{12b_2}{b_1}, c_1=\frac{12b_2}{b_1}, c_0=-\frac{2b_2}{b_1}, a_2=\frac{a_1b_2}{b_1}, b_2=b_0, c=a_0-a_2.$$

Case (19)

$$c_2=-\frac{12b_2}{b_1}, c_1=\frac{12b_2}{b_1}, c_0=0, a_2=a_1=0, b_2=-b_0, c=a_0.$$

Case (20)

$$c_2=-\frac{12b_0}{5a_1}, c_1=c_0=0, a_2=\frac{b_0}{5}, b_1=b_2=0, c=a_0-\frac{6b_0}{5}.$$

Case (21)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = -\frac{2(29a_2b_0b_2 + 6b_0^2b_2 + 29a_2b_2^2 + 102b_0b_2^2 + 92b_2^3)}{b_1(a_2b_0 + 45a_2b_2 + 29b_0b_2 + 29b_2^2)},$$

$$c_0 = \frac{12(a_2b_0 + 45a_2b_2 + 29b_0b_2 + 29b_2^2)}{b_1(29a_2 + 6b_0 + 96b_2)},$$

$$c = \frac{29a_0a_2 + 6a_0b_0 + 35a_2b_0 + 6b_0^2 + 96a_0b_2 + 299a_2b_2 + 276b_0b_2 + 270b_2^2}{29a_2 + 6b_0 + 96b_2},$$

$$a_1 = \frac{35a_2b_0b_1 + 6b_0^2b_1 + 299a_2b_1b_2 + 276b_0b_1b_2 + 270b_1b_2^2}{6(a_2b_0 + 45a_2b_2 + 29b_0b_2 + 29b_2^2)}, \quad b_0 = \frac{a_2^2 - 6a_2b_2 - 6b_2^2}{6b_2}.$$

Case (22)

$$c_2 = -\frac{12a_2}{a_1}, c_1 = \frac{12a_2}{a_1}, c_0 = -\frac{2a_2}{a_1}, b_0 = b_1 = b_2 = 0, c = a_0 - a_2.$$

Case (23)

$$c_2 = -\frac{12b_2}{b_1}, c_1 = \frac{48b_2}{b_1}, c_0 = -\frac{36b_2}{b_1}, a_1 = -3b_1, a_2 = -18b_2, b_0 = 71b_2, c = a_0 + 54b_2.$$

For the sake of simplicity, we consider only the solution with respect to Case (1), the other solutions can be obtained in a similar way:

$$u(x,t) = \frac{12a_2}{a_1(1 + e^{x-(a_0+a_2)t}\rho)} - \frac{12a_2}{a_1(1 + e^{x-(a_0+a_2)t}\rho)^2},$$

and $b_0 = b_1 = b_2 = 0.$

Exact solutions of the generalized Kuramoto-Sivashinsky equation

Consider the following generalized Kuramoto-Sivashinsky equation [13]:

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{13}$$

let’s assume the traveling wave solution of eq.(13) in the form

$$u(x,t) = u(\xi), \qquad \xi = x - ct,, \tag{14}$$

where c is a arbitrary constant. Using the wave variable (14), the eq. (13) is carried to

$$-cu' + uu' + \alpha u'' + \beta u^{(3)} + \gamma u^{(4)} = 0, \tag{15}$$

integrating eq.(15) once with respect to ξ and setting the integration constant as zero,we get:

$$-cu + \frac{1}{2}u^2 + \alpha u' + \beta u'' + \gamma u^{(3)} = 0, \tag{16}$$

suppose that the solution of ODE (16) can be expressed:

$$u(\xi) = \sum_{i=0}^{N} c_i Q(\xi)^i, \tag{17}$$

where $c_i (0 \le i \le N)$ are constants to be determined, such that $c_N \ne 0$.

Consider the homogeneous balance between the highest order derivative $u^{(3)}$ and nonlinear term u^2 appearing in (16), we have $N = 3$, we then suppose that eq.(16) has the following solutions:

$$u(\xi) = c_0 + c_1 Q(\xi) + c_2 Q(\xi)^2 + c_3 Q(\xi)^3, \qquad c_3 \ne 0, \tag{18}$$

substituting eq.(18) along with eq.(6) into eq.(16) and collecting all the terms with the same power of $Q(\xi)$ together, equating each coefficient to zero, yields a set of algebraic equations, which is large and difficult to solve, with the aid of the characteristic set algorithm, we can distinguish the different cases namely:

Case (1)

$$c_0 = 180\gamma, c_1 = -480\gamma, c_2 = 420\gamma, c_3 = -120\gamma, c = 90\gamma, \alpha = 73\gamma, \beta = -16\gamma.,$$

Case (2)

$$c_0 = -12\gamma, c_1 = 0, c_2 = 120\gamma, c_3 = -120\gamma, c = -6\gamma, \alpha = \gamma, \beta = 4\gamma.,$$

Case (3)

$$c_0 = -60\gamma, c_1 = 0, c_2 = 180\gamma, c_3 = -120\gamma, c = -30\gamma, \alpha = -19\gamma, \beta = 0.,$$

Case (4)

$$c_0 = 0, c_1 = -\frac{720}{11}\gamma, c_2 = 180\gamma, c_3 = -120\gamma, c = -\frac{30}{11}\gamma, \alpha = \frac{19}{11}\gamma, \beta = 0.,$$

Case (5)

$$c_0 = 180\gamma, c_1 = 0, c_2 = -60\gamma, c_3 = -120\gamma, c = 90\gamma, \alpha = 73\gamma, \beta = 16\gamma.$$

Case (6)

$$c_0 = 0, c_1 = -120\gamma, c_2 = 240\gamma, c_3 = -120\gamma, c = -6\gamma, \alpha = \gamma, \beta = -4\gamma.$$

Case (7)

$$c_0 = c_1 = c_2 = 0, c_3 = -120\gamma, c = -60\gamma, \alpha = 47\gamma, \beta = 12\gamma.$$

Case (8)

$$c_0 = c_1 = 0, c_2 = 180\gamma, c_3 = -120\gamma, c = 30\gamma, \alpha = -19\gamma, \beta = 0.$$

Case (9)

$$c_0 = c_1 = 0, c_2 = 120\gamma, c_3 = -120\gamma, c = 6\gamma, \alpha = \gamma, \beta = -4\gamma.$$

Case (10)

$$c_0 = c_1 = 0, c_2 = -60\gamma, c_3 = -120\gamma, c = -90\gamma, \alpha = 73\gamma, \beta = 16\gamma.$$

Case (11)

$$c_0 = 12\gamma, c_1 = -120\gamma, c_2 = 240\gamma, c_3 = -120\gamma, c = 6\gamma, \alpha = \gamma, \beta = -4\gamma.$$

Case (12)

$$c_0 = 120\gamma, c_1 = -360\gamma, c_2 = 360\gamma, c_3 = -120\gamma, c = 60\gamma, \alpha = 47\gamma, \beta = -12\gamma.$$

Case (13)

$$c_0 = 120\gamma, c_1 = c_2 = 0, c_3 = -120\gamma, c = 60\gamma, \alpha = 47\gamma, \beta = 12\gamma.$$

Case (14)

$$c_0 = 0, c_1 = -480\gamma, c_2 = 420\gamma, c_3 = -120\gamma, c = -90\gamma, \alpha = 73\gamma, \beta = -16\gamma.$$

Case (15)

$$c_0 = -\frac{8(11\beta\gamma - 28\gamma^2)}{7\beta + 44\gamma}, c_1 = \frac{240(\beta\gamma - 18\gamma^2)}{7\beta + 44\gamma}, c_2 = -15(\beta - 12\gamma),$$

$$c_3 = -120\gamma, c = -\frac{4(11\beta\gamma - 28\gamma^2)}{7\beta + 44\gamma}, \alpha = -\gamma, \beta^2 + 16\gamma^2 = 0.$$

Case (16)

$$c_0 = 0, c_1 = -360\gamma, c_2 = 360\gamma, c_3 = -120\gamma, c = -60\gamma, \alpha = 47\gamma, \beta = -12\gamma.$$

Case (17)

$$c_0 = 0, c_1 = \frac{240(\beta\gamma - 18\gamma^2)}{7\beta + 44\gamma}, c_2 = -15(\beta - 12\gamma),$$

$$c_3 = -120\gamma, c = \frac{4(11\beta\gamma - 28\gamma^2)}{7\beta + 44\gamma}, \alpha = -\gamma, \beta^2 + 16\gamma^2 = 0.$$

Case (18)

$$\mathrm{w}\, c_0 = \frac{60}{11}\gamma, c_1 = -\frac{720}{11}\gamma, c_2 = 180\gamma, c_3 = -120\gamma, c = \frac{30}{11}\gamma, \alpha = \frac{19}{11}\gamma, \beta = 0.$$

For the sake of simplicity, we consider only the solution with respect to Case (1), the other solutions can be obtained in a similar way:

$$u(x,t) = 180\gamma - \frac{120\gamma}{(1 + e^{x-90\gamma t}\rho)^3} + \frac{420\gamma}{(1 + e^{x-90\gamma t}\rho)^2} - \frac{480\gamma}{1 + e^{x-90\gamma t}\rho},$$

and $\alpha = 73\gamma, \beta = -16\gamma$.

Conclusions

In this paper, we use the Kudryashov method combined with characteristic set algorithm to solve the surface wave equation and the generalized Kuramoto-Sivashinsky equation which are arising in fluid mechanics, this process can be reduced to solve a large system of algebraic equations, which is hard to solve, then we use characteristic set algorithm to solve the algebraic equations. The results show the effective of this method.

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Nomenclature

$u(x,t)$ - speed of travelling wave, [m/s] x -space, [m]
t - time, [s]

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