DERIVATION AND SOLITON DYNAMICS OF A NEW NON-ISOSPECTRAL AND VARIABLE-COEFFICIENT SYSTEM

by

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Under investigation in this paper is a new and more general non-isospectral and variable-coefficient non-linear integrodifferential system. Such a system is Lax integrable because of its derivation from the compatibility condition of a generalized linear non-isospectral problem and its accompanied time evolution equation which is generalized in this paper by embedding four arbitrary smooth enough functions. Soliton solutions of the derived system are obtained in the framework of the inverse scattering transform method with a time-varying spectral parameter. It is graphically shown the dynamical evolutions of the obtained soliton solutions possess time-varying amplitudes and that the inelastic collisions can happen between two-soliton solutions.

Key words: *Soliton solution, dynamical evolution, non-isospectral and variable-coefficient integrodifferenail system, inverse scattering transform method*

Introduction

Non-linear partial differential equations (PDEs) are often related to some non-linear natural phenomena, for example the celebrated Korteweg–de Vries (KdV) shallow-water wave equation which is used to describe the soliton phenomena first observed by J. Scott Russell in 1834 [1]. In soliton theory, the non-isospectral PDEs are a kind of non-linear equations describing the solitary waves in a certain type of non-uniform media, while the isospectral PDEs often describe solitary waves in lossless and uniform media. Recently, the investigation on derivations and solutions of non-isospectral PDEs has attached much attentions [2-15]. In 2017, by introducing a new spectral parameter $ik_t = 0.5 + ik - 2k^2$, Zhang and Hong [11] generalized the linear isospectral problem [3]:

$$\phi_x = M\phi\,,\quad M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix},\quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{1}$$

and its time evolution equation:

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$$\phi_t = N\phi\,,\quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2}$$

where i is the imaginary unit, k is the spectral parameter independent of x, $q = q(x,t)$, $r = r(x,t)$ and their derivatives of any order with respect to x and t are smooth functions which vanish as x tends to infinity, and A, B, C are undetermined functions of x, t, q, r and k.

Starting from eqs. (1) and (2) equipped with the parameter $ik_t = 0.5 + ik - 2k^2$, Zhang and Hong [11] derived a non-isospectral integrodifferential system:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -2q_x - xq_{xx} + 2q\partial^{-1}qr + 2xq^2r + q + xq_x - xq - tq \\ 2r_x + xr_{xx} - 2r\partial^{-1}qr - 2xqr^2 + r + xr_x + xr + tr \end{pmatrix} \tag{3}$$

In the present paper, we would like to consider a new and more general non-isospectral and variable-coefficient integrodifferential system:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} \alpha q_x + \beta_2(-2q_x - xq_{xx} + 2q\partial^{-1}qr + 2xq^2r) + \beta_1(q + xq_x) - \beta_0 xq \\ \alpha r_x + \beta_2(2r_x + xr_{xx} - 2r\partial^{-1}qr - 2xqr^2) + \beta_1(r + xr_x) + \beta_0 xr \end{pmatrix} \tag{4}$$

where $\alpha = \alpha(t)$, $\beta_1 = \beta_1(t)$, $\beta_2 = \beta_2(t)$ and $\beta_3 = \beta_3(t)$ are arbitrary smooth enough functions of t. There are maily three starting points of this paper: the first one is to derive system (5) by equipping eqs. (3) and (4) with the generalized spectral parameter k [12]:

$$ik_t = \frac{1}{2}\sum_{s=0}^{2}\beta_s(2ik)^s \tag{5}$$

and the new generalized function A:

$$A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}\alpha(2ik) - \frac{1}{2}[\sum_{s=0}^{2}\beta_s(2ik)^s]x \tag{6}$$

the second one is to extend the inverse scattering transform (IST) with the time-varying spectral parameter (5) for constructing soliton solutions of system (4); the last one is to gain some insights into the dynamical evolutions of the obtained soliton solutions. To the best of our knowledge, such a system (4) is new and the IST has not been extended to system (4).

Derivation

Firstly, from the compatibility condition of eqs. (1) and (2) we have:

$$A_x = qC - rB - ik_t\,,\quad q_t = B_x + 2ikB + 2qA\,,\quad r_t = C_x - 2ikC - 2rA \tag{7}$$

which can be simplified as:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -B \\ C \end{pmatrix} - 2ik\begin{pmatrix} -B \\ C \end{pmatrix} + \alpha(2ik)\begin{pmatrix} -q \\ r \end{pmatrix} + [\sum_{s=0}^{2}\beta_s(2ik)^s]\begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{8}$$

by using eqs. (5) and (6).

Secondly, we suppose that:

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{s=1}^{2}\begin{pmatrix} -b_s \\ {}_2c_s \end{pmatrix}(2ik)^{2-s} \tag{9}$$

and substitute eq. (10) into eq. (9) and then have:

$$(2ik)^0: \begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} + \beta_0\begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{10}$$

$$(2ik)^1: \begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} = L\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} + \beta_1\begin{pmatrix} -xq \\ xr \end{pmatrix} + \alpha\begin{pmatrix} -q \\ r \end{pmatrix} \tag{11}$$

$$(2ik)^2: \begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \beta_2\begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{12}$$

by comparing the coefficients of $2ik$ in eq. (8). Here the operator L is defined by:

$$L = \sigma\partial + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r,q), \quad \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x}\mathrm{d}x - \int_{x}^{+\infty}\mathrm{d}x\right), \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

Finally, from eqs. (10)-(12) we obtain system (4), which is Lax integrable.

Soliton solutions

Based on eqs. (1), (2) and (5) and the results in [2, 8-13], we have exact solutions:

$$q(x,t) = -2K_1(x,x,t), \quad r(x,t) = \frac{K_{2x}(x,x,t)}{K_1(x,x,t)} \tag{14}$$

where $K(x,y,t) = (K_1(x,y,t), K_2(x,y,t))^T$ satisfies Gel'fand–Levitan–Marchenko integral equation:

$$\begin{aligned} K(x,y,t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\bar{F}(x+y,t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\int_x^\infty F(z+x,t)\bar{F}(z+y,t)\mathrm{d}z \\ + \int_x^\infty K(x,s,t)\int_x^\infty F(z+s,t)\bar{F}(z+y,t)\mathrm{d}z\mathrm{d}s = 0 \end{aligned} \tag{15}$$

and

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)e^{ikx}\mathrm{d}k + \sum_{j=1}^{n} c_j^2 e^{i\kappa_j x}, \quad \bar{F}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{R}(k,t)e^{-ikx}\mathrm{d}k - \sum_{j=1}^{\bar{n}} \bar{c}_j^2 e^{-i\bar{\kappa}_j x} \tag{16}$$

are determined by the scattering data:

$$\left\{\kappa_j(t), c_j(t), R(k,t) = \frac{a(k,t)}{b(k,t)}, j = 1,2,\cdots,n\right\}, \quad \left\{\bar{\kappa}_m(t), \bar{c}_m(t), \bar{R}(k,t) = \frac{\bar{a}(k,t)}{\bar{b}(k,t)}, m = 1,2,\cdots,\bar{n}\right\} \tag{17}$$

which possess the following time-dependence:

$$\kappa_{jt}(t) = -\frac{i}{2}\sum_{s=0}^{2}\beta_s(2i\kappa_j(t))^s, \quad i\bar{\kappa}_{mt}(t) = -\frac{i}{2}\sum_{s=0}^{2}\beta_s(2i\bar{\kappa}_m(t))^s \tag{18}$$

$$c_j^2(t) = c_j^2(0)\mathrm{e}^{\int_0^t[\beta_1(w)+2i\kappa_j(w)(\alpha(w)+2\beta_2(w))]\mathrm{d}w}, \quad \bar{c}_m^2(t) = \bar{c}_m^2(0)\mathrm{e}^{\int_0^t[\beta_1(w)+2i\bar{\kappa}_j(w)(\alpha(w)+2\beta_2(w))]\mathrm{d}w} \tag{19}$$

$$a(k,t) = a(k,0 \ , \quad b(k,t) = b(k,0 \ , \quad \bar{a}(k,t) = \bar{a}(k,0 \ , \quad \bar{b}(k,t) = \bar{b}(k,0 \tag{20}$$

Here $\kappa_j^2(0)$, $\bar{\kappa}_m^2(0)$, $c_j^2(0)$, $\bar{c}_m^2(0)$, $R(k,0) = b(k,0)/a(k,0)$ and $\bar{R}(k,0) = \bar{b}(k,0)/\bar{a}(k,0)$ are the scattering data of the generalized spectral problem (4) in the case of $(q(x,0),r(x,0))^T$.

To construct soliton solutions, as did in [2] we set $R(k,t)=\bar{R}(k,t)=0$ and have

$$K_1(x,y,t)=-\mathrm{tr}(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(y,t)),\quad K_2(x,y,t)=i\mathrm{tr}(W^{-1}(x,t)E(x,t)\Lambda(x,t)\bar{\Lambda}^T(y,t)) \tag{21}$$

Then we obtain the following *n*-soliton solutions of system (5):

$$q(x,t)=2\mathrm{tr}(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(x,t)) \tag{22}$$

$$r(x,t)=-\frac{\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}(W^{-1}(x,t)P(x,t)\frac{\mathrm{d}}{\mathrm{d}x}P^T(x,t))}{\mathrm{tr}(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(x,t))} \tag{23}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a given matrix, E is a $\bar{n}\times\bar{n}$ unit matrix and

$$\Lambda=(c_1(t)e^{i\kappa_1 x},c_2(t)e^{i\kappa_2 x},\cdots,c_n(t)e^{i\kappa_n x})^T,\quad \bar{\Lambda}=(\bar{c}_1(t)e^{-i\bar{\kappa}_1 x},\bar{c}_2(t)e^{-i\bar{\kappa}_2 x},\cdots,\bar{c}_n(t)e^{-i\bar{\kappa}_{\bar{n}} x})^T \tag{24}$$

$$W(x,t)=E+P(x,t)P^T(x,t),\quad P(x,t)=\left(\frac{c_j(t)\bar{c}_m(t)}{\kappa_j-\bar{\kappa}_m}e^{i(\kappa_j-\bar{\kappa}_m)x}\right)_{\bar{n}\times n} \tag{25}$$

Particularly, when $n=\bar{n}=1$ eqs. (22) and (23) give the one-soliton solutions:

$$q=\frac{2\bar{c}_1^{\,2}(0)\mathrm{e}^{-2i\bar{\kappa}_1(t)x-\int_0^t[\beta_1(w)+2i\bar{\kappa}_j(w)(\alpha(w)+2\beta_2(w))]\mathrm{d}w}}{1+\frac{c_1^2(0)\bar{c}_1^{\,2}(0)}{(\kappa_1(t)-\bar{\kappa}_1(t))^2}\mathrm{e}^{2i(\kappa_1(t)-\bar{\kappa}_1(t))x+\int_0^t[2i(\kappa_1(w)-\bar{\kappa}_j(w))(\alpha(w)+2\beta_2(w))]\mathrm{d}w}} \tag{26}$$

$$r=\frac{2c_1^2(0)\mathrm{e}^{2i\kappa_1 x+\int_0^t[\beta_1(w)+2i\kappa_j(w)(\alpha(w)+2\beta_2(w))]\mathrm{d}w}}{1+\frac{c_1^2(0)\bar{c}_1^{\,2}(0)}{(\kappa_1(t)-\bar{\kappa}_1(t))^2}\mathrm{e}^{2i(\kappa_1(t)-\bar{\kappa}_1(t))x+\int_0^t[2i(\kappa_j(w)-\bar{\kappa}_1(w))(\alpha(w)+2\beta_2(w))]\mathrm{d}w}} \tag{27}$$

where $\kappa_1(t)$ and $\bar{\kappa}_1(t)$ are respectively determined in eq. (18).

In figs. 1-5, the spatial structures and dynmical evolutions of the one-soliton solutions (26) and (27) are shown by selecting $\kappa_1(0)=1$, $\bar{\kappa}_1(0)=-0.5$, $c_1(0)=-0.1$, $\bar{c}_1(0)=0.2$, $\alpha=t^2$, $\beta_0=1$, $\beta_1=1$ and $\beta_2=1$. The spatial structures and dynmical evolutions of the two-soliton solutions determined by eqs. (22) and (23) are shown in figs. 6-10, where $\kappa_1(0)=1$, $\bar{\kappa}_1(0)=0.2$, $\kappa_2(0)=-1.2$, $\bar{\kappa}_2(0)=-0.5$, $c_1(0)=-0.3$, $\bar{c}_1(0)=-0.5$, $c_2(0)=1$, $\bar{c}_2(0)=2$ and the others are same as figs 1-5. It is easy to see from figs. 1-10 that the one-soliton solutions and the two-soliton solutions possess time-varying amplitudes. Besides, the inelastic collisions happen between two-soliton solutions.

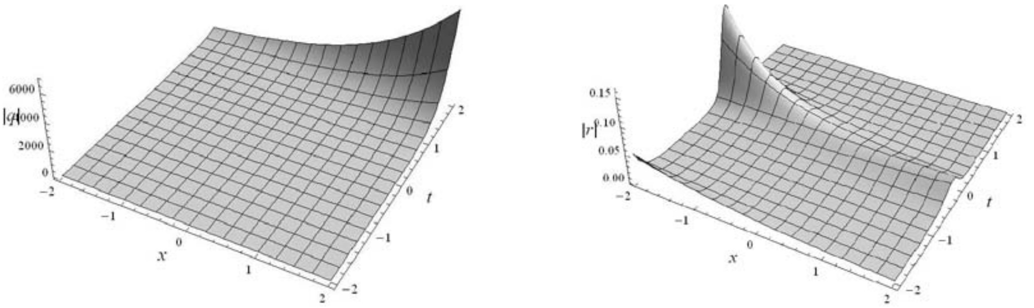
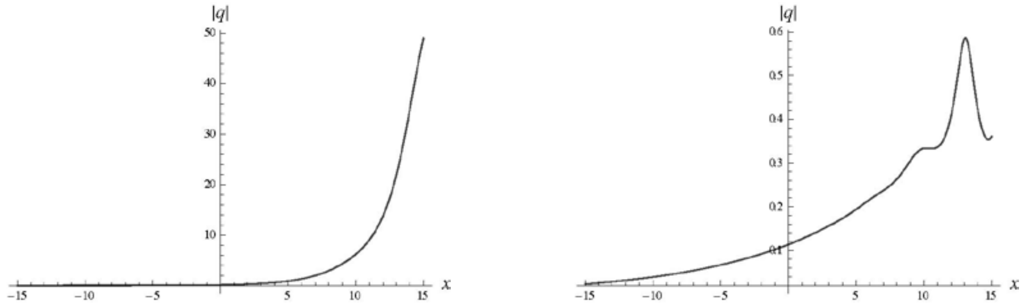
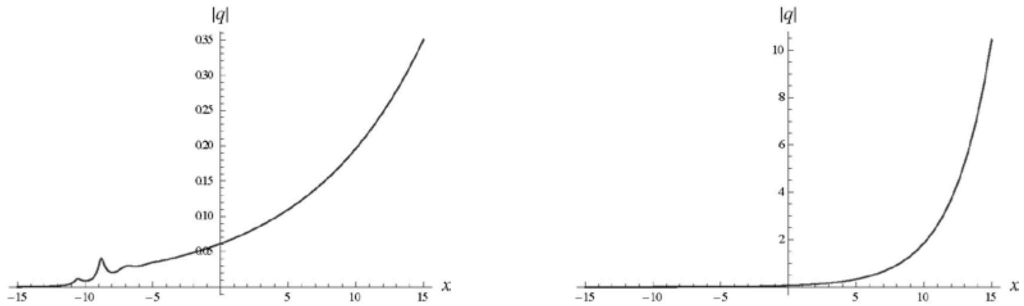


Figure 1. Spatial structures of one-soliton solutions (26) and (27).



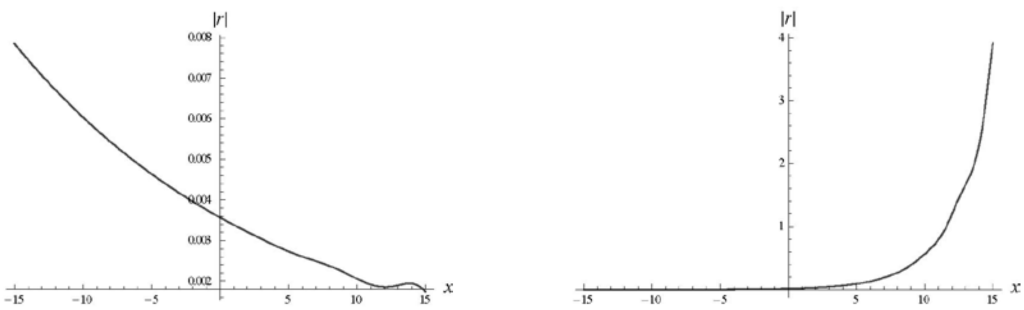
(a) (b)

Figure 2. Dynamical evolutions of one-soliton solution (26) at times: (a) $t = -1$; (b) $t = -0.4$.



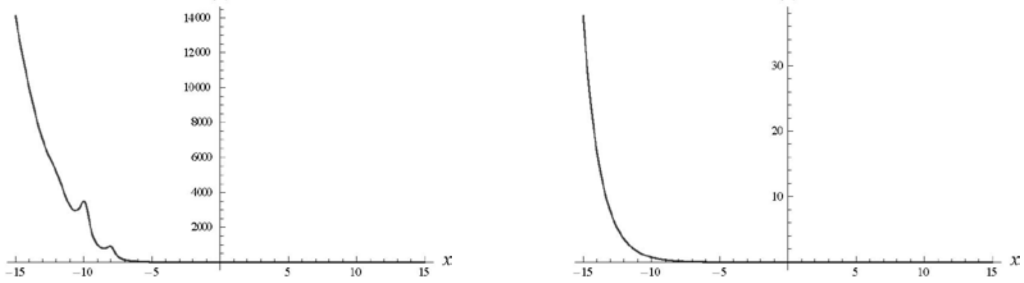
(a) (b)

Figure 3. Dynamical evolutions of one-soliton solution (26) at times: (a) $t = 0.3$; (b) $t = 0.5$.



(a) (b)

Figure 4. Dynamical evolutions of one-soliton solution (27) at times: (a) $t = -1$, $t = -0.2$.



(a) (b)

Figure 5. Dynamical evolutions of one-soliton solution (27) at times: (a) $t = 0.5$; (b) $t = 1.5$.

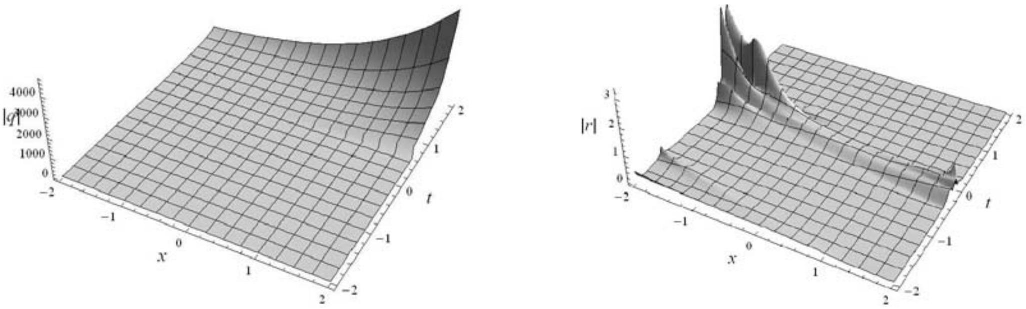
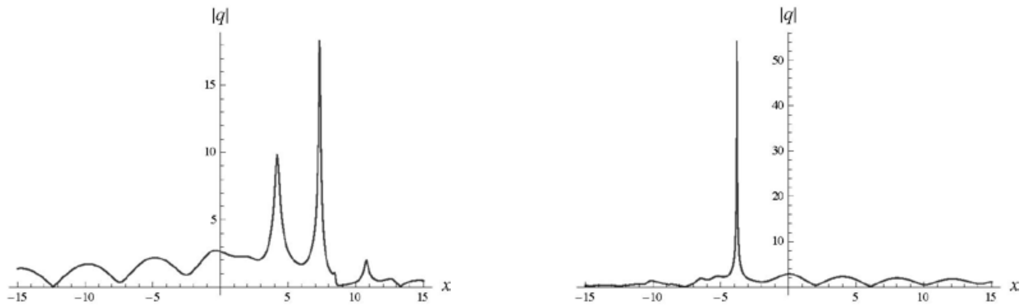
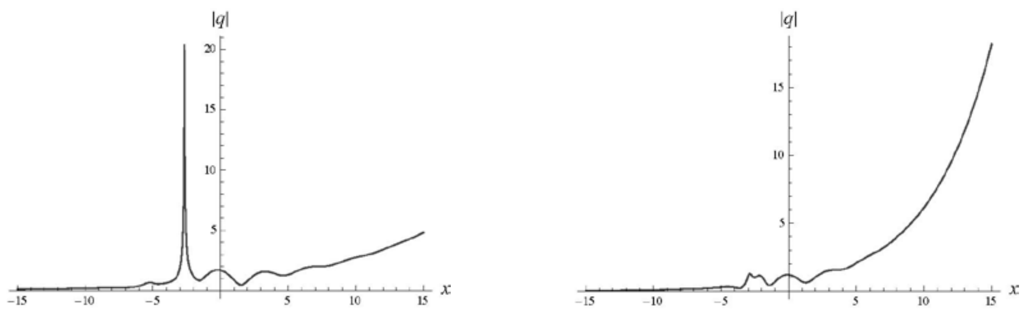


Figure 6. Spatial structures of two-soliton solutions determined by (22) and (23).



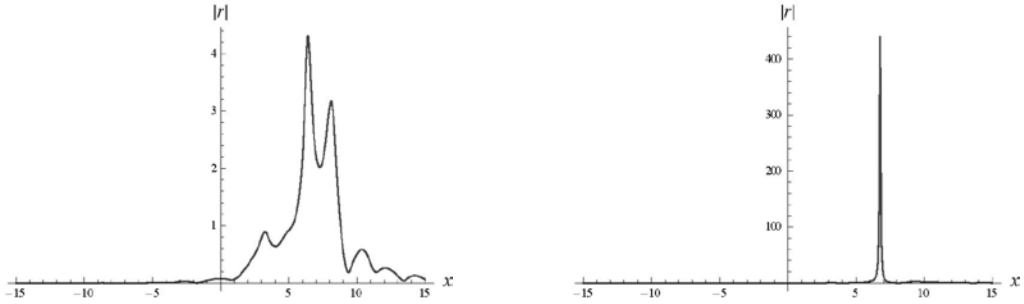
(a) (b)

Figure 7. Dynamical evolutions of two-soliton solution determined by (25) at times: (a) $t = -0.1$; (b) $t = 0.1$.



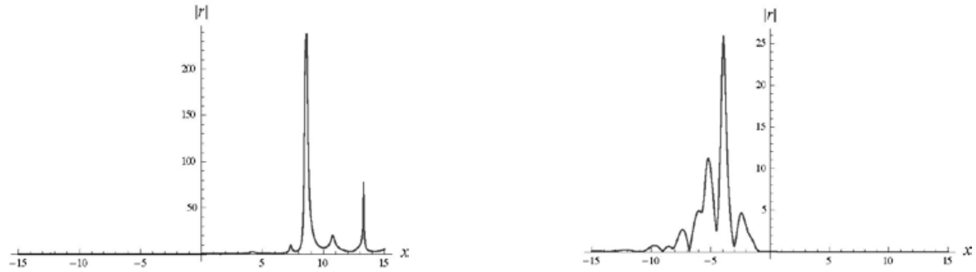
(a) (b)

Figure 8. Dynamical evolutions of two-soliton solution determined by (22) at times: (a) $t = 0.3$; (b) $t = 0.4$.



(a) (b)

Figure 9. Dynamical evolutions of two-soliton solution determined by (23) at times: (a) $t = -0.3$; (b) $t = -0.2$.



(a) (b)

Figure 10. Dynamical evolutions of two-soliton solution determined by (23) at times: (a) $t = -0.1$; (b) $t = 0.5$.

Conclusion

In summary, we have derived and solved the non-isospectral and variable-coefficient integrodifferential system (4). The arbitraryness of the smooth functions $\alpha = \alpha(t)$, $\beta_1 = \beta_1(t)$, $\beta_2 = \beta_2(t)$ and $\beta_3 = \beta_3(t)$ in the obtained n-soliton solutions provides with freedom to disscuss the dynamical evolutions of solutions. It is shown when $n = 1$ and $n = 2$ the one-soliton solutions and the two-soliton solutions possess time-varying amplitudes and that the inelastic collisions can happen between two-soliton solutions. To the best of our knowledge, system (4) and solutions (14), (22) and (23) have not been reported in literatures. How to extend the IST method to the non-linear PDEs with non-integer derivatives [16-20] is worthy of study.

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Nomenclature

d/dt —the first derivative, [–]

$n, \bar{n}$ —positive integers, [–]

e	—the base of natural logarithms, [–]	s	—positive integer, [–]
E	— $\overline{n}\times\overline{n}$ unit matrix, [–]	t	—time, [s]
i	—imaginary unit, [–]	T	—transposition, [–]
j	—natural number, [–]	x, y, z	—displacements, [m]
k	—spectral parameter, [–]	π	—circumference ratio, [–]
M, N	—matrices, [–]		

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