APPLICATION OF THE KUDRYSOV METHOD
WITH CHARACTERISTIC SET ALGORITHM TO SOLVE
SOME PARTIAL DIFFERENTIAL EQUATIONS IN FLUID MECHANICS

by

Yi TIAN∗

College of Data Science and Application, Inner Mongolia University of
Technology, Hohhot 010080, China

In this paper, we pay attention to the analytical method named, the
Kudryashov method combined with characteristic set algorithm for
finding the exact travelling solutions of two nonlinear partial differential
equations in fluid mechanics, which named surface wave equation and
the generalized Kuramoto-Sivashinsky equation. The solution procedure
of the Kudryashov method can be reduced to solve a large system of
algebraic equations, which is hard to solve, then we use characteristic
set algorithm to solve this problem. The obtained results show that the
Kudryashov method combined with characteristic set algorithm is effective.

Key words: Kudryashov method, characteristic set algorithm,
surface wave equation, generalized Kuramoto-Sivashinsky
equation

Introduction

The partial differential equations (PDEs) arising in many physical fields like the
condense matter physics, fluid mechanics, plasma physics and optics, etc. The investigation of
the exact solutions plays an important role in the study of physical systems, and finding exact
solutions of the PDEs is one of the central themes in mathematics and physics. In the past
decades, a wealth of methods have been developed to obtain exact solutions of PDEs. Some
of the most important methods are the homotopy perturbation method [1], variational iteration
method[2], Riccati differential equation method [3], and other methods[4-9].

The objective of this article is to look for new study for relating to the
Kudryashov method to explore exact travelling wave solution for the surface wave equation
and the generalized Kuramoto- Sivashinsky equation. The solution procedure of the
Kudryashov method can be reduced to solve a large system of algebraic equations, which is
hard to solve, then we use the characteristic set algorithm to solve this problem. This
application displays the simplicity, efficiency and effectiveness of the Kudryashov method
with characteristic set algorithm [10]. To the best of our knowledge that the Kudryashov

∗ Corresponding author; e-mail: ttxsun@163.com
method has not been applied to the above mentioned equation in previous literature.

The Kudryashov method

Let us introduce the Kudryashov method as follows [10].

Consider the nonlinear partial differential equation in the following form:

\[ Q(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0, \]  

where \( u = u(x,t) \) is an unknown function, \( Q \) is a polynomial of \( u(x,t) \) and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved and the subscripts stands for the partial derivatives.

**Step-1:** We familiarize the travelling wave transformation

\[ u(x,t) = u(\xi), \quad \xi = x - ct, \]  

where \( c \) is the speed of travelling wave, the travelling wave transformation eq.(2) transform eq.(1) into an ordinary differential equation (ODE) for \( u = u(\xi) \):

\[ \Theta(u, u', -cu'', \ldots) = 0, \]  

where \( \Theta \) is a polynomial of \( u \) and its derivatives and the superscripts specify the ordinary derivatives with respect to \( \xi \).

**Step-2:** We look for exact solution of eq. (3) in the form

\[ u(\xi) = \sum_{i=0}^{N} a_i Q(\xi)^i, \]  

where \( a_i (0 \leq i \leq N) \) are constants to be determined, such that \( a_N \neq 0 \), while \( Q(\xi) \) has the form

\[ Q(\xi) = \frac{1}{1 + \rho \exp(\xi)}, \]  

a solution to the Riccati equation

\[ Q'(\xi) = Q(\xi)^2 - Q(\xi), \]  

where \( \rho \) is arbitrary constant.

**Step-3:** By balancing the highest order derivative terms with the nonlinear terms of the highest order come out in eq.(4), we can evaluated the value of the positive integer \( N \).

**Step-4:** By substituting eq.(4) along with eq.(6) into eq.(3) and equating all the coefficients of same power of \( Q(\xi) \) to zero, we obtained a system of algebraic equations. The obtaining system can be solved to find the value of \( c \), \( a_i (0 \leq i \leq N) \) substituting these terms into eq.(4) along with (5), the determination of solutions of eq.(1) will be completed.

**Characteristic set algorithm**

Let us give the characteristic set algorithm as follows [11].

A characteristic set \( CS \) of a polynomial system \( PS \) will be determined according to the following algorithm.

**Input:** A polynomial system \( PS \).
Output: A characteristic set $CS$ of $PS$.

**Step 1:** Set $PS_0 ← PS$.

**Step 2:** Take a basic set $BS$ of $PS_0$.

**Step 3:** Form $RS = \text{Rem}(PS_0 \setminus BS, BS) \setminus \{0\}$.

**Step 4:** If $RS = \emptyset$ then $CS ← BS$ and return. Otherwise $PS_0 = PS + BS + RS$ and go to Step 2.

The Step 2 will be achieved by the algorithm below:

**Input:** A polynomial system $PS$.

**Output:** A basic set $BS$ of $PS$.

**Step 1:** Set $PS' = PS$ and $BS = \emptyset$.

**Step 2:** If $PS' = \emptyset$ then return $BS$. Otherwise take from $PS'$ a polynomial $B$ of least class and least degree and set $BS ← BS + \{B\}$.

**Step 3:** Set $PS ← \{\text{polynomials in } PS' \text{ which are reduced w.r.t. } B\}$.

**Step 4:** Go to Step 2.

(Well-Ordering Principle) Let $CS$ be a characteristic set of a polynomials system $PS$. Then:

$$\text{Zero}(PS / IP) = \text{Zero}(CS / IP),$$

$$\text{Zero}(CS / IP) \subseteq \text{Zero}(PS) \subseteq \text{Zero}(CS),$$

$$\text{Zero}(PS) = \text{Zero}(CS / IP) \cup \text{Zero}(PS + \{I_i\}),$$

where $I_i$ are initials in $CS$, and $IP$ is the initial-product of $CS$.

**Exact solutions of a surface wave equation in convecting fluid**

Consider the following surface wave equation [12]:

$$u_t + a_0 u_x + a_1 uu_x + a_2 u_{xxx} + b_0 u_{xx} + b_1 (uu_x)_x + b_2 u_{xxxx} = 0, \quad (7)$$

which describes oscillatory Rayleigh-Marangoni instability in a liquid layer with free boundary. Let’s assume the traveling wave solution of eq. (7) in the form

$$u(x,t) = u(\xi), \quad \xi = x - ct, \quad (8)$$

where $c$ is a arbitrary constant. Using the wave variable (8), the eq. (7) is carried to

$$a_0 u' - cu' + a_1 uu' + b_0 u'' + b_1 (u'^2 + uu'') + a_2 u'^{(3)} + b_2 u'^{(4)} = 0, \quad (9)$$

integrating eq. (9) once with respect to $\xi$ and setting the integration constant as zero, we get:

$$(a_0 - c)u + \frac{a_1}{2} u^2 + (b_0 + b_1) u' + a_2 u'^{(3)} + b_2 u'^{(4)} = 0, \quad (10)$$

suppose that the solution of ODE (10) can be expressed:

$$u(\xi) = \sum_{i=0}^{N} c_i Q(\xi)^i, \quad (11)$$

where $c_i (0 \leq i \leq N)$ are constants to be determined, such that $c_N \neq 0$.

Consider the homogeneous balance between the highest order derivative $u'^{(3)}$ and nonlinear term $uu'$ appearing in (10), we have $N = 2$, we then suppose that eq. (10) has the following solutions:
\[ u(\xi) = c_0 + c_1 Q(\xi) + c_2 Q(\xi)^2, \quad c_2 \neq 0, \quad (12) \]

Substituting eq. (12) along with eq. (6) into eq. (10) and collecting all the terms with the same power of \( Q(\xi) \) together, equating each coefficient to zero, yields a set of algebraic equations, which is large and difficult to solve, with the aid of the characteristic set algorithm, we can distinguish the different cases namely:

Case (1)

\[ c_2 = \frac{12a_2}{a_1}, \quad c_1 = \frac{12a_2}{a_1}, \quad c_0 = 0, \quad b_0 = b_1 = b_2 = 0, \quad c = a_0 + a_2. \]

Case (2)

\[ c_2 = \frac{12b_2}{b_1}, \quad c_1 = \frac{24b_1}{b_1}, \quad c_0 = 0, \quad a_1 = -25b_1, \quad a_2 = -30b_2, \quad b_0 = 119b_2, \quad c = a_0 - 150b_2. \]

Case (3)

\[ c_2 = \frac{12b_2}{b_1}, \quad c_1 = \frac{24b_1}{b_1}, \quad c_0 = -\frac{12b_2}{b_1}, \quad a_1 = -13b_1, \quad a_2 = -18b_2, \quad b_0 = 71b_2, \quad c = a_0 + 78b_2. \]

Case (4)

\[ c_2 = \frac{12b_2}{b_1}, \quad c_1 = \frac{12b_2}{b_1}, \quad c_0 = 0, \quad a_1 = \frac{a_2}{b_2}, \quad b_0 = -b_2, \quad c = a_0 + a_2. \]

Case (5)

\[ c_2 = \frac{12b_2}{b_1}, \quad c_1 = \frac{12b_2}{b_1}, \quad c_0 = 0, \quad a_1 = -6b_1, \quad a_2 = -6b_2, \quad b_0 = -b_2, \quad c = a_0 - 6b_2. \]

Case (6)

\[ c_2 = \frac{12b_0}{5a_1}, \quad c_1 = \frac{24b_0}{5a_1}, \quad c_0 = 0, \quad a_1 = -\frac{b_0}{5}, \quad b_1 = b_2 = 0, \quad c = a_0 - \frac{6}{5} b_0. \]

Case (7)

\[ c_2 = \frac{12b_2}{b_1}, \quad c_1 = \frac{24b_1}{b_1}, \quad c_0 = -\frac{12b_2}{b_1}, \quad a_1 = \frac{b_0 + 19b_2}{5}, \quad a_2 = -\frac{b_0 b_1 + 6b_2}{5b_2}, \quad c = a_0 + \frac{6b_0 - 36b_2}{5}. \]

Case (8)

\[ c_2 = \frac{12b_2}{b_1}, \quad c_1 = \frac{12b_2}{b_1}, \quad c_0 = \frac{b_0 + b_2}{b_1}, \quad a_1 = 0, \quad c = a_0. \]

Case (9)

\[ c_2 = \frac{12b_0}{5a_1}, \quad c_1 = \frac{24b_0}{5a_1}, \quad c_0 = -\frac{12b_0}{5a_1}, \quad a_1 = -\frac{b_0}{5}, \quad b_1 = b_2 = 0, \quad c = a_0 + \frac{6}{5} b_0. \]

Case (10)

\[ c_2 = \frac{12b_2}{b_1}, \quad c_1 = 0, \quad a_2 = \frac{b_0 + 19b_2}{5}, \quad a_1 = \frac{b_0 b_1 - 6b_2}{5b_2}, \quad c = a_0 - \frac{6b_0 - 36b_2}{5}. \]
Case (11)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = -\frac{2(a_2 b_0 - 29 a_2 b_2 + 24 b_2^2 - 156 b_2^2)}{b_1 (5 a_2 + b_0 + 3 b_2)}, c_0 = 0,\]
\[a_i = \frac{a_2 b_0 b_1 + a_2 b_1 b_2 + 30 b_1 b_2 b_2 + 30 b_2 b_2^2}{6 b_2 (5 a_2 + b_0 + 3 b_2)}, b_0 = \frac{a_2^2 + 6 a_2 b_2 - 6 b_2^2}{6 b_2}, c = a_0 + a_2 - b_0 - b_2.\]

Case (12)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = \frac{24 b_2}{b_1}, c_0 = 0, a_2 = -6 b_2, a_1 = -b_1, b_0 = -b_2, c = a_0 - 6 b_2.\]

Case (13)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = \frac{22 b_2}{b_1}, c_0 = -\frac{10 b_2}{b_1}, a_2 = -5 b_2, a_1 = -\frac{5}{6} b_1, b_0 = \frac{49}{6} b_2, c = a_0 + \frac{25}{6} b_2.\]

Case (14)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = \frac{24 b_2}{b_1}, c_0 = -\frac{6 b_2}{b_1}, a_2 = -5 b_2, a_1 = 0, c = a_0.\]

Case (15)

\[c_2 = \frac{12 b_2}{5 a_1}, c_1 = \frac{12 b_2}{5 a_1}, a_2 = \frac{b_2}{5}, b_1 = b_2 = 0, c = a_0 + \frac{6}{5} b_0.\]

Case (16)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = \frac{72 b_2}{b_1}, c_0 = 0, a_2 = -30 b_2, a_1 = -5 b_1, b_0 = 119 b_2, c = a_0 - 150 b_2.\]

Case (17)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = \frac{24 b_2}{b_1}, c_0 = 0, a_2 = -\frac{b_2 + 31 b_2}{5}, a_1 = -\frac{b_2 h_1 - 6 b_2 b_2}{5 b_2}, c = a_0 - \frac{6 b_0 + 36 b_2}{5}.\]

Case (18)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = \frac{12 b_2}{b_1}, c_0 = \frac{2 b_2}{b_1}, a_2 = \frac{a_2 h_2}{b_2}, b_2 = b_1, c = a_0 - a_2.\]

Case (19)

\[c_2 = \frac{12 b_2}{b_1}, c_1 = \frac{12 b_2}{b_1}, c_0 = 0, a_2 = a_1 = 0, b_2 = -b_0, c = a_0.\]

Case (20)

\[c_2 = \frac{12 b_0}{5 a_1}, c_1 = c_0 = 0, a_2 = \frac{b_0}{5}, b_1 = b_2 = 0, c = a_0 - \frac{6 b_0}{5}.\]

Case (21)
\[ c_2 = \frac{12b_2}{b_1}, c_1 = -\frac{2(29a_2b_0b_2 + 6b_0b_2 + 29a_2b_2^2 + 102b_0b_2^2 + 92b_2^4)}{b_1(a_2b_0 + 45a_2b_2 + 29b_0b_2 + 29b_2^2)}, \]
\[ c_0 = \frac{12(a_2b_0 + 45a_2b_2 + 29b_0b_2 + 29b_2^2)}{b_1(29a_2 + 6b_0 + 96b_2)} , \]
\[ c = \frac{29a_0a_2 + 6a_0b_0 + 35a_0b_2 + 6b_0^2 + 96a_0b_2 + 299a_2b_2 + 276b_0b_2 + 270b_2^2}{29a_2 + 6b_0 + 96b_2} , \]
\[ a_1 = \frac{35a_2b_2b_1 + 6b_0^2b_1 + 299a_2b_2b_2 + 276b_0b_2b_2 + 270b_2b_2}{6(a_2b_0 + 45a_2b_2 + 29b_0b_2 + 29b_2^2)}, \quad b_0 = \frac{a_0^2 - 6a_0b_2 - 6b_2^2}{6b_2} .
\]

Case (22)
\[ c_2 = \frac{12a_2}{a_1}, c_1 = \frac{12a_2}{a_1}, c_0 = -\frac{2a_2}{a_1}, b_0 = b_1 = b_2 = 0, c = a_0 - a_2 .
\]

Case (23)
\[ c_2 = \frac{12b_2}{b_1}, c_1 = \frac{48b_2}{b_1}, c_0 = -\frac{36b_2}{b_1}, a_1 = -3b_1, a_2 = -18b_2, b_0 = 71b_2, c = a_0 + 54b_2 .
\]

For the sake of simplicity, we consider only the solution with respect to Case (1), the other solutions can be obtained in a similar way:
\[ u(x, t) = \frac{12a_2}{a_1(1 + e^{-(a_0 + a_2)t})} - \frac{12a_2}{a_1(1 + e^{-(a_0 + a_2)t})^2} , \]
and \( b_0 = b_1 = b_2 = 0 .
\]

**Exact solutions of the generalized Kuramoto-Sivashinsky equation**

Consider the following generalized Kuramoto-Sivashinsky equation [13]:
\[ u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0 , \quad (13) \]
let’s assume the traveling wave solution of eq.(13) in the form
\[ u(x, t) = u(\xi), \quad \xi = x - ct , , \quad (14) \]
where \( c \) is a arbitrary constant. Using the wave variable (14), the eq. (13) is carried to
\[ -cu^t + uu^t + \alpha u^t + \beta u^{(3)} + \gamma u^{(4)} = 0 , \quad (15) \]
integrating eq.(15) once with respect to \( \xi \) and setting the integration constant as zero,we get:
\[ -cu + \frac{1}{2} u^2 + \alpha u^t + \beta u^t + \gamma u^{(3)} = 0 , \quad (16) \]
suppose that the solution of ODE (16) can be expressed:
\[ u(\xi) = \sum_{i=0}^{N} c_i Q(\xi)^i , \quad (17) \]
where \( c_i (0 \leq i \leq N) \) are constants to be determined, such that \( c_N \neq 0 \).

Consider the homogeneous balance between the highest order derivative \( u^{(3)} \) and nonlinear term \( u^2 \) appearing in (16), we have \( N = 3 \), we then suppose that eq.(16) has the following solutions:

\[
u(\xi) = c_0 + c_1 Q(\xi) + c_2 Q(\xi)^2 + c_3 Q(\xi)^3, \quad c_3 \neq 0,
\]

substituting eq.(18) along with eq.(6) into eq.(16) and collecting all the terms with the same power of \( Q(\xi) \) together, equating each coefficient to zero, yields a set of algebraic equations, which is large and difficult to solve, with the aid of the characteristic set algorithm, we can distinguish the different cases namely:

Case (1)
\[c_0 = 180\gamma, c_1 = -480\gamma, c_2 = 420\gamma, c_3 = -120\gamma, c = 90\gamma, \alpha = 73\gamma, \beta = -16\gamma.,\]

Case (2)
\[c_0 = -12\gamma, c_1 = 0, c_2 = 120\gamma, c_3 = -120\gamma, c = -6\gamma, \alpha = \gamma, \beta = 4\gamma.,\]

Case (3)
\[c_0 = -60\gamma, c_1 = 0, c_2 = 180\gamma, c_3 = -120\gamma, c = -30\gamma, \alpha = -19\gamma, \beta = 0.,\]

Case (4)
\[c_0 = 0, c_1 = -\frac{720}{11} \gamma, c_2 = 180\gamma, c_3 = -120\gamma, c = -\frac{30}{11} \gamma, \alpha = \frac{19}{11} \gamma, \beta = 0.,\]

Case (5)
\[c_0 = 180\gamma, c_1 = 0, c_2 = -60\gamma, c_3 = -120\gamma, c = 90\gamma, \alpha = 73\gamma, \beta = 16\gamma.\]

Case (6)
\[c_0 = 0, c_1 = -120\gamma, c_2 = 240\gamma, c_3 = -120\gamma, c = -6\gamma, \alpha = \gamma, \beta = -4\gamma.\]

Case (7)
\[c_0 = c_1 = c_2 = 0, c_3 = -120\gamma, c = -60\gamma, \alpha = 47\gamma, \beta = 12\gamma.\]

Case (8)
\[c_0 = c_1 = 0, c_2 = 180\gamma, c_3 = -120\gamma, c = 30\gamma, \alpha = -19\gamma, \beta = 0.\]

Case (9)
\[c_0 = c_1 = 0, c_2 = 120\gamma, c_3 = -120\gamma, c = 6\gamma, \alpha = \gamma, \beta = -4\gamma.\]

Case (10)
\[c_0 = c_1 = 0, c_2 = -60\gamma, c_3 = -120\gamma, c = -90\gamma, \alpha = 73\gamma, \beta = 16\gamma.\]

Case (11)
\[c_0 = 12\gamma, c_1 = -120\gamma, c_2 = 240\gamma, c_3 = -120\gamma, c = 6\gamma, \alpha = \gamma, \beta = -4\gamma.\]

Case (12)
\[c_0 = 120\gamma, c_1 = -360\gamma, c_2 = 360\gamma, c_3 = -120\gamma, c = 60\gamma, \alpha = 47\gamma, \beta = -12\gamma.\]
Case (13)
\[ c_0 = 120\gamma, c_1 = c_2 = 0, c_3 = -120\gamma, c = 60\gamma, \alpha = 47\gamma, \beta = 12\gamma. \]

Case (14)
\[ c_0 = 0, c_1 = -480\gamma, c_2 = 420\gamma, c_3 = -120\gamma, c = 90\gamma, \alpha = 73\gamma, \beta = -16\gamma. \]

Case (15)
\[ c_0 = -\frac{8(11\beta\gamma - 28\gamma^3)}{7\beta + 44\gamma}, c_1 = \frac{240(\beta\gamma - 18\gamma^2)}{7\beta + 44\gamma}, c_2 = -15(\beta - 12\gamma), \]
\[ c_3 = -120\gamma, c = -\frac{4(11\beta\gamma - 28\gamma^3)}{7\beta + 44\gamma}, \alpha = -\gamma, \beta^2 + 16\gamma^2 = 0. \]

Case (16)
\[ c_0 = 0, c_1 = -360\gamma, c_2 = 360\gamma, c_3 = -120\gamma, c = -60\gamma, \alpha = 47\gamma, \beta = -12\gamma. \]

Case (17)
\[ c_0 = 0, c_1 = \frac{240(\beta\gamma - 18\gamma^2)}{7\beta + 44\gamma}, c_2 = -15(\beta - 12\gamma), \]
\[ c_3 = -120\gamma, c = \frac{4(11\beta\gamma - 28\gamma^3)}{7\beta + 44\gamma}, \alpha = -\gamma, \beta^2 + 16\gamma^2 = 0. \]

Case (18)
\[ \omega c_0 = \frac{60}{11} \gamma, c_1 = -\frac{720}{11} \gamma, c_2 = 180\gamma, c_3 = -120\gamma, c = \frac{3}{11} \gamma, \alpha = \frac{19}{11} \gamma, \beta = 0. \]

For the sake of simplicity, we consider only the solution with respect to Case (1), the other solutions can be obtained in a similar way:
\[ u(x,t) = 180\gamma - \frac{120\gamma}{(1 + e^{-989\gamma})}, + \frac{420\gamma}{(1 + e^{-989\gamma})}, - \frac{480\gamma}{1 + e^{-989\gamma}}, \]
and \( \alpha = 73\gamma, \beta = -16\gamma. \)

Conclusions

In this paper, we use the Kudryashov method combined with characteristic set algorithm to solve the surface wave equation and the generalized Kuramoto-Sivashinsky equation which are arising in fluid mechanics, this process can be reduced to solve a large system of algebraic equations, which is hard to solve, then we use characteristic set algorithm to solve the algebraic equations. The results show the effective of this method.

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Nomenclature

\( u(x,t) \) - speed of travelling wave, [m/s]  
\( x \) - space, [m]  
\( t \) - time, [s]

References


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