APPROXIMATE ANALYTICAL SOLUTIONS OF NON-LINEAR LOCAL FRACTIONAL HEAT EQUATIONS

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Consider the non-linear local fractional heat equation. The fractional complex transform method and the Adomian decomposition method are used to solve the equation. The approximate analytical solutions are obtained.

Key words: *local fractional heat equation, fractional complex transform, Adomian decomposition method*

Introduction

In present investigation, we consider the following non-linear local fractional heat equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - k^2 u \left(\frac{\partial u}{\partial x} \right)^2 + f(x,t) \tag{1}$$

with the conditions:

$$u(x,0) = \varphi(x) \tag{2}$$

where k is a constant, $\partial^{\alpha} u / \partial t^{\alpha}$ and $\partial^{2\beta} u / \partial x^{2\beta}$ are the local fractional derivatives [1-5] ($0 < \alpha \le 1$, $0 < \beta \le 1$), $\varphi(x)$ and $f(x, t)$ are given functions.

The classical heat equation is one of the most important PDE to model problems in mathematical physics [6-15]. The non-linear local fractional heat equation can be used to model the fractal electromagnetic radiation, the fractal seismology, the fractal acoustics and so on [1-5].

The linear heat equation involving local fractional derivative operators have been investigated over the last decade. In case of $k = 0$, $\alpha = \beta$, the eq. (1) have been solved by applying the local fractional series expansion method and the local fractional variational iteration method [1-3]. The main objective of the present paper is to solve the problems (1)-(2) by means of the complex transform and Adomian decomposition method (ADM) [16, 17].

Preliminaries

Local fractional derivative

In this section, we give some definitions and properties of local fractional derivative, for more detail see [1-5].

Definition 1. For arbitrary $\varepsilon > 0$, assume that the relation below exists:

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \tag{3}$$

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with $|x - x_0| < \delta$. Then $f(x)$ is called local fractional continuous at x_0 which is denoted by $\lim_{x \to x_0} f(x) = f(x_0)$. If $f(x)$ is local fractional continuous on the interval (a, b), it is denoted:

$$f(x) \in C_\alpha(a,b)$$

Definition 2. Let $f(x) \in C_\alpha(a, b)$ In fractal space, the local fractional derivative of $f(x)$ of order at the point $x = x_0$ is given by:

$$D_x^\alpha f(x_0) = \frac{d^\alpha}{dx^\alpha} f(x)\bigg|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^\alpha[f(x) - f(x_0)]}{(x - x_0)^\alpha} \tag{4}$$

where

$$\Delta[f(x) - f(x_0)] \cong \Gamma(\alpha + 1)[f(x) - f(x_0)]$$

Local fractional partial derivative of high order is defined in the form:

$$\frac{\partial^{k\alpha} f(x,t)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x,t) \tag{5}$$

The following formula on local fractional derivative hold true:

$$\frac{d^\alpha f[g(x)]}{dx^\alpha} = f'[g(x)]g^{(\alpha)}(x) \tag{6}$$

where there exist $f'[g(x)]$ and $g^{(\alpha)}(x)$.

Adomian decomposition method

To illustrate Adomian decomposition method [16], consider the following equation:

$$L(u) + N(u) = f(x) \tag{7}$$

where L is a linear operator, N – a non-linear operator, and $f(x)$ is a given function.

We can solve the eq. (7) by defining the unknown function:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{8}$$

where the components $u_n(x)$ are usually determined recurrently. The non-linear operator, $N(u)$, can be decomposed into the following result:

$$N(u) = \sum_{n=0}^{\infty} A_n \tag{9}$$

where A_n are called Adomian's polynomials of $u_0, u_1, u_2...u_n$ defined:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0} , \quad n = 0,1,2,3... \tag{10}$$

Substituting eqs. (8) and (9) into (7):

$$\sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} A_i = f(x) \tag{11}$$

Thus, the components $u_i(x, t)$ of the solution $u(x, t)$ can be computed by using the recursive relation:

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$$Lu_0(x) = f(x)$$
$$Lu_1(x) + A_0(u_0) = 0$$
$$Lu_2(x) + A_1(u_0, u_1) = 0 \tag{12}$$
$$\vdots$$

Finally, the k-term approximate solution of eq. (7) is given by:

$$u = u_0 + u_1 + \cdots + u_{k-1}$$

Solution of the problem (1)-(2)

In this section, we consider the following initial value problem of non-linear local fractional heat equation:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = \dfrac{\partial^{2\beta} u}{\partial x^{2\beta}} - k^2 u \left(\dfrac{\partial u}{\partial x} \right)^2 + f(x,t) \\ u(x,0) = \varphi(x) \end{cases} \tag{13}$$

Where we assume that the functions $f(x, t)$ and and $\varphi(x)$ are local fractional continuous.

To solve this eq. (13), we use the following fractional complex transform [17]:

$$X = \frac{x^\beta}{\Gamma(1+\beta)}, \quad T = \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{14}$$

By eq. (14), the problem (13) becomes:

$$\begin{cases} \dfrac{\partial u}{\partial T} = \dfrac{\partial^2 u}{\partial X^2} - k^2 u \left(\dfrac{\partial u}{\partial X} \right)^2 + f(X,T) \\ u(X,0) = \varphi(X) \end{cases} \tag{15}$$

Next, we present the solutions of non-linear fractional heat eq. (17) by an application of the Adomian decomposition method.

For eq. (13), we have:

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \tag{16}$$

$$Nu = -k^2 u \left(\frac{\partial u}{\partial x} \right)^2 \tag{17}$$

Then, by eq. (12), we obtain:

$$u_0(X,T) = \varphi(X), \quad \frac{\partial u_1}{\partial T} = \frac{\partial^2 u_0}{\partial X^2} - k^2 A_0 + f(X,T)$$

$$\frac{\partial u_n}{\partial T} = \frac{\partial^2 u_{n-1}}{\partial X^2} - k^2 A_{n-1}, \quad (n = 2,3,\cdots) \tag{18}$$

where

$$A_0 = u_0 \left(\frac{\partial u_0}{\partial X} \right)^2, \quad A_1 = 2u_0 \left(\frac{\partial u_0}{\partial X} \right) \left(\frac{\partial u_1}{\partial X} \right) + u_1 \left(\frac{\partial u_0}{\partial X} \right)^2$$

$$A_2 = 2u_0 \left(\frac{\partial u_0}{\partial X} \right) \left(\frac{\partial u_2}{\partial X} \right) + u_0 \left(\frac{\partial u_1}{\partial X} \right)^2 + u_2 \left(\frac{\partial u_0}{\partial X} \right)^2 + 2u_1 \left(\frac{\partial u_0}{\partial X} \right) \left(\frac{\partial u_1}{\partial X} \right)$$

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and so on.

Thus the n – term approximate solution of eq. (15):

$$u(X,T) = u_0(X,T) + u_1(X,T) + u_2(X,T) + \cdots u_n(X,T)$$

From eq. (14), we get the solution of eq. (15):

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots u_n(x,t) + \cdots$$

Example 1. Consider eq. (1) in the form:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = \dfrac{\partial^{2\beta} u}{\partial x^{2\beta}} - k^2 u \left(\dfrac{\partial u}{\partial x} \right)^2 \\ u(x,0) = \exp\left(-\dfrac{x^{2\beta}}{\Gamma^2(1+\beta)} \right) \end{cases} \tag{19}$$

By the relations (18), we obtain:

$$u_0(X,T) = \exp(-X^2)$$

$$u_1(X,T) = \left[4X^2 \exp(-X^2) - 2\exp(-X^2) - 4x^2k^2 \exp(-3X^2) \right]T$$

$$u_2(X,T) = \left[\left(8x^4 - 24x^2 + 6 \right)\exp(-x^2) + k^2\left(28x^2 - 24x^4 \right)\exp(-3x^2) + 56k^2x^4 \exp(-5x^2) \right]T^2$$

$$\vdots$$

Thus, by eq. (14), we obtain:

$$u_0(x,t) = \exp\left[-\frac{x^{2\beta}}{\Gamma^2(1+\beta)} \right]$$

$$u_1(x,t) = \left\{ \frac{4x^{2\beta}}{\Gamma^2(1+\beta)}\exp\left[\frac{-x^{2\beta}}{\Gamma^2(1+\beta)} \right] - 2\exp\left[-\frac{x^{2\beta}}{\Gamma^2(1+\beta)} \right] - \frac{4k^2x^{2\beta}}{\Gamma^2(1+\beta)}\exp\left[\frac{-3x^{2\beta}}{\Gamma^2(1+\beta)} \right] \right\} \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x,t) = \left\{ \left[8\frac{x^{4\beta}}{\Gamma^4(1+\beta)} - 24\frac{x^{2\beta}}{\Gamma^2(1+\beta)} + 6 \right]\exp\left[\frac{-x^{2\beta}}{\Gamma^2(1+\beta)} \right] \right\} \frac{t^{2\alpha}}{\Gamma^2(1+\alpha)} +$$

$$+ \left\{ k^2\left[\frac{28x^{2\beta}}{\Gamma^2(1+\beta)} - \frac{24x^{4\beta}}{\Gamma^4(1+\beta)} \right]\exp\left[\frac{-3x^{2\beta}}{\Gamma^2(1+\beta)} \right] + k^2\frac{56x^{4\beta}}{\Gamma^4(1+\beta)}\exp\left[\frac{-5x^{2\beta}}{\Gamma^2(1+\beta)} \right] \right\} \frac{t^{2\alpha}}{\Gamma^2(1+\alpha)}$$

$$\vdots$$

Finally, the solution of eq. (19) is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + \cdots$$

When $k = 0$, $\alpha = \beta = 1$, we have:

$$u(x,t) = e^{-x^2} + 2(2x^2 - 1)e^{-x^2}t + 2(4x^4 - 12x^2 + 3)e^{-x^2}t^2 + \cdots$$

which is close to the exact solution [18]:

$$u(x,t) = \frac{e^{\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$$

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Conclusion

In this paper, we consider a non-linear local fractional heat equation.The fractional complex transform and Adomian decomposition method are used to solve the equation. The approximate analytical solutions are obtained .We believe that for engineers and scientists the approximate analytical solutions would be quite useful to analyze the properties of the aforementioned non-linear local fractional heat equation.

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