ANALYTICAL SOLUTIONS OF BIHARMONIC EQUATION BY THE FOURIER-YANG INTEGRAL TRANSFORM

by

Lin GAO [a b], Zhizhen ZHANG [a*] and Yan XING [a b]

[a] School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou 221116, China

[b] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China

The biharmonic equation (BE) are frequently encountered in computational fluid dynamics. In this investigation, the BE in the semi-infinite domains are addressed using a new Fourier-like integral transform proposed in [Therm. Sci.21 (2017, 79-87)]. The properties of the new Fourier-like integral transform are expanded in this article. Meanwhile, the analytical solutions for the BE in the semi-infinite domains are found. This demonstrates the new Fourier-like integral transform is an efficient and accurate method to clarify mathematical physics problems described by partial differential equations.

Key words*: biharmonic equation, Laplace equation, Fourier-Yang integral transform, fluid dynamics, partial differential equation.*

Introduction

The biharmonic equation (BE) derived from the mathematical model in fluid dynamics have an extensive application background in science research and engineering technology. Accordingly, there is growing concern about the improvement on calculation method for BE [2-7]. Mathieu [8] suggested the method of successive approximations for solving the BE and proved its convergence for a square plate. Using the Jacobi-Galerkin approximations, Doha and Bhrawy [9] solved the BE under the first and second boundary conditions. The two-dimensional BE for the semi-infinite elastic medium in the absence of the body force was settled in [10]. Those methods effectively solved the BE for certain boundary conditions. However, some boundary conditions are still challenging to deal with, thus the method of solving BE under certain boundary conditions is also need to continuously improve and innovate.

The integral transform (IT) has become an increasingly widespread application in solving the partial differential equation (PDEs). Currently, the conventional ITs of three types, e.g., Laplace transform (LT) [11, 12], Fourier transform [13, 14] and Sumudu transform (ST) [11, 15], were widely used in solving the PDEs arising from mathematical physics. In recent years, many scholars also applied ITs to solve the BE [10.16.17]. Still, there are unresolved boundary question for actual

* Corresponding author; e-mail: zzzcumt@163.com

engineering. In view of the above-mentioned facts, we have focused on a new Fourier-like IT termed Fourier-Yang IT proposed in [1] to handle partial differential equation on steady heat transfer. The proposed IT was subsequently applied to solved the 1-D heat diffusion equation [18]. Those applications demonstrate that the Fourier-Yang integral transform can serve as a new effective method for solution of the PDEs. Therefore, the object of this paper is to solve the BE in semi-infinite medium based on the Fourier-Yang integral transform.

This paper is organized as follows. In Section 2, the properties of the Fourier-Yang integral transform are deduced and perfected. In Section 3, the BE is solved by means of the Fourier-Yang integral transform and the analytical solutions are given under specific boundary conditions. Finally, a brief summary is drawn in Section 4.

The new Fourier-like integral transforms

The Fourier-like integral transform of $\theta(t)$, denoted by $\tilde{\vartheta}(\kappa)$, is defined as (see [1]):

$$\Gamma\left[\theta(t)\right]=\kappa\int_{-\infty}^{+\infty}\theta(t)e^{-i\kappa t}dt, \tag{1}$$

where Γ is called the Fourier-like integral transform operator or the Fourier-Yang transform operator.

The inverse Fourier-like integral transform of $\tilde{\vartheta}(\kappa)$ is defined by

$$\Gamma^{-1}\left[\tilde{\vartheta}_2(\kappa)\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\vartheta}_2(\kappa)}{\kappa}e^{i\kappa t}d\kappa. \tag{2}$$

The properties of the Fourier-like integral transform are presented as follows.

(T1) If $\tilde{\vartheta}_1(\kappa)=\Gamma\left[\vartheta_1(t)\right]$ and $\tilde{\vartheta}_2(\kappa)=\Gamma\left[\vartheta_2(t)\right]$, then we have [1]:

$$\Gamma\left[a\vartheta_1(t)+b\vartheta_2(t)\right]=a\tilde{\vartheta}_1(\kappa)+b\tilde{\vartheta}_2(\kappa), \tag{3}$$

where a and b are constants.

(T2) Let $\Gamma\left[\vartheta(t)\right]=\tilde{\vartheta}(\kappa)$. Then we have:

$$\Gamma\left[\vartheta(at)\right]=\tilde{\vartheta}\left(\frac{\kappa}{a}\right), \tag{4}$$

where a is a constant greater than zero.

Proof. We present

$$\Gamma\left[\vartheta(at)\right]=\kappa\int_{-\infty}^{+\infty}\vartheta(at)e^{-i\kappa t}dt. \tag{5}$$

Taking $x=at$, we have:

$$\Gamma\left[\vartheta(at)\right]=\frac{\kappa}{a}\int_{-\infty}^{+\infty}\vartheta(x)e^{-i\frac{\kappa}{a}x}dx=\tilde{\vartheta}\left(\frac{\kappa}{a}\right). \tag{6}$$

For $a<0$, it is easily obtained that

$$\Gamma\left[\vartheta\left(at\right)\right]=-\tilde{\vartheta}\left(\frac{\kappa}{a}\right). \tag{7}$$

(T3) Let $\Gamma\left[\vartheta\left(t\right)\right]=\tilde{\vartheta}\left(\kappa\right)$. Then we have:

$$\Gamma\left[\vartheta\left(t\pm t_0\right)\right]=e^{\pm i\kappa t_0}\tilde{\vartheta}\left(\kappa\right). \tag{8}$$

Proof. We have

$$\Gamma\left[\vartheta\left(t\pm t_0\right)\right]=\kappa\int_{-\infty}^{+\infty}\vartheta\left(t\pm t_0\right)e^{-i\kappa t}d\quad . \tag{9}$$

Taking $x=at$ gives

$$\begin{aligned}\Gamma\left[\vartheta\left(t\pm t_0\right)\right]&=\kappa\int_{-\infty}^{+\infty}\vartheta\left(x\right)e^{-i\kappa\left(x\pm t_0\right)}dx\\&=e^{\pm i\kappa t_0}\kappa\int_{-\infty}^{+\infty}\vartheta\left(x\right)e^{-i\kappa x}dx.\\&=e^{\pm i\kappa t_0}\tilde{\vartheta}\left(\kappa\right)\end{aligned} \tag{10}$$

(T4) Let $\Gamma\left[\vartheta\left(t\right)\right]=\tilde{\vartheta}\left(\kappa\right)$ and $\lim\limits_{t\to-\infty}f^{(m)}\left(t\right)=0\left(m=0,1,\cdots n-1\right)$. Then

$$\Gamma\left[\vartheta^{(n)}\left(t\right)\right]=\left(i\kappa\right)^n\tilde{\vartheta}\left(\kappa\right), \tag{11}$$

where $\vartheta^{(n)}\left(t\right)$ is the derivative of ϑ of order n with respect to t.

Proof. For $n=1$, we have [18]:

$$\begin{aligned}\Gamma\left[\vartheta^{(1)}\left(t\right)\right]&=\kappa\int_{-\infty}^{+\infty}\vartheta^{(1)}\left(t\right)e^{-i\kappa t}dt\\&=\kappa\vartheta\left(t\right)e^{-i\kappa t}\Big|_{-\infty}^{+\infty}+i\kappa^2\int_{-\infty}^{+\infty}\vartheta\left(t\right)e^{-i\kappa t}dt\cdot\\&=i\kappa\tilde{\vartheta}\left(\kappa\right)\end{aligned} \tag{12}$$

Generally,

$$\Gamma\left[\vartheta^{(n)}\left(t\right)\right]=i\kappa\times\Gamma\left[\vartheta^{(n-1)}\left(t\right)\right]=\left(i\kappa\right)^2\times\Gamma\left[\vartheta^{(n-2)}\left(t\right)\right]=\left(i\kappa\right)^n\tilde{\vartheta}\left(\kappa\right). \tag{13}$$

(T5) Suppose $\Gamma\left[\vartheta\left(t\right)\right]=\tilde{\vartheta}\left(\kappa\right)$ and $\theta\left(t\right)=\int_{-\infty}^{t}\vartheta\left(t\right)dt$. Then we have:

$$\Gamma\left[\theta\left(t\right)\right]=\frac{1}{ik}\tilde{\vartheta}\left(\kappa\right). \tag{14}$$

Proof. We get

$$\begin{aligned}\Gamma\left[\theta\left(t\right)\right]&=\kappa\int_{-\infty}^{+\infty}\theta\left(t\right)e^{-i\kappa t}dt\\&=-\frac{1}{i}\theta\left(t\right)e^{-i\kappa t}\Big|_{-\infty}^{+\infty}+\frac{1}{i}\int_{-\infty}^{+\infty}\vartheta\left(t\right)e^{-i\kappa t}dt.\\&=\frac{1}{ik}\tilde{\vartheta}\left(\kappa\right)\end{aligned} \tag{15}$$

(T6) If $\tilde{\vartheta}_1(\kappa)=\Gamma\left[\vartheta_1(t)\right]$ and $\tilde{\vartheta}_2(\kappa)=\Gamma\left[\vartheta_2(t)\right]$, then we have :

$$\Gamma\left[\left(\vartheta_1(t)*\vartheta_2(t)\right)\right]=\frac{1}{\kappa}\tilde{\vartheta}_1(\kappa)\tilde{\vartheta}_2(\kappa), \tag{16}$$

where the convolution of $\vartheta_1(t)$ and $\vartheta_2(t)$ is defined as

$$\vartheta_1(t)*\vartheta_2(t)=\int_{-\infty}^{+\infty}\vartheta_1(x)\vartheta_2(t-x)\,dx. \tag{17}$$

Proof. We have

$$\begin{aligned}\Gamma\left[\left(\vartheta_1(t)*\vartheta_2(t)\right)\right]&=\kappa\int_{-\infty}^{+\infty}\left(\vartheta_1*\vartheta_2\right)(t)e^{-i\kappa t}dt\\&=\kappa\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\vartheta_1(x)\vartheta_2(t-x)\,dx\right]e^{-i\kappa t}dt\\&=\kappa\int_{-\infty}^{+\infty}\vartheta_1(x)\left[\int_{-\infty}^{+\infty}\vartheta_2(t-x)e^{-i\kappa t}dt\right]dx\\&=\int_{-\infty}^{+\infty}\vartheta_1(x)\tilde{\vartheta}_2(\kappa)e^{-i\kappa x}dx\\&=\frac{1}{\kappa}\tilde{\vartheta}_1(\kappa)\tilde{\vartheta}_2(\kappa)\end{aligned} \tag{18}$$

(T7) Let $\vartheta(t)=\delta(t)$. Then (see[1]):

$$\tilde{\vartheta}_1(\kappa)=\Gamma\left[\delta(t)\right]=k, \tag{19}$$

where $\delta(t)$ is the Dirac function.

(T8) If $\vartheta(t)=\operatorname{sgn}(t)$, then we have:

$$\tilde{\vartheta}_1(\kappa)=\Gamma\left[\operatorname{sgn}(t)\right]=-2i, \tag{20}$$

where $\operatorname{sgn}(t)$ respects the sign function defined by [10]

$$\operatorname{sgn}(t)=\begin{cases}1 & t>0\\ -1 & t<0\end{cases}. \tag{21}$$

Proof. With the aid of exponential decay function defined by [19]

$$f(t)=e^{at}u(t)=\begin{cases}e^{-at} & t>0\\ 0 & t<0\end{cases},\qquad a>0, \tag{22}$$

it is easily obtained:

$$\Gamma\left(f(t)\right)=k\int_0^{+\infty}e^{-(a+ik)t}dt=\frac{k}{a+ik}. \tag{23}$$

Meanwhile, the sign function $\operatorname{sgn}(t)$ can be written as (see [20])

$$\operatorname{sgn}(t)=\lim_{a\to 0}e^{-at}u(t)-\lim_{a\to 0}e^{-at}u(-t). \tag{24}$$

Hence,

$$\Gamma\left(\operatorname{sgn}(t)\right)=\lim_{a\to 0}\frac{k}{a+ik}-\lim_{a\to 0}\frac{k}{a-ik}=\frac{2}{i}. \tag{25}$$

Solving PDEs in semi-infinite domains

We consider the BE in the semi-infinite domain (see [10]) as:

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2 y^2}+\frac{\partial^4\psi}{\partial y^4}=0, \qquad x>0, \tag{26}$$

which can be presented symbolically as

$$\nabla^4\psi=0, \tag{27}$$

subject to the boundary conditions

$$\frac{\partial^2\psi}{\partial y^2}=-p(y), \qquad \frac{\partial^2\psi}{\partial xy}=0 \ \text{ on } \ x=0. \tag{28}$$

By taking the Fourier-Yang IT with respect to y, we present in the form:

$$\left(\frac{d^2}{dx^2}-\kappa^2\right)^2\psi=0, \tag{29}$$

subject to the initial value:

$$\kappa^2\tilde{\psi}(0,\kappa)=\tilde{p}(k), \ (i\kappa)\left(\frac{d\tilde{\psi}}{dx}\right)_{x=0}=0, \tag{30}$$

where $\Gamma\left[\psi(x,y)\right]=\tilde{\psi}(x,\kappa)$ and $\Gamma\left[p(y)\right]=\tilde{p}(k)$.

In view of eq. (29), we obtain its bounded solution in the following form:

$$\tilde{\psi}(x,\kappa)=(A+Bx)\exp\left(-|\kappa|x\right), \tag{31}$$

where A and B are constants to be determined.

It follows from eq. (30) that

$$A=\frac{\tilde{p}(k)}{\kappa^2}, \qquad B=\frac{\tilde{p}(k)}{|\kappa|}. \tag{32}$$

Hence, eq. (31) can be expressed as

$$\tilde{\psi}(x,\kappa)=\frac{\tilde{p}(k)}{\kappa^2}\left(1+|\kappa|x\right)\exp\left(-|\kappa|x\right). \tag{33}$$

Application of the inverse transform of the new IT for eq. (33) gives

$$\psi(x,\mathrm{y})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{p}(k)}{\kappa^3}\left(1+|\kappa|x\right)\exp\left(i\kappa y-|\kappa|x\right)d\kappa. \tag{34}$$

In particular, when

$$p(y) = P_0\delta(y), \tag{35}$$

it is easily obtained from Eq. (19)

$$\tilde{p}(k) = P_0 k, \tag{36}$$

where P_0 is a constant.

Substituting this value into Eq. (34), we obtain

$$\begin{aligned}\psi(x, \mathrm{y}) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P_0}{\kappa^2}\left(1+|\kappa|x\right)\exp\left(i\kappa y-|\kappa|x\right)d\kappa \\ &= \frac{P_0}{\pi}\int_0^{+\infty}\frac{1}{\kappa^2}\left(1+\kappa x\right)\cos\left(ky\right)\exp\left(-\kappa x\right)d\kappa\end{aligned}. \tag{37}$$

This result is agreement with the solution in [10].

Another special case is

$$p(y) = \mathrm{sgn}(y), \tag{38}$$

so that

$$\tilde{p}(k) = -2i. \tag{39}$$

In view of Eq. (34), the solution becomes

$$\begin{aligned}\psi(x, \mathrm{y}) &= \frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\kappa^2}\left(1+|\kappa|x\right)\exp\left(i\kappa y-|\kappa|x\right)d\kappa \\ &= \frac{2i}{\pi}\int_0^{+\infty}\frac{1}{\kappa^2}\left(1+\kappa x\right)\cos\left(ky\right)\exp\left(-\kappa x\right)d\kappa\end{aligned}. \tag{40}$$

Conclusion

In this paper, some properties of the new IT were developed. It was used to find the analytical solutions of the BE in semi-infinite domains. The results are in agreement with those of Fourier transform and Laplace transform. The technology can solve the PDEs in mathematical physics accurately and efficiently.

Acknowledgement

This work is financially supported by the Fundamental Research Funds for the Central Universities (No. 2018BSCXC44) and the Postgraduate Research & Practice Innovation Program of Jiangsu Province (KYCX18_1984).

Nomenclature

a - the space coordinate, [m]

t - time, [s]

$\mathrm{sgn}(t)$ - the signal function

y - the space coordinate, [m]

x - the space coordinate, [m] $\delta(t)$ - the Dirac function

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Paper submitted: May 10, 2018
Paper revised: June 25, 2018
Paper accepted: August 25, 2018