NEW HIGH-ORDER CONSERVATIVE DIFFERENCE SCHEME FOR RLW EQUATION WITH RICHARDSON EXTRAPOLATION

by

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Numerical solution for the regularized long wave equation is considered by a new three-level conservative implicit finite difference scheme coupled with Richardson extrapolation which has the accuracy of $O(\tau^2 + h^4)$. The scheme is a linear system of equations solved without iteration. The conservation properties of the algorithm are verified by computing the discrete mass and discrete energy. Existence and uniqueness of the numerical solution are proved. Convergence and stability of the scheme are also derived using energy method. The results of numerical experiments show that our proposed scheme is efficiency.

Key words: RLW equation, conservative difference scheme, Richardson extrapolation, stability, convergence

Introduction

Consider the following initial-boundary value problem for the Regularized Long Wave (RLW) equation,

$$u_t + u_x + uu_x - u_{xxt} = 0 \quad (x, t) \in (x_L, x_R) \times (0, T)$$

(1)

with an initial condition

$$u(x, 0) = u_0(x) \quad x \in [x_L, x_R]$$

(2)

and boundary condition

$$u(x_L, t) = u(x_R, t) = 0 \quad t \in [0, T]$$

(3)

where Error! Reference source not found. is a given known function. The RLW equation is originally introduced to describe the behavior of the undular bore by Peregrine [1] and plays a major role in the study of nonlinear dispersive waves [2] because of its description to a larger number of important physical phenomena, such as shallow water waves and ion acoustic plasma waves.

Mathematical theory for the equation was developed in [3]. Due to nonlinear nature of the RLW equation, few exact solutions exist in the literature [4,5]. Studies mainly consider numerical solution of the problem. These include variational iteration method[6,7], finite difference methods [8-15] and various finite element methods such as the Galerkin method [16-20], the least squares method [21-23] and collocation method with quadratic B-splines[24], cubic B-splines [25] and recent septic splines [26].

The problem (1)-(3) has two conserved quantities: mass and energy, i.e.,

$$Q(t) = \int_{x_L}^{x_R} u(x, t) dx = \int_{x_L}^{x_R} u_0(x) dx = Q(0)$$

(4)
and

\[ E(t) = \|u(t)\|^2_{L^2} + \|u(t)\|^2_{L^2} = \|u(0)\|^2_{L^2} + \|(u_0)\|^2_{L^2} = E(0), \tag{5} \]

where Error! Reference source not found. and Error! Reference source not found. are two positive constants which relate to the initial condition. Zhang et al. pointed out [27] that the conservative difference schemes perform better than the non-conservative ones, and the non-conservative difference schemes may easily show nonlinear "blow-up". In [28], Li and Vu-Quoc pointed out that "in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation". Thus, the purpose of this paper is to present a conservative difference scheme for the initial-boundary value problem (1)-(3). By the Richardson extrapolation, the scheme has the accuracy of Error! Reference source not found. without refined mesh. Moreover, the resulting scheme is a linear system of equations, and it can be solved easily without any iterations.

The paper is organized as follows. In section 2, we give the three-level conservative implicit difference scheme and the two discrete conserved quantities are presented. In section 3, we prove the existence and uniqueness of the scheme. Priori estimate, convergence and stability are proved in section 4 and numerical results are reported in section 5.

**Finite difference scheme**

Let Error! Reference source not found. be any positive integers and Error! Reference source not found. be the step size for the spatial grid such that Error! Reference source not found.. Let \( \tau \) be the step size for the temporal direction, \( T_n = n \tau \ (n = 0, 1, 2, L, \ N) \). Denote Error! Reference source not found. and

\[ Z_0^n = \{ u = (u_j) | u_{j-1} = u_0 = u_j = u_{j+1} = 0, \ j = -1, 0, 1, 2, L, J, J + 1 \}. \]

Define

\[ (u^n_j)_x = \frac{u^n_{j+1} - u^n_j}{h} \quad (u^n_j)_t = \frac{u^n_j - u^n_{j+1}}{2h} \quad (u^n_j)_{xx} = \frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{4h} \]

\[ (u^n_j)_x = \frac{u^n_{j+1} - u^n_{j-1}}{2} \quad (u^n_j)_t = \frac{u^n_j - u^n_{j+1}}{2\tau} \quad \bar{u}^n_j = \frac{u^n_j + u^n_{j+1}}{2} \]

\[ \langle u^n, v^n \rangle = h \sum_{j=1}^{J+1} u^n_j v^n_j \quad \|u^n\|^2 = \langle u^n, u^n \rangle \quad \|u^n\|_\infty = \max_{1 \leq j \leq J+1} |u^n_j| \]

and in the paper, Error! Reference source not found. denotes a general positive constant which may have different values in different occurrences.

**Lemma 1.** For a mesh function Error! Reference source not found., by Cauchy-Schwarz inequality, we have

\[ \|u_0\|^2 \leq \|u_1\|^2 \leq \|u_N\|^2 \]

The following conservative difference scheme for the problem (1)-(3) is considered,

\[ (u^n_j)_t - \frac{4}{3} (u^n_j)_{xx} + \frac{1}{3} (u^n_j)_{x} + \frac{4}{3} (\bar{u}^n_j)_{x} + \frac{1}{3} (\bar{u}^n_j)_{x} = \frac{4}{9} [u^n_j (\bar{u}^n_j)_{x} + (u^n_j \bar{u}^n_j)] - \frac{1}{9} [u^n_j (\bar{u}^n_j)_{x} + (u^n_j \bar{u}^n_j)] = 0 \quad j = 1, 2, L, J - 1; \quad n = 1, 2, L, N - 1 \tag{6} \]
\[ u_0^0 = u_0(x_j) \quad j = 0, 1, 2, L, J \quad (7) \]
\[ u_j^0 = \frac{4}{3} (u^0_j)^x + \frac{1}{3} (u^0_j)^{xx} = u_0(x_j) - \frac{\partial^2 u_0}{\partial x^2}(x_j) - \tau \frac{\partial u_0}{\partial x}(x_j) - \tau u_0(x_j) \frac{\partial u_0}{\partial x}(x_j) \quad j = 1, 2, L, J - 1 \quad (8) \]
\[ u^n \in Z^n_0 \quad n = 0, 1, 2, L, N \quad (9) \]

Based on the scheme (6)-(9), the discrete versions of (4) and (5) are obtained as follows,

**Theorem 1.** The scheme (6)-(9) admits the following invariant,

\[ Q^n = \frac{h}{2} \sum_{j=1}^{J-1} (u^{n+1}_j + u^n_j) + \frac{2h \tau}{9} \sum_{j=1}^{J-1} u^n_j (u^{n+1})_z - \frac{h \tau}{18} \sum_{j=1}^{J-1} u^n_j (u^{n+1})_z \mathbf{z} = 0 \quad (10) \]
\[ E^n = \frac{1}{2} \left( \|u^{n+1}\|^2_z + \frac{4}{3} \|u^{n+1}\|^2_{z z} - \frac{1}{3} \|u^n\|^2_z + \frac{4}{3} \|u^n\|^2_{z z} - \frac{1}{3} \|u^n\|^2 \right) \]
\[ = E^{n-1} = L = E^0. \quad (11) \]

For **Error! Reference source not found.**

**Proof.** Multiplying (6) with **Error! Reference source not found.**, then summing up for **Error! Reference source not found.** from 1 to **Error! Reference source not found.**, by the boundary condition (9) and formula of summation by parts [29], we have

\[ \frac{h}{2} \sum_{j=1}^{J-1} (u^{n+1}_j - u^{n-1}_j) + \frac{4h \tau}{9} \sum_{j=1}^{J-1} u^n_j (u^{n+1})_z - \frac{h \tau}{18} \sum_{j=1}^{J-1} u^n_j (u^{n+1})_z \mathbf{z} = 0 \quad (12) \]

Again since

\[ h \sum_{j=1}^{J-1} u^n_j (\overline{u}_j^n)_z = h \sum_{j=1}^{J-1} u^n_j (u^{n+1})_z - h \sum_{j=1}^{J-1} (u^{n+1})_z \mathbf{z} \quad (13) \]
\[ h \sum_{j=1}^{J-1} u^n_j (\overline{u}_j^n)_z = h \sum_{j=1}^{J-1} u^n_j (u^{n+1})_z - h \sum_{j=1}^{J-1} (u^{n+1})_z \mathbf{z} \quad (14) \]

Substitute (13) and (14) into (12), then (10) is obtained.

Taking the inner product of (6) with **Error! Reference source not found.**, according to boundary condition (9) and formula of summation by parts [29], we get

\[ \|u^n\|^2_z + \frac{4}{3} \|u^n\|^2_{z z} - \frac{1}{3} \|u^n\|^2_z + \frac{4}{3} \|u^n\|^2_{z z} - \frac{1}{3} \|u^n\|^2 \]
\[ + 2 \langle \varphi(u^n_j, \overline{u}_j^n), \overline{u}_j^n \rangle - 2 \langle \kappa(u^n_j, \overline{u}_j^n), \overline{u}_j^n \rangle = 0, \quad (15) \]

where

\[ \varphi(u^n_j, \overline{u}_j^n) = \frac{4}{9} [u^n_j (\overline{u}_j^n)_z + (u^n_j \overline{u}_j^n)_z], \]
\[ \kappa(u^n_j, \overline{u}_j^n) = \frac{1}{9} [u^n_j (\overline{u}_j^n)_z + (u^n_j \overline{u}_j^n)_z] \]

Considering

\[ \langle \overline{u}_j^n, \overline{u}_j^n \rangle = 0 \quad (16) \]
\[ \langle \varphi(u^n_j, \overline{u}_j^n), \overline{u}_j^n \rangle = 0 \quad (17) \]

and

\[ \langle \kappa(u^n_j, \overline{u}_j^n), \overline{u}_j^n \rangle = 0 \quad (18) \]

Substituting (16)-(18) into (15), we have
\[
\frac{1}{2\tau} \left( \|u^{n+1}_x\|^2 - \|u^n_x\|^2 \right) + \frac{2}{3\tau} \left( \|u^{n+1}\|^2 - \|u^n\|^2 \right) - \frac{1}{6\tau} \left( \|u^{n+1}_x\|^2 - \|u^n_x\|^2 \right) = 0 \tag{19}
\]

By the definition of Error! Reference source not found., (11) is gotten from (19).

Solvability

Next, we are going to prove the solvability of the finite difference scheme (6)-(9).

**Theorem 1.** The difference scheme (6)-(9) is uniquely solvable.

**Proof.** Use the mathematical induction. It is obvious that Error! Reference source not found. and Error! Reference source not found. are uniquely determined by (7) and (8). Now suppose Error! Reference source not found. be solved uniquely. Consider the equation of (6) for Error! Reference source not found.,

\[
\frac{1}{2\tau} \|u^{n+1}_x\|^2 + \frac{2}{3\tau} \|u^{n+1}\|^2 - \frac{1}{6\tau} \|u^{n+1}_x\|^2 = \frac{1}{2} \langle \phi(u^n, u^{n+1}), u^{n+1} \rangle - \frac{1}{2} \langle \kappa(u^n, u^{n+1}), u^{n+1} \rangle = 0 \tag{21}
\]

Since

\[
\langle u^{n+1}_x, u^{n+1} \rangle = 0, \quad \langle u^{n+1}, u^{n+1} \rangle = 0 \tag{22}
\]

\[
\langle \phi(u^n, u^{n+1}), u^{n+1} \rangle = 0 \tag{23}
\]

and

\[
\langle \kappa(u^n, u^{n+1}), u^{n+1} \rangle = 0 \tag{24}
\]

Substituting (22)-(24) into (21), by Lemma 1, we have

\[
\|u^{n+1}_x\|^2 + \|u^{n+1}\|^2 \leq 0.
\]

That is, (20) has only a trivial solution. Therefore, (6) determines Error! Reference source not found. uniquely. This completes the proof.

Convergence and stability

Let Error! Reference source not found. be the solution of problem (1)-(3) and Error! Reference source not found., then the truncation error of the scheme (6)-(9) is derived as follows,

\[
r^{n}_j = (v^n)_i - \frac{4}{3} (v^n)_{i\pm 1} + \frac{1}{3} (v^n)_{i\pm 2} + \frac{1}{3} (\overline{v}^n)_{i\pm 1} - \frac{1}{3} (\overline{v}^n)_{i\pm 2} + \phi(v^n, \overline{v}^n) - \kappa(v^n, \overline{v}^n)
\]

\[
j = 1, 2, L, J - 1; \quad n = 1, 2, L, N - 1 \tag{25}
\]

\[
v^{0}_j = u_0(x_j) \quad j = 0, 1, 2, L, J \tag{26}
\]

\[
v^{1}_j = u_0(x_j) - \frac{\partial^2 u_0}{\partial x^2}(x_j) - \tau \frac{\partial u_0}{\partial x}(x_j) - \tau u_0(x_j) \frac{\partial u_0}{\partial x}(x_j) + r^{0}_j
\]

\[
j = 1, 2, L, J - 1 \tag{27}
\]
According to Taylor expansion, we obtain that
\[ |r_j^n| = O(\tau^2 + h^4) \]  \hspace{1cm} (29)
holds as Error! Reference source not found..

For the difference solution of the scheme (2.1)-(2.4), we have the following priori estimates.

**Lemma 1.** Suppose Error! Reference source not found., then the solution of the initial-boundary value problem (1)-(3) satisfies
\[ \|u\|_{L_2} \leq C \quad \|u_x\|_{L_2} \leq C \quad \|u\|_{L_{\infty}} \leq C . \]

**Proof.** It follows from (5) that
\[ E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 = E(0) = C , \]
which yields
\[ \|u\|_{L_2} \leq C \quad \|u_x\|_{L_2} \leq C . \]

By Sobolev inequality, we have
Error! Reference source not found..

**Lemma 2.** Suppose Error! Reference source not found., then the solution of the scheme (6)-(9) satisfies
\[ \|u^n\| \leq C \quad \|u^n_x\| \leq C \quad \|u^n\|_{L_{\infty}} \leq C . \]

for Error! Reference source not found..

**Proof.** It follows from Theorem 1 and Lemma 1 that
\[ \frac{1}{2} (\|u^{n+1}\|^2 + \|u^{n+1}_x\|^2 + \|u^n\|^2 + \|u^n_x\|^2) \leq E^n = E^0 = C . \]
that is,
\[ \|u^n\| \leq C \quad \|u^n_x\| \leq C . \]

By discrete Sobolev inequality[29], we have
Error! Reference source not found..

**Theorem 1.** Suppose Error! Reference source not found., then the solution Error! Reference source not found. of the difference scheme (6)-(9) converges to the solution of the problem (1)-(3) with order Error! Reference source not found. by the Error! Reference source not found. norm.

**Proof.** Letting
\[ e^n_j = v^n_j - u^n_j \]
and subtracting (6)-(9) from (25)-(28), respectively, we have
\[ r^n_j = (e^n_j)_{j,i} - \frac{4}{3} (e^n_j)_{i,j} + \frac{1}{3} (e^n_j)_{i,i} + \frac{4}{3} (\bar{e}^n_j)_{i,j} - \frac{1}{3} (\bar{e}^n_j)_{i,i} = \varphi(v^n_j, \bar{v}^n_j) - \varphi(u^n_j, \bar{u}^n_j) - \kappa(v^n_j, \bar{v}^n_j) + \kappa(u^n_j, \bar{u}^n_j) \]
\[ -j = 1,2,\ldots,J - 1 \quad n = 1,2,\ldots,N - 1 \quad (30) \]
\[ e^0_j = 0 \quad j = 0,1,2,\ldots,J \quad (31) \]
\[ e^j_j - \frac{4}{3} (e^j)_{i,j} + \frac{1}{3} (e^j)_{i,i} = r^0_j \quad j = 1,2,\ldots,J - 1 \quad (32) \]
\[ e^n \in Z^n_n \quad n = 0,1,2,\ldots,N \quad (33) \]
Computing the inner product of (32) with Error! Reference source not found., and
using the boundary condition (33), we get
\[ \|e^1\|^2 + \frac{4}{3} \|e^2\|^2 - \frac{1}{3} \|e^3\|^2 = \langle r^0, e \rangle \]  
(34)

From (29), Cauchy-Schwarz inequality and Lemma 1, we obtain
\[ \|e^1\|^2 + \|e^2\|^2 \leq O(\tau^2 + h^4) \]  
(35)

Computing the inner product of (29) with Error! Reference source not found., and using (33) again, we have
\[ \langle r^n, 2\overline{e}^n \rangle = \|e^n\|^2 + \frac{4}{3} \|e'^n\|^2 - \frac{1}{3} \|e'^{n+1}\|^2 + \frac{2}{3} \langle \overline{e}^n, \overline{e}^n \rangle - 2\langle \phi(v^n, \overline{v}^n) - \phi(u^n, \overline{u}^n), \overline{e}^n \rangle - 2\langle \kappa(v^n, \overline{v}^n) - \kappa(u^n, \overline{u}^n), \overline{e}^n \rangle \]  
(36)

Similarly to (16), we have
\[ \langle \overline{e}^n, \overline{e}^n \rangle = 0 \quad \langle \overline{e}^0, \overline{e}^n \rangle = 0 \]  
(37)

According to Lemma 1, Lemma 2, Theorem 1 and Cauchy-Schwarz inequality, we get
\[ \langle \phi(v^n, \overline{v}^n) - \phi(u^n, \overline{u}^n), \overline{e}^n \rangle = \frac{4}{9} h \sum_{j=1}^{n} (e^n_j (\overline{v}^n_j)_j + u^n_j (\overline{u}^n_j)_j) \overline{e}^n_j - \frac{4}{9} h \sum_{j=1}^{n} (e'^n_j (\overline{v}^n_j)_j + u'^n_j (\overline{u}^n_j)_j) \overline{e}^n_j \] \leq C(\|e^n\|^2 + \|e'^n\|^2 + \|e'^{n+1}\|^2)
\leq C(\|e^n\|^2 + \|e'^n\|^2 + \|e'^{n+1}\|^2 + \|e'^{n-1}\|^2)
(38)

\[ \langle \kappa(v^n, \overline{v}^n) - \kappa(u^n, \overline{u}^n), \overline{e}^n \rangle = \frac{1}{9} h \sum_{j=1}^{n} (v^n_j (\overline{v}^n_j)_j - u^n_j (\overline{u}^n_j)_j) \overline{e}^n_j + \frac{1}{9} h \sum_{j=1}^{n} (v'^n_j (\overline{v}^n_j)_j - u'^n_j (\overline{u}^n_j)_j) \overline{e}^n_j \] \leq C(\|e^n\|^2 + \|e'^n\|^2 + \|e'^{n+1}\|^2 + \|e'^{n-1}\|^2)
\leq C(\|e^n\|^2 + \|e'^n\|^2 + \|e'^{n+1}\|^2 + \|e'^{n-1}\|^2)
(39)

and
\[ \langle r^n, 2\overline{e}^n \rangle = \langle r^n, e'^{n+1} + e'^{-1} \rangle \leq \|r^n\|^2 + \|e'^{n+1}\|^2 + \|e'^{-1}\|^2 \]  
(40)

Substituting (37)-(40) into (36), we get
\[ \|e^n\|^2 + \frac{4}{3} \|e'^n\|^2 - \frac{1}{3} \|e'^{n+1}\|^2 \leq \|r^n\|^2 + C(\|e'^{n+1}\|^2 + \|e'^n\|^2 + \|e'^{n+1}\|^2 + \|e'^{n-1}\|^2) \]  
(41)

Letting
\[ B^n = \|e'^{n+1}\|^2 + \|e'^n\|^2 + \frac{4}{3} \|e'^{n+1}\|^2 + \frac{4}{3} \|e'^n\|^2 - \frac{1}{3} \|e'^{n+1}\|^2 - \frac{1}{3} \|e'^n\|^2 \]
and summing up (41) from Error! Reference source not found. to Error! Reference source not found., we have
\[ B^n \leq B^0 + C\tau \sum_{l=1}^{n} \|r^l\|^2 + C\tau \sum_{l=0}^{n} (\|e^l\|^2 + \|e'^l\|^2) \]  
(42)

Noticing
\[ \tau \sum_{l=1}^{n} \|r^l\|^2 \leq n\tau \max_{1\leq l\leq n} \|r^l\|^2 \leq T \cdot O(\tau^2 + h^4)^2. \]

From (31) and (35), we have Error! Reference source not found.. Hence, from (42), and Lemma 1, we get
\[ \|e^{n+1}\| + \|e^n\| + \|e^{n+1}\| + \|e^n\| \leq B^n \leq O(\tau^2 + h^4) + C\tau \sum_{j=0}^{n+1} (\|e^j\| + \|e^j\|) \]

By discrete Gronwall inequality [29], we have
\[ \|e^n\| \leq O(\tau^2 + h^4) \]
\[ \|e^n\| \leq O(\tau^2 + h^4) \]

Finally, by discrete Sobolev inequality [29], we get
\[ \|e^n\| \leq O(\tau^2 + h^4) . \]

This completes the proof of Theorem 1.

Similarly, we can prove the stability of the difference solution.

Theorem 2. Under the conditions of Theorem 1, the solution of the scheme (6)-(9) is stable by the \( \| \cdot \|_\infty \) norm.

Numerical experiments

The single solitary-wave solution of RLW equation (1) is given by,
\[ u(x,t) = A \sec h^2 (kx - \omega t + \delta) , \]
where
\[ A = \frac{3a^2}{1 - a^2} \quad k = \frac{a}{2} \quad \omega = \frac{a}{2(1 - a^2)} , \]
and \( a \) and \( \delta \) are constants.

The scheme (6)-(9) is a linear system of equations which can be solved without iteration.

Take \( a = \frac{1}{2} \), \( \delta = 0 \) and the initial function of the problem (1)-(3) is rewritten as
\[ u(x,0) = \sec h^2 (\frac{1}{4} x) . \]

In the numerical experiments, we take \( x_i = -50 \), \( x_k = 50 \), and \( T = 20 \). The errors in the sense of \( L_\infty \)-norm and \( L_2 \)-norm of the numerical solutions are listed on Table 1 under different mesh steps \( \tau \) and \( h \). Table 2 shows that the computational and the theoretical orders of the scheme are very close to each other. Table 3 shows the value of \( E^n \) and \( Q^n \) at different time. It indicates that the conservation of the scheme (6)-(9) is very good and it is suitable for long-term computation.

Table 1. The errors estimates of numerical solution with various \( \tau \) and \( h \).

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( h )</th>
<th>( |e^n|_\infty )</th>
<th>( |e^n| )</th>
<th>( |e^n|_\infty )</th>
<th>( |e^n| )</th>
<th>( |e^n|_\infty )</th>
<th>( |e^n| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.1</td>
<td>1.293731e-2</td>
<td>2.473412e-2</td>
<td>3.472503e-2</td>
<td>4.338962e-2</td>
<td>5.260183e-2</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>6.219891e-3</td>
<td>1.12209e-2</td>
<td>1.505045e-2</td>
<td>1.828892e-2</td>
<td>2.520183e-2</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>1.0</td>
<td>8.406610e-4</td>
<td>1.575704e-3</td>
<td>2.206277e-3</td>
<td>2.765299e-3</td>
<td>4.483281e-3</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>1.0</td>
<td>4.019759e-4</td>
<td>7.156453e-4</td>
<td>9.588093e-4</td>
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<tr>
<td>0.025</td>
<td>0.2</td>
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<td>9.871933e-5</td>
<td>1.381956e-4</td>
<td>1.740699e-4</td>
<td>7.302556e-4</td>
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</tr>
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</table>

Table 2. The numerical verification of theoretical accuracy \( O(\tau^2 + h^4) \).

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( h )</th>
<th>( |e^n(h,\tau)| / |e^n(h,\tau/2,\tau/4)| )</th>
<th>( |e^n(h,\tau)| / |e^n(h,\tau/2,\tau/4)| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.1</td>
<td>0.0125</td>
<td>0.0125</td>
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<tr>
<td>0.05</td>
<td>0.2</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

7
| t=5  | —       | 15.389452 | 15.963269 | —   | 15.473292 | 15.950260 |
| t=10 | —       | 15.697181 | 15.961462 | —   | 15.681087 | 15.962535 |
| t=15 | —       | 15.739194 | 15.964880 | —   | 15.697027 | 15.967716 |
| t=20 | 15.690750 | 15.976137 | —       | —   | 15.680645 | 15.971630 |

Table 3. Discrete mass and discrete energy with various $\tau$ and $h$.  

<table>
<thead>
<tr>
<th>$\tau$ = 0.2</th>
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<td>$E^n$</td>
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<td>$E^n$</td>
</tr>
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From these computational results, it shows that our proposed algorithm is efficient and reliable.

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**References**  


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