NEW HIGH-ORDER CONSERVATIVE DIFFERENCE SCHEME FOR RLW EQUATION WITH RICHARDSON EXTRAPOLATION

by

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Numerical solution for the regularized long wave equation is considered by a new three-level conservative implicit finite difference scheme coupled with Richardson extrapolation which has the accuracy of $O(\tau^2 + h^4)$ *. The scheme is a linear system of equations solved without iteration. The conservation properties of the algorithm are verified by computing the discrete mass and discrete energy. Existence and uniqueness of the numerical solution are proved. Convergence and stability of the scheme are also derived using energy method. The results of numerical experiments show that our proposed scheme is efficiency.*

Key words: RLW equation, conservative difference scheme, Richardson extrapolation, stability, convergence

Introduction

Consider the following initial-boundary value problem for the Regularized Long Wave(RLW) equation,

$$u_t + u_x + uu_x - u_{xxt} = 0 \quad (x,t) \in (x_L, x_R) \times (0,T) \tag{1}$$

with an initial condition

$$u(x,0) = u_0(x) \quad x \in [x_L, x_R] \tag{2}$$

and boundary condition

$$u(x_L,t) = u(x_R,t) = 0 \quad t \in [0,T] \tag{3}$$

where $u_0(x)$ is a given known function. The RLW equation is originally introduced to describe the behavior of the undular bore by Peregrine [1] and plays a major role in the study of nonlinear dispersive waves [2] because of its description to a larger number of important physical phenomena, such as shallow water waves and ion acoustic plasma waves.

Mathematical theory for the equation was developed in [3]. Due to nonlinear nature of the RLW equation, few exact solutions exist in the literature [4,5]. Studies mainly consider numerical solution of the problem. These include variational iteration method[6,7], finite difference methods [8-15] and various finite element methods such as the Galerkin method [16-20], the least squares method [21-23] and collocation method with quadratic B-splines[24], cubic B-splines [25] and recent septic splines [26].

The problem (1)-(3) has two conserved quantities: mass and energy, *i.e.*,

$$Q(t) = \int_{x_L}^{x_R} u(x,t)dx = \int_{x_L}^{x_R} u_0(x)dx = Q(0) \tag{4}$$

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and

$$E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 = \|u_0\|_{L_2}^2 + \|(u_0)_x\|_{L_2}^2 = E(0)\,, \tag{5}$$

where $Q(0)$ and $E(0)$ are two positive constants which relate to the initial condition. Zhang et al. pointed out [27] that the conservative difference schemes perform better than the non-conservative ones, and the non-conservative difference schemes may easily show nonlinear "blow-up". In [28], Li and Vu-Quoc pointed out that *"in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation"*. Thus, the purpose of this paper is to present a conservative difference scheme for the initial-boundary value problem (1)-(3). By the Richardson extrapolation, the scheme has the accuracy of $O(\tau^2 + h^2)$ without refined mesh. Moreover, the resulting scheme is a linear system of equations, and it can be solved easily without any iterations.

The paper is organized as follows. In section 2, we give the three-level conservative implicit difference scheme and the two discrete conserved quantities are presented. In section 3, we prove the existence and uniqueness of the scheme. Priori estimate, convergence and stability are proved in section 4 and numerical results are reported in section 5.

Finite difference scheme

Let N, J be any positive integers and $h = \dfrac{x_R - x_L}{J}$ be the step size for the spatial grid such that $x_j = x_L + jh$ ($j = -1, 0, 1, 2, \cdots, J, J+1$). Let τ be the step size for the temporal direction, $t_n = n\tau$ ($n = 0, 1, 2, \cdots, N$) $N = \left[\dfrac{T}{\tau}\right]$.

Denote $u_j^n \approx u(x_j, t_n)$ and

$$Z_h^0 = \{u = (u_j) \,|\, u_{-1} = u_0 = u_J = u_{J+1} = 0,\ \ j = -1, 0, 1, 2, \cdots, J, J+1\}\,.$$

Define

$$(u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h} \qquad (u_j^n)_{\bar x} = \frac{u_j^n - u_{j-1}^n}{h} \qquad (u_j^n)_{\hat x} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

$$(u_j^n)_{\ddot x} = \frac{u_{j+2}^n - u_{j-2}^n}{4h} \qquad (u_j^n)_{\hat t} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau} \qquad \bar u_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2}$$

$$\langle u^n, v^n \rangle = h\sum_{j=1}^{J-1} u_j^n v_j^n \qquad \|u^n\|^2 = \langle u^n, u^n \rangle \qquad \|u^n\|_\infty = \max_{1 \le j \le J-1} |u_j^n|$$

and in the paper, C denotes a general positive constant which may have different values in different occurrences.

Lemma 1. *For a mesh function* $u \in Z_h^0$, *by Cauchy-Schwarz inequality, we have*

$$\|u_{\ddot x}\|^2 \le \|u_{\hat x}\|^2 \le \|u_x\|^2$$

The following conservative difference scheme for the problem (1)-(3) is considered,

$$(u_j^n)_{\hat t} - \frac{4}{3}(u_j^n)_{x\bar x\hat t} + \frac{1}{3}(u_j^n)_{\hat x\hat x t} + \frac{4}{3}(\bar u_j^n)_{\hat x} - \frac{1}{3}(\bar u_j^n)_{\ddot x} + \frac{4}{9}[u_j^n(\bar u_j^n)_x + (u_j^n \bar u_j^n)_x]$$

$$-\frac{1}{9}[u_j^n(\bar u_j^n)_{\ddot x} + (u_j^n \bar u_j^n)_{\ddot x}] = 0 \quad j = 1, 2, \cdots, J-1\,; \quad n = 1, 2, \cdots, N-1 \tag{6}$$

$$u_j^0 = u_0(x_j) \qquad j = 0, 1, 2, \cdots, J \tag{7}$$

$$u_j^1-\frac{4}{3}(u_j^1)_{x\bar{x}}+\frac{1}{3}(u_j^1)_{\hat{x}\hat{x}}=u_0(x_j)-\frac{\partial^2 u_0}{\partial x^2}(x_j)-\tau\frac{\partial u_0}{\partial x}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j)$$

$$j=1,2,\cdots,J-1 \tag{8}$$

$$u^n\in Z_h^0 \quad n=0,1,2,\cdots,N \tag{9}$$

Based on the scheme (6)-(9), the discrete versions of (4) and (5) are obtained as follows,

Theorem 1. The scheme (6)-(9) admits the following invariant,

$$Q^n=\frac{h}{2}\sum_{j=1}^{J-1}(u_j^{n+1}+u_j^n)+\frac{2h}{9}\tau\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_{\hat{x}}-\frac{h}{18}\tau\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_{\ddot{x}}$$

$$=Q^{n-1}=\cdots=Q^0 \tag{10}$$

$$E^n=\frac{1}{2}(\|u^{n+1}\|^2+\frac{4}{3}\|u_x^{n+1}\|^2-\frac{1}{3}\|u_{\hat{x}}^{n+1}\|^2+\|u^n\|^2+\frac{4}{3}\|u_x^n\|^2-\frac{1}{3}\|u_x^n\|^2)$$

$$=E^{n-1}=\cdots=E^0. \tag{11}$$

for $n=1,2,\cdots,N-1$.

Proof. Multiplying (6) with h, then summing up for j from 1 to $J-1$, by the boundary condition (9) and formula of summation by parts [29], we have

$$h\sum_{j=1}^{J-1}\frac{(u_j^{n+1}-u_j^{n-1})}{2\tau}+\frac{4}{9}h\sum_{j=1}^{J-1}u_j^n(\bar{u}_j^n)_{\hat{x}}-\frac{1}{9}h\sum_{j=1}^{J-1}u_j^n(\bar{u}_j^n)_{\ddot{x}}=0 \tag{12}$$

Again since

$$h\sum_{j=1}^{J-1}u_j^n(\bar{u}_j^n)_{\hat{x}}=\frac{h}{2}\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_x-\frac{h}{2}\sum_{j=1}^{J-1}u_j^{n-1}(u_j^n)_x \tag{13}$$

$$h\sum_{j=1}^{J-1}u_j^n(\bar{u}_j^n)_{\ddot{x}}=\frac{h}{2}\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_{\ddot{x}}-\frac{h}{2}\sum_{j=1}^{J-1}u_j^{n-1}(u_j^n)_{\ddot{x}} \tag{14}$$

substitute (13) and (14) into (12), then (10) is obtained.

Taking the inner product of (6) with $2\bar{u}^n$ (that is, $u^{n+1}+u^{n-1}$), according to boundary condition (9), we get

$$\|u^n\|_{\hat{t}}^2+\frac{4}{3}\|u_x^n\|_t^2-\frac{1}{3}\|u_{\hat{x}}^n\|_t^2+\frac{8}{3}\langle\bar{u}_x^n,\bar{u}^n\rangle-\frac{2}{3}\langle\bar{u}_{\ddot{x}}^n,\bar{u}^n\rangle$$

$$+2\langle\varphi(u_j^n,\bar{u}_j^n),\bar{u}^n\rangle-2\langle\kappa(u_j^n,\bar{u}_j^n),\bar{u}^n\rangle=0, \tag{15}$$

where

$$\varphi(u_j^n,\bar{u}_j^n)=\frac{4}{9}[u_j^n(\bar{u}_j^n)_{\hat{x}}+(u_j^n\bar{u}_j^n)_x],$$

$$\kappa(u_j^n,\bar{u}_j^n)=\frac{1}{9}[u_j^n(\bar{u}_j^n)_{\ddot{x}}+(u_j^n\bar{u}_j^n)_{\ddot{x}}].$$

Considering

$$\langle\bar{u}_{\hat{x}}^n,\bar{u}^n\rangle=0 \qquad \langle\bar{u}_{\ddot{x}}^n,\bar{u}^n\rangle=0 \tag{16}$$

$$\langle\varphi(u^n,\bar{u}^n),\bar{u}^n\rangle=0 \tag{17}$$

and

$$\langle\kappa(u^n,\bar{u}^n),\bar{u}^n\rangle=0 \tag{18}$$

Substituting (16)-(18) into (15), we have

$$\frac{1}{2\tau}(\|u^{n+1}\|^2-\|u^{n-1}\|^2)+\frac{2}{3\tau}(\|u_x^{n+1}\|^2-\|u_x^{n-1}\|^2)-\frac{1}{6\tau}(\|u_{\hat{x}}^{n+1}\|^2-\|u_x^{n-1}\|^2)=0 \tag{19}$$

By the definition of E^n, (11) is gotten from (19).

Solvability

Next, we are going to prove the solvability of the finite difference scheme (6)-(9).

Theorem 1. The difference scheme (6)-(9) is uniquely solvable.

Proof. Use the mathematical induction. It is obvious that u^0 and u^1 are uniquely determined by (7) and (8). Now suppose $u^0, u^1, \cdots, u^{n-1}, u^n$ be solved uniquely. Consider the equation of (6) for u^{n+1},

$$\frac{1}{2\tau}u_j^{n+1} - \frac{2}{3\tau}(u_j^{n+1})_{x\bar{x}} + \frac{1}{6\tau}(u_j^{n+1})_{\hat{x}\hat{x}} + \frac{2}{3}(u_j^{n+1})_{x} - \frac{1}{6}(u_j^{n+1})_{\ddot{x}} + \frac{1}{2}\varphi(u_j^n, u_j^{n+1})$$
$$-\frac{1}{2}\kappa(u_j^n, u_j^{n+1}) = 0 \qquad j = 1, 2, \cdots, J-1; \quad n = 1, 2, \cdots, N-1 \quad (20)$$

Computing the inner product of (20) with u^{n+1}, using (9), we obtain

$$\frac{1}{2\tau}\|u^{n+1}\|^2 + \frac{2}{3\tau}\|u_x^{n+1}\|^2 - \frac{1}{6\tau}\|u_{\hat{x}}^{n+1}\|^2 + \frac{2}{3}\langle u_x^{n+1}, u^{n+1}\rangle - \frac{1}{6}\langle u_{\ddot{x}}^{n+1}, u^{n+1}\rangle$$
$$\frac{1}{2}\langle \varphi(u^n, u^{n+1}), u^{n+1}\rangle - \frac{1}{2}\langle \kappa(u^n, u^{n+1}), u^{n+1}\rangle = 0 \qquad (21)$$

Since

$$\langle u_{\hat{x}}^{n+1}, u^{n+1}\rangle = 0 \qquad \langle u_{\ddot{x}}^{n+1}, u^{n+1}\rangle = 0 \qquad (22)$$

$$\langle \varphi(u^n, u^{n+1}), u^{n+1}\rangle = 0 \qquad (23)$$

and

$$\langle \kappa(u^n, u^{n+1}), u^{n+1}\rangle = 0 \qquad (24)$$

Substituting (22)-(24) into (21), by Lemma 1, we have

$$\|u^{n+1}\|^2 + \|u_x^{n+1}\|^2 \le 0 .$$

That is, (20) has only a trivial solution. Therefore, (6) determines u_j^{n+1} uniquely. This completes the proof.

Convergence and stability

Let $v(x,t)$ be the solution of problem (1)-(3) and $v_j^n = u(x_j, t_n)$, then the truncation error of the scheme (6)-(9) is derived as follows,

$$r_j^n = (v_j^n)_{\hat{t}} - \frac{4}{3}(v_j^n)_{x\bar{x}\hat{t}} + \frac{1}{3}(v_j^n)_{\hat{x}\hat{x}t} + \frac{4}{3}(\bar{v}_j^n)_{\hat{x}} - \frac{1}{3}(\bar{v}_j^n)_{\ddot{x}} + \varphi(v_j^n, \bar{v}_j^n) - \kappa(v_j^n, \bar{v}_j^n)$$
$$j = 1, 2, \cdots, J-1; \quad n = 1, 2, \cdots, N-1 \qquad (25)$$

$$v_j^0 = u_0(x_j) \qquad j = 0, 1, 2, \cdots, J \qquad (26)$$

$$v_j^1 - \frac{4}{3}(v_j^1)_{x\bar{x}} + \frac{1}{3}(v_j^1)_{\hat{x}\hat{x}} = u_0(x_j) - \frac{\partial^2 u_0}{\partial x^2}(x_j) - \tau\frac{\partial u_0}{\partial x}(x_j) - \tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j) + r_j^0$$
$$j = 1, 2, \cdots, J-1 \qquad (27)$$

$$v^n \in Z_h^0 \qquad n = 0, 1, 2, \cdots, N \qquad (28)$$

According to Taylor expansion, we obtain that

$$|r_j^n| = O(\tau^2 + h^4) \qquad (29)$$

holds as $h, \tau \to 0$.

For the difference solution of the scheme (2.1)-(2.4), we have the following priori estimates.

Lemma 1. Suppose $u_0 \in H_0^1[x_L, x_R]$*, then the solution of the initial-boundary value problem (1)-(3) satisfies*

$$\|u\|_{L_2} \le C \quad \|u_x\|_{L_2} \le C \quad \|u\|_{L_\infty} \le C .$$

Proof. It follows from (5) that

$$E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 = E(0) = C ,$$

which yields

$$\|u\|_{L_2} \le C \quad \|u_x\|_{L_2} \le C .$$

By Sobolev inequality, we have

$$\|u\|_{L_\infty} \le C .$$

Lemma 2. Suppose $u_0 \in H_0^1[x_L, x_R]$*, then the solution of the scheme (6)-(9) satisfies*

$$\|u^n\| \le C \quad \|u_x^n\| \le C \quad \|u^n\|_\infty \le C .$$

for $n = 0,1,2,\cdots,N$.

Proof. It follows from Theorem 1 and Lemma 1 that

$$\frac{1}{2}(\|u^{n+1}\|^2 + \|u_x^{n+1}\|^2 + \|u^n\|^2 + \|u_x^n\|^2) \le E^n = E^0 = C .$$

that is,

$$\|u^n\| \le C \quad \|u_x^n\| \le C$$

By discrete Sobolev inequality[29], we have

$$\|u^n\|_\infty \le C .$$

Theorem 1. *Suppose* $u_0 \in H_0^1[x_L, x_R]$*, then the solution* u^n *of the difference scheme (6)-(9) converges to the solution of the problem (1)-(3) with order* $O(\tau^2 + h^4)$ *by the* $\|\cdot\|_\infty$ *norm.*

Proof. Letting

$$e_j^n = v_j^n - u_j^n$$

and subtracting (6)-(9) from (25)-(28), respectively, we have

$$r_j^n = (e_j^n)_{\hat{t}} - \frac{4}{3}(e_j^n)_{x\bar{x}\hat{t}} + \frac{1}{3}(e_j^n)_{\hat{x}\hat{x}t} + \frac{4}{3}(\overline{e}_j^n)_{\hat{x}} - \frac{1}{3}(\overline{e}_j^n)_{\ddot{x}} + \varphi(v_j^n, \overline{v}_j^n) - \varphi(u_j^n, \overline{u}_j^n)$$
$$-\kappa(v_j^n, \overline{v}_j^n) + \kappa(u_j^n, \overline{u}_j^n) \qquad j = 1,2,\cdots,J-1 \quad n = 1,2,\cdots,N-1 \qquad (30)$$

$$e_j^0 = 0 \qquad j = 0,1,2,\cdots,J \qquad (31)$$

$$e_j^1 - \frac{4}{3}(e_j^1)_{x\bar{x}} + \frac{1}{3}(e_j^1)_{\hat{x}\hat{x}} = r_j^0 \qquad j = 1,2,\cdots,J-1 \qquad (32)$$

$$e^n \in Z_h^0 \qquad n = 0,1,2,\cdots,N \qquad (33)$$

Computing the inner product of (32) with e^1, and using the boundary condition (33), we get

$$\|e^1\|^2 + \frac{4}{3}\|e_x^1\|^2 - \frac{1}{3}\|e_{\hat{x}}^1\|^2 = \langle r^0, e^1 \rangle \qquad (34)$$

From (29), Cauchy-Schwarz inequality and Lemma 1, we obtain

$$\|e^1\|^2 + \|e_x^1\|^2 \le O(\tau^2 + h^4)^2 \qquad (35)$$

Computing the inner product of (29) with $2\overline{e}^n$, and using (33) again, we have

$$\langle r^n, 2\bar{e}^n \rangle = \|e^n\|_{\hat{t}}^2 + \frac{4}{3}\|e_x^n\|_t^2 - \frac{1}{3}\|e_{\hat{x}}^n\|_t^2 + \frac{8}{3}\langle \bar{e}_x^n, \bar{e}^n \rangle - \frac{2}{3}\langle \bar{e}_{\ddot{x}}^n, \bar{e}^n \rangle$$
$$+2\langle \varphi(v^n, \bar{v}^n) - \varphi(u^n, \bar{u}^n), \bar{e}^n \rangle - 2\langle \kappa(v^n, \bar{v}^n) - \kappa(u^n, \bar{u}^n), \bar{e}^n \rangle \tag{36}$$

Similarly to (16), we have

$$\langle \bar{e}_{\hat{x}}^n, \bar{e}^n \rangle = 0 \quad \langle \bar{e}_{\ddot{x}}^n, \bar{e}^n \rangle = 0 \tag{37}$$

According to Lemma 1, Lemma 2, Theorem 1 and Cauchy-Schwartz inequality, we get

$$\begin{aligned} \langle \varphi(v^n, \bar{v}^n) - \varphi(u^n, \bar{u}^n), \bar{e}^n \rangle &= \frac{4}{9}h\sum_{j=1}^{J-1}[e_j^n(\bar{v}_j^n)_{\hat{x}} + u_j^n(\bar{e}_j^n)_x]\bar{e}_j^n - \frac{4}{9}h\sum_{j=1}^{J-1}(e_j^n\bar{v}_j^n + u_j^n\bar{e}_j^n)(\bar{e}_j^n)_x \\ &\le C(\|e^n\|^2 + \|\bar{e}^n\|^2 + \|\bar{e}_{\hat{x}}^n\|^2) \\ &\le C(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2) \end{aligned} \tag{38}$$

$$\begin{aligned} \langle \kappa(v^n, \bar{v}^n) - \kappa(u^n, \bar{u}^n), \bar{e}^n \rangle &= \frac{1}{9}h\sum_{j=1}^{J-1}[v_j^n(\bar{v}_j^n)_{\ddot{x}} - u_j^n(\bar{u}_j^n)_{\ddot{x}}]\bar{e}_j^n + \frac{1}{9}h\sum_{j=1}^{J-1}[(v_j^n\bar{v}_j^n)_{\ddot{x}} - (u_j^n\bar{u}_j^n)_{\ddot{x}}]\bar{e}_j^n \\ &\le C(\|e^n\|^2 + \|\bar{e}^n\|^2 + \|\bar{e}_{\ddot{x}}^n\|^2) \\ &\le C(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2) \end{aligned} \tag{39}$$

and

$$\langle r^n, 2\bar{e}^n \rangle = \langle r^n, e^{n+1} + e^{n-1} \rangle \le \|r^n\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}\|^2 \tag{40}$$

Substituting (37)-(40) into (36), we get

$$\|e^n\|_{\hat{t}}^2 + \frac{4}{3}\|e_x^n\|_t^2 - \frac{1}{3}\|e_{\hat{x}}^n\|_t^2 \le \|r^n\|^2 + C(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2) \tag{41}$$

Letting

$$B^n = \|e^{n+1}\|^2 + \|e^n\|^2 + \frac{4}{3}\|e_x^{n+1}\|^2 + \frac{4}{3}\|e_x^n\|^2 - \frac{1}{3}\|e_{\hat{x}}^{n+1}\|^2 - \frac{1}{3}\|e_x^n\|^2$$

and summing up (41) from 1 to n, we have

$$B^n \le B^0 + C\tau\sum_{l=1}^{n}\|r^l\|^2 + C\tau\sum_{l=0}^{n}(\|e^l\|^2 + \|e_x^l\|^2) \tag{42}$$

Noticing

$$\tau\sum_{l=1}^{n}\|r^l\|^2 \le n\tau\max_{1\le l\le n}\|r^l\|^2 \le T \cdot O(\tau^2 + h^4)^2.$$

From (31) and (35), we have $B^0 = O(\tau^2 + h^4)^2$. Hence, from (42), and Lemma 1, we get

$$\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 \le B^n \le O(\tau^2 + h^4)^2 + C\tau\sum_{l=0}^{n+1}(\|e^l\|^2 + \|e_x^l\|^2)$$

By discrete Gronwall inequality [29], we have

$$\|e^n\| \le O(\tau^2 + h^4) \quad \|e_x^n\| \le O(\tau^2 + h^4)$$

Finally, by discrete Sobolev inequality [29], we get

$$\|e^n\|_\infty \le O(\tau^2 + h^4).$$

This completes the proof of Theorem 1.

Similarly, we can prove the stability of the difference solution.

Theorem 2. Under the conditions of Theorem 1, the solution of the scheme (6)-(9) is stable by the $\|\cdot\|_\infty$

norm.

Numerical experiments

The single solitary-wave solution of RLW equation (1) is given by,

$$u(x,t) = A\sec h^2(kx - \omega t + \delta),$$

where

$$A = \frac{3a^2}{1-a^2} \quad k = \frac{a}{2} \quad \omega = \frac{a}{2(1-a^2)},$$

and a and δ are constants.

The scheme (6)-(9) is a linear system of equations which can be solved without iteration. Take $a = \frac{1}{2}$, $\delta = 0$ and the initial function of the problem (1)-(3) is rewritten as

$$u(x,0) = \sec h^2(\frac{1}{4}x).$$

In the numerical experiments, we take $x_L = -50$, $x_R = 50$, and $T = 20$. The errors in the sense of L_∞-norm and L_2-norm of the numerical solutions are listed on Table 1 under different mesh steps τ and h. Table 2 shows that the computational and the theoretical orders of the scheme are very close to each other. Table 3 shows the value of E^n and Q^n at different time. It indicates that the conservation of the scheme (6)-(9) is very good and it is suitable for long-term computation.

Table 1. The errors estimates of numerical solution with various τ and h.

	$\tau = 0.2 \quad h = 0.1$		$\tau = 0.05 \quad h = 0.05$		$\tau = 0.0125 \quad h = 0.025$	
	$\lVert e^n \rVert$	$\lVert e^n \rVert_\infty$	$\lVert e^n \rVert$	$\lVert e^n \rVert_\infty$	$\lVert e^n \rVert$	$\lVert e^n \rVert_\infty$
t=5	1.293731e-2	6.219891e-3	8.406610e-4	4.019759e-4	5.266221e-5	2.520183e-5
t=10	2.473412e-2	1.122209e-2	1.575704e-3	7.156453e-4	9.871933e-5	4.483281e-5
t=15	3.472503e-2	1.505045e-2	2.206277e-3	9.588093e-4	1.381956e-4	6.004674e-5
t=20	4.338962e-2	1.828892e-2	2.765299e-3	1.166337e-3	1.740699e-4	7.302556e-5

Table 2. The numerical verification of theoretical accuracy $O(\tau^2 + h^4)$.

	$\lVert e^n(h,\tau)\rVert \Big/ \left\lVert e^{4n}\left(\frac{h}{2},\frac{\tau}{4}\right)\right\rVert$			$\lVert e^n(h,\tau)\rVert_\infty \Big/ \left\lVert e^{4n}\left(\frac{h}{2},\frac{\tau}{4}\right)\right\rVert_\infty$		
	$\tau = 0.2$ $h = 0.1$	$\tau = 0.05$ $h = 0.05$	$\tau = 0.0125$ $h = 0.025$	$\tau = 0.2$ $h = 0.1$	$\tau = 0.05$ $h = 0.05$	$\tau = 0.0125$ $h = 0.025$
t=5	—	15.389452	15.963269	—	15.473292	15.950260
t=10	—	15.697181	15.961462	—	15.681087	15.962535
t=15	—	15.739194	15.964880	—	15.697027	15.967716
t=20		15.690750	15.976137		15.680645	15.971630

Table 3. Discrete mass and discrete energy with various τ and h.

	$\tau = 0.2 \quad h = 0.1$		$\tau = 0.05 \quad h = 0.05$		$\tau = 0.0125 \quad h = 0.025$	
	Q^n	E^n	Q^n	E^n	Q^n	E^n
t=0	8.0023652	5.5999999	8.0001481	5.5999999	8.0000090	5.5999999
t=5	8.0023653	5.5999999	8.0001481	5.5999999	8.0000092	5.5999999

t=10	8.0023652	5.5999999	8.0001480	5.5999999	8.0000091	5.5999999
t=15	8.0023623	5.5999999	8.0001451	5.5999999	8.0000062	5.5999999
t=20	8.0022799	5.5999999	8.0000633	5.5999999	8.0000037	5.5999999

From these computational results, it shows that our proposed algorithm is efficient and reliable.

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