NEW HIGH-ORDER CONSERVATIVE DIFFERENCE SCHEME FOR REGULARIZED LONG WAVE EQUATION WITH RICHARDSON EXTRAPOLATION

by

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Numerical solution for the regularized long wave equation is considered by a new three-level conservative implicit finite difference scheme coupled with Richardson extrapolation which has the accuracy of $O(\tau + h^4)$. The scheme is a linear system of equations solved without iteration. The conservation properties of the algorithm are verified by computing the discrete mass and discrete energy. Existence and uniqueness of the numerical solution are proved. Convergence and stability of the scheme are also derived using energy method. The results of numerical experiments show that our proposed scheme is efficiency.

Key words: regularized long wave equation, conservative difference scheme, Richardson extrapolation, stability, convergence

Introduction

Consider the following initial-boundary value problem for the regularized long wave (RLW) equation:

$$u_t + uu_x + u_{xxt} - u_{xxx} = 0, \quad (x, t) \in (x_L, x_R) \times (0, T)$$

(1)

with an initial condition:

$$u(x, 0) = u_0(x), \quad x \in [x_L, x_R]$$

(2)

and boundary condition:

$$u(x_L, t) = u(x_R, t) = 0, \quad t \in [0, T]$$

(3)

where $u_0(x)$ is a given known function. The RLW equation is originally introduced to describe the behavior of the undular bore by Peregrine [1] and plays a major role in the study of non-linear dispersive waves [2] because of its description a larger number of important physical phenomena, such as shallow water waves and ion acoustic plasma waves.

Mathematical theory for the equation was developed in [3]. Due to non-linear nature of the RLW equation, few exact solutions exist in the literature [4, 5]. Studies mainly consider numerical solution of the problem. These include variational iteration method [6, 7], finite difference methods [8-15] and various finite element methods such as the Galerkin method [16-20],

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the least squares method [21-23] and collocation method with quadratic B-splines [24], cubic
B-splines [25], and recent septic splines [26].

The problem (1)-(3) has two conserved quantities: mass and energy, i.e.,

\[ Q(t) = \int_{x_L}^{x_R} u(x, t) \, dx = \int_{x_L}^{x_R} u_0(x) \, dx = Q(0) \]  

(4)

and

\[ E(t) = \| u \|_1^2 + \| u_t \|_2^2 = \| u_0 \|_1^2 + \| u_0 \|_2^2 = E(0) \]  

(5)

where \( Q(0) \) and \( E(0) \) are two positive constants which relate to the initial condition. Zhang
et al. [27] pointed out that the conservative difference schemes perform better than the non-conservative
ones, and the non-conservative difference schemes may easily show non-linear blow-up. Li and Vu-Quoc
[28] pointed out that in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion judge the success of a numerical simulation. Thus, the purpose of this paper is to present a conservative difference scheme for
the initial-boundary value problem (1)-(3). By the Richardson extrapolation, the scheme has the
accuracy of \( O(\tau^2 + h^4) \) without refined mesh. Moreover, the resulting scheme is a linear system of
equations, and it can be solved easily without any iterations.

**Finite difference scheme**

Let \( N, J \) be any positive integers and \( h = (x_R - x_L)/J \) be the step size for the grid
such that \( x_j = x_L + jh \) \((j = -1, 2, ..., J, J + 1)\). Let \( \tau \) be the step for the temporal direction,
\( t_n = n\tau \) \((n = 0, 1, 2, ..., N), N = \lceil T/\tau \rceil \).

Denote \( u_j^n \equiv u(x_j, t_n) \) and:

\[ Z_h^0 = \{ u = (u_j) \mid u_{-1} = u_0 = u_J = u_{J+1} = 0, \ j = -1, 0, 1, 2, ..., J, J+1 \} \]

Define:

\[ (u_{j+1}^n)_x = \frac{u_j^n - u_{j+1}^n}{h}, \quad (u_j^n)_x = \frac{u_j^n - u_{j-1}^n}{h}, \quad (u_j^n)_t = \frac{u_{j+1}^n - u_{j-1}^n}{2h} \]

\[ (u_j^n)_x = \frac{u_j^n - u_{j-1}^n}{h}, \quad (u_j^n)_t = \frac{u_j^n - u_{j-1}^n}{h\tau}, \quad \| u_j^n \|_c = \max_{|x_j| \leq \max} | u_j^n | \]

and in the paper, \( C \) denotes a general positive constant which may have different values in dif-
f erent occurrences.

**Lemma 1.** For a mesh function by Cauchy-Schwarz inequality:

\[ \| u \|_1^2 \leq \| u_t \|_2^2 \leq \| u \|_1^2 \]

The following conservative difference scheme for the problem (1)-(3) is considered:

\[ (u_j^n)_t - \frac{4}{3} (u_j^n)_x + \frac{1}{3} (u_j^n)_x + \frac{4}{9} (u_j^n)_x = \frac{1}{3} (u_j^n)_x + \frac{4}{9} \left[ u_j^n (u_j^n)_x + (u_j^n (u_j^n)_x) \right] \]

\[ - \frac{1}{9} \left[ u_j^n (u_j^n)_x + (u_j^n (u_j^n)_x) \right] = 0, \quad j = 1, 2, ..., J - 1, \ n = 1, 2, ..., N - 1 \]  

(6)

\[ u_j^0 = u_0(x_j), \quad j = 0, 1, 2, ..., J \]  

(7)
Based on the scheme (6)-(9), the discrete versions of (4) and (5) are obtained: 

**Theorem 1.** The scheme (6)-(9) admits the following invariant,

\[
Q^n = \frac{h^2}{2} \sum_{j=1}^{J-1} (u_{j+1}^n + u_j^n) + \frac{2h}{9} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_t - \frac{h}{18} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_s, = Q^{n+1} = \cdots = Q^0
\]

for \( n = 1, 2, \ldots, N - 1 \).

**Proof.** Multiplying (6) with \( h \), then summing up for \( j \) from 1 to \( J - 1 \), by the boundary condition (9) and formula of summation by parts [29]:

\[
h \sum_{j=1}^{J-1} \left( u_j^{n+1} - u_j^n \right) + \frac{4h}{9} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_t - \frac{h}{9} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_s = 0
\]

Again since:

\[
h \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_t = \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_t - \frac{h}{2} \sum_{j=1}^{J-1} u_j^{n-1}(u_j^n)_t
\]

\[
h \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_s = \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_s - \frac{h}{2} \sum_{j=1}^{J-1} u_j^{n-1}(u_j^n)_s
\]

(13)

(14)

substitute (13) and (14) into eq. (12), then eq. (10) is obtained.

Taking the inner product of (6) with \( 2\bar{u}^n \), that is \( u^{n+1} + u^n \), according to boundary condition (9):

\[
\|e\|^2 + \frac{4}{3} \|e\|^2 - \frac{8}{3} \langle \bar{u}_1^n, \bar{u}^n \rangle + \frac{8}{3} \langle \bar{u}_1^n, \bar{u}^n \rangle - \frac{2}{3} \langle \bar{u}_1^n, \bar{u}^n \rangle + \langle \phi(u_j^n, \bar{u}_j^n), \bar{u}^n \rangle - 2 \langle \kappa(u_j^n, \bar{u}_j^n), \bar{u}^n \rangle = 0
\]

(15)

where

\[
\phi(u_j^n, \bar{u}_j^n) = \frac{4}{9}[u_j^n (\bar{u}_j^n)_t + (u_j^n \bar{u}_j^n)_t]
\]

\[
\kappa(u_j^n, \bar{u}_j^n) = \frac{1}{9}[u_j^n (\bar{u}_j^n)_t + (u_j^n \bar{u}_j^n)_t]
\]

Considering

\[
\langle \bar{u}_1^n, \bar{u}^n \rangle = 0, \langle \bar{u}_1^n, \bar{u}^n \rangle = 0
\]

(16)

(17)

and

\[
\langle \kappa(u_j^n, \bar{u}_j^n), \bar{u}^n \rangle = 0
\]

(18)

Substituting (16)-(18) into eq. (15), we have:

\[
\frac{1}{2\tau} \left( \|u^{n+1}_1\|^2 - \|u^n_1\|^2 \right) + \frac{2}{3\tau} \left( \|u^{n+1}_1\|^2 - \|u^n_1\|^2 \right) - \frac{1}{6\tau} \left( \|u^{n+1}_1\|^2 - \|u^n_1\|^2 \right) = 0
\]

(19)

By the definition of \( E^n \), (11) is gotten from (19).
Solvability

Next, we are going to prove the solvability of the finite difference scheme (6)-(9).

Theorem 1. The difference scheme (6)-(9) is uniquely solvable.

Proof. Use the mathematical induction. It is obvious that $u^0$ and $u^1$ are uniquely determined by eqs. (7) and (8). Now suppose $u^0, u^1, \ldots, u^{n-1}, u^n$ be solved uniquely. Consider the equation of eq. (6) for $u^{n+1}$:

$$
\frac{1}{2\tau} u^{n+1}_j - \frac{2}{3\tau} \left( u^{n+1}_j \right)_z + \frac{1}{6\tau} \left( u^{n+1}_j \right)_{zz} + \frac{2}{3} \left( u^{n+1}_j \right)_{\tau} - \frac{1}{6} \left( u^{n+1}_j \right)_{\tau\tau} + \frac{1}{2} \varphi(u^*, u^{n+1}_j) - \frac{1}{2} \kappa(u^n, u^{n+1}_j) = 0, \quad j = 1, 2, \ldots, J - 1, \quad n = 1, 2, \ldots, N - 1
$$

(20)

Computing the inner product of (20) with $u^{n+1}_j$, using (9), we obtain:

$$
\frac{1}{2\tau} \left\| u^{n+1}_j \right\|^2 + \frac{2}{3\tau} \left\| u^{n+1}_j \right\|^2 - \frac{1}{6\tau} \left\| u^{n+1}_j \right\| + \frac{2}{3} \langle u^{n+1}_j, u^{n+1}_j \rangle - \frac{1}{6} \langle u^{n+1}_j, u^{n+1}_j \rangle
$$

$$
= \frac{1}{2} \langle \varphi(u^*, u^{n+1}_j), u^{n+1}_j \rangle - \frac{1}{2} \langle \kappa(u^n, u^{n+1}_j), u^{n+1}_j \rangle = 0
$$

(21)

Since:

$$
\langle u^{n+1}_j, u^{n+1}_j \rangle = 0
$$

$$
\langle \varphi(u^*, u^{n+1}_j), u^{n+1}_j \rangle = 0
$$

$$
\langle \kappa(u^n, u^{n+1}_j), u^{n+1}_j \rangle = 0
$$

(22)-(24)

Substituting (22)-(24) into (21), by Lemma 1, we have:

$$
\left\| u^{n+1}_j \right\|^2 + \left\| u^{n+1}_j \right\|^2 \leq 0
$$

That is, (20) has only a trivial solution. Therefore, (6) determines $u^{n+1}_j$ uniquely. This completes the proof.

Convergence and stability

Let $v(x, t)$ be the solution of problem (1)-(3) and $v_j = u(x_j, t_n)$, the the truncation error of the scheme (6)-(9) is derived:

$$
r_j = (v_j^*) - \frac{4}{3} (v_j^*)_{zz} + \frac{1}{3} (v_j^*)_{zzz} + \frac{4}{3} (v_j^*)_{zz} + \frac{1}{3} (v_j^*)_{zzz} + \frac{1}{3} \varphi(v_j^*, v_j^*) - \frac{1}{3} \kappa(v_j^*, v_j^*)
$$

$$
\quad j = 1, 2, \ldots, J - 1, \quad n = 1, 2, \ldots, N - 1
$$

(25)

$$
v_j = u(x_j), \quad j = 0, 1, 2, \ldots, J
$$

(26)

$$
v_j^* = \frac{4}{3} (v_j^*)_{zz} + \frac{1}{3} (v_j^*)_{zzz} = u_0(x_j) - \frac{\partial^2 u_0}{\partial x^2}(x_j) - \tau \frac{\partial u_0}{\partial x}(x_j) - \tau x u_0(x_j) \frac{\partial u_0}{\partial x}(x_j) + r_j^*,
$$

$$
\quad j = 1, 2, \ldots, J - 1
$$

(27)

$$
v_n^* \in Z_n^+ , \quad n = 0, 1, 2, \ldots, N
$$

(28)
According to Taylor expansion, we obtain that:

$$|r_j^t| = O(t^2 + h^4)$$  \hspace{1cm} (29)$$

holds as \( h, \tau \to 0 \).

For the difference solution of the scheme (6)-(9), we have the following priori estimates.

**Lemma 1.** Suppose \( u_0 \in H^1_0[x_L, x_R] \) then the solution of the initial-boundary value problem (1)-(3) satisfies:

$$\| u_n \|_{L^2} \leq C \quad \| u_n \|_{L^2} \leq C \quad \| u_n \|_{L^2} \leq C$$

*Proof.* It follows from (5) that:

$$E(t) = \|u_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 = E(0) = C$$

which yields:

$$\| u_n \|_{L^2} \leq C \quad \| u_n \|_{L^2} \leq C$$

By Sobolev inequality, we have:

$$\| u_n \|_{L^2} \leq C$$

**Lemma 2.** Suppose \( u_0 \in H^1_0[x_L, x_R] \) then the solution of the scheme (6)-(9) satisfies:

$$\| u_n \| \leq C \quad \| u_n \| \leq C \quad \| u_n \| \leq C$$

for \( n = 0, 1, 2, ..., N \).

*Proof.* It follows from Theorem 1 and Lemma 1 that:

$$\frac{1}{2} \left( \|e^{n+1}\|_{L^2}^2 + \|e^{n+1}\|_{L^2}^2 + \|e^{n+1}\|_{L^2}^2 + \|e^{n+1}\|_{L^2}^2 \right) \leq E^o = E^o = C$$

that is:

$$\| u_n \| \leq C \quad \| u_n \| \leq C$$

By discrete Sobolev inequality \([29]\), we have:

$$\| u_n \| \leq C$$

**Theorem 1.** Suppose \( u_0 \in H^1_0[x_L, x_R] \) then the solution \( u_n^o \) of the difference scheme (6)-(9) converges to the solution of the problem (1)-(3) with order \( O(t^2 + h^4) \) by the \( \|\cdot\|_{\infty} \) norm.

*Proof.* Letting:

$$e_j^t = v_j^t - u_j^t$$

and subtracting (6)-(9) from (25)-(28), respectively, we have:

$$r_j^t = (e_j^t)(\tau) - \frac{4}{3}(e_j^t)(\tau) + \frac{1}{3}(e_j^t)(\tau) + \frac{4}{3} (e_j^t)(\tau) - \frac{1}{3} (e_j^t)(\tau) + \phi(v_j^t, v_j^t) - \phi(u_j^t, \phi_j^t) -$$

$$- \kappa(v_j^t, v_j^t) + \kappa(u_j^t, \phi_j^t), \quad j = 1, 2, ..., J - 1, \quad n = 1, 2, ..., N - 1$$  \hspace{1cm} (30)$$

$$e_j^t = 0, \quad j = 0, 1, 2, ..., J$$  \hspace{1cm} (31)$$

$$e_j^t = \frac{4}{3} (e_j^t)(\tau) + \frac{1}{3} (e_j^t)(\tau) = r_j^t, \quad j = 1, 2, ..., J - 1$$  \hspace{1cm} (32)$$

$$e^t \in Z_n^0, \quad n = 0, 1, 2, ..., N$$  \hspace{1cm} (33)$$
Computing the inner product of (32) with $e^1$, and using the boundary condition (33), we get:

$$\|e^1\| + \frac{4}{3} \|e^3\| - \frac{1}{3} \|e^2\| = \langle r^0, e^1 \rangle$$  \hspace{1cm} (34)

From (29), Cauchy-Schwarz inequality and Lemma 1, we obtain:

$$\|e^1\| + \|e^3\| \leq O(\tau^2 + h^*)^2$$  \hspace{1cm} (35)

Computing the inner product of (29) with $2e^3$ and using (33) again, we have:

$$\langle r^0, 2e^3 \rangle = \|e^1\|^2 + \frac{4}{3} \|e^3\|^2 - \frac{1}{3} \|e^2\|^2 + \frac{8}{3} \langle \tau^3_j, e^3 \rangle - \frac{2}{3} \langle \tau^3_j, \tau'^3 \rangle +$$

$$+ 2(\phi(v^0, \tau^3) - \phi(u^0, \tau^3), e^3) - 2(\kappa(v^0, \tau^3) - \kappa(u^0, \tau^3), \tau'^3)$$  \hspace{1cm} (36)

Similarly to (16), we have:

$$\langle \tau^3_j, e^3 \rangle = 0, \quad \langle \tau^3_j, \tau'^3 \rangle = 0$$  \hspace{1cm} (37)

According to Lemma 1, Lemma 2, Theorem 1 and Cauchy-Schwarz inequality, we get:

$$\langle \phi(v^0, \tau^3) - \phi(u^0, \tau^3), e^3 \rangle = \frac{4}{9} h \sum_{j=0}^{n-1} [e^3_j (\tau^3_j + u^0_j (\tau^3_j)), \tau^3_j] - \frac{4}{9} h \sum_{j=0}^{n-1} (e^3_j (\tau^3_j + u^0_j (\tau^3_j)), \tau^3_j) \leq$$

$$\leq C \left( \|e^3\|^2 + \|\tau^3\|^2 + \|\tau'^3\|^2 \right) \leq C \left( \|e^3\|^2 + \|\tau^3\|^2 + \|\tau'^3\|^2 + \|e^{s-1}\|^2 + \|e^{s+1}\|^2 \right)$$  \hspace{1cm} (38)

$$\langle \kappa(v^0, \tau^3) - \kappa(u^0, \tau^3), \tau'^3 \rangle = \frac{1}{9} h \sum_{j=0}^{n-1} [(v^0_j (\tau^3_j) - u^0_j (\tau^3_j)), \tau^3_j] + \frac{1}{9} h \sum_{j=0}^{n-1} [(v^0_j (\tau^3_j) - u^0_j (\tau^3_j)), \tau^3_j] \leq$$

$$\leq C \left( \|e^3\|^2 + \|\tau^3\|^2 + \|\tau'^3\|^2 \right) \leq C \left( \|e^3\|^2 + \|\tau^3\|^2 + \|\tau'^3\|^2 + \|e^{s-1}\|^2 + \|e^{s+1}\|^2 \right)$$  \hspace{1cm} (39)

and

$$\langle r^0, 2e^3 \rangle = \langle r^0, e^{s+1} + e^{s-1} \rangle \leq \|r^0\|^2 + \|e^{s+1}\|^2 + \|e^{s-1}\|^2$$  \hspace{1cm} (40)

Substituting (37)-(40) into (36), we get:

$$\|e^1\|^2 + \frac{4}{3} \|e^3\|^2 - \frac{1}{3} \|e^2\|^2 \leq \|e^{s+1}\|^2 + C \left( \|e^{s+1}\|^2 + \|e^{s-1}\|^2 + \|e^3\|^2 + \|e^{s+1}\|^2 + \|e^3\|^2 + \|e^{s-1}\|^2 \right)$$  \hspace{1cm} (41)

Letting:

$$B^* = \|e^{s+1}\|^2 + \|e^3\|^2 + \frac{4}{3} \|e^{s+1}\|^2 + \frac{4}{3} \|e^3\|^2 - \frac{1}{3} \|e^2\|^2$$

and summing up (41) from 1 to $n$, we have:

$$B^* \leq B_0 + C \tau^2 \sum_{j=1}^{n-1} \|r^j\|^2 + C \tau \sum_{j=0}^{n-1} (\|e^1\|^2 + \|e^3\|^2)$$  \hspace{1cm} (42)

Noticing:

$$\tau \sum_{j=1}^{n} \|r^j\|^2 \leq n \tau \max_{j \in \mathbb{N}} \|r^j\|^2 \leq TO(\tau^2 + h^*)^3$$

From (31) and (35), we have $B_0 = O(\tau^2 + h^*)^2$. Hence, from (42), and Lemma 1, we get:

$$\|e^{s+1}\|^2 + \|e^3\|^2 + \|e^{s-1}\|^2 \leq B^* \leq O(\tau^2 + h^*)^2 + C \tau^2 \sum_{j=0}^{n-1} (\|e^1\|^2 + \|e^3\|^2)$$
By discrete Gronwall inequality [29], we have:
\[ \|e^n\| \leq O(\tau^2 + h^4) \quad \|e^n\|_{\infty} \leq O(\tau^2 + h^4) \]

Finally, by discrete Sobolev inequality [29], we get:
\[ \|e^n\|_{\infty} \leq O(\tau^2 + h^4) \]

This completes the proof of Theorem 1.

Similarly, we can prove the stability of the difference solution.

Theorem 2. Under the conditions of Theorem 1, the solution of the scheme (6)-(9) is stable by the \( \| \cdot \|_{\infty} \) norm.

Numerical experiments

The single solitary-wave solution of RLW eq. (1) is given by:
\[ u(x,t) = A \sec h^4(kx - \omega t + \delta) \]
where
\[ A = \frac{3a^2}{1-a^2}, \quad k = \frac{a}{2}, \quad \omega = \frac{a}{2(1-a^2)} \]
and \( a \) and \( \delta \) are constants.

The scheme (6)-(9) is a linear system of equations which can be solved without iteration. Take and the initial function of the problem (1)-(3) is re-written:
\[ u(x,0) = \sec h^4 \left( \frac{1}{4} \right) \]

In the numerical experiments, we take \( x_L = -50, x_R = 50 \), and \( T = 20 \). The errors in the sense of \( L^\infty \)-norm and \( L^2 \)-norm of the numerical solutions are listed on tab. 1 under different mesh steps \( \tau \) and \( h \). Table 2 shows that the computational and the theoretical orders of the scheme are very close to each other. Table 3 shows the value of \( E^n \) and \( Q^n \) at different time. It indicates that the conservation of the scheme (6)-(9) is very good and it is suitable for long-term computation.

Table 1. The errors estimates of numerical solution with various \( \tau \) and \( h \)

| \( \tau \) | \( h \) | \( ||e^n|| \) | \( ||e^n||_{\infty} \) | \( ||e^n|| \) | \( ||e^n||_{\infty} \) | \( ||e^n|| \) | \( ||e^n||_{\infty} \) |
|-------|-----|---------|---------|---------|---------|---------|---------|
| 0.2   | 0.01| 1.293731 | 1.293731 | 6.219891 | 6.219891 | 8.406610 | 8.406610 |
| 0.05  | 0.05| 1.505045 | 1.505045 | 1.575704 | 1.575704 | 7.156453 | 7.156453 |
| 0.0125| 0.025| 1.828892 | 1.828892 | 2.206277 | 2.206277 | 1.166337 | 1.166337 |
| 0.01  | 0.025| 2.765299 | 2.765299 | 2.765299 | 2.765299 | 1.166337 | 1.166337 |

Table 2. The numerical verification of theoretical accuracy \( O(\tau^2 + h^4) \)

| \( \tau \) | \( h \) | \( ||e^n(\tau,\tau)\|_{\infty} \) | \( ||e^n(\tau,\tau)|_{\tau} \) | \( ||e^n(\tau,\tau)\|_{\infty} \) | \( ||e^n(\tau,\tau)|_{\tau} \) |
|-------|-----|---------|---------|---------|---------|
| 0.2   | 0.1 | 15.389452 | 15.389452 | 15.963269 | 15.963269 |
| 0.05  | 0.05| 15.961462 | 15.961462 | 15.950260 | 15.950260 |
| 0.0125| 0.025| 15.967027 | 15.967027 | 15.967716 | 15.967716 |
| 0.01  | 0.025| 15.971630 | 15.971630 | 15.971630 | 15.971630 |
Table 3. Discrete mass and discrete energy with various $\tau$ and $h$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tau = 0.2$</th>
<th>$h = 0.1$</th>
<th>$\tau = 0.05$</th>
<th>$h = 0.05$</th>
<th>$\tau = 0.0125$</th>
<th>$h = 0.025$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>8.0023652</td>
<td>5.59999999</td>
<td>8.0001481</td>
<td>5.5999999</td>
<td>8.00000090</td>
<td>5.5999999</td>
</tr>
<tr>
<td>$t = 5$</td>
<td>8.0023653</td>
<td>5.59999999</td>
<td>8.0001480</td>
<td>5.5999999</td>
<td>8.00000092</td>
<td>5.5999999</td>
</tr>
<tr>
<td>$t = 10$</td>
<td>8.0023652</td>
<td>5.59999999</td>
<td>8.0001451</td>
<td>5.5999999</td>
<td>8.00000623</td>
<td>5.5999999</td>
</tr>
<tr>
<td>$t = 15$</td>
<td>8.0022799</td>
<td>5.59999999</td>
<td>8.0000633</td>
<td>5.5999999</td>
<td>8.00000370</td>
<td>5.5999999</td>
</tr>
<tr>
<td>$t = 20$</td>
<td>8.0022799</td>
<td>5.59999999</td>
<td>8.0000633</td>
<td>5.5999999</td>
<td>8.00000370</td>
<td>5.5999999</td>
</tr>
</tbody>
</table>

From these computational results, it shows that our proposed algorithm is efficient and reliable.

Acknowledgment

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References


