APPROXIMATE ANALYTICAL SOLUTIONS OF NONLINEAR LOCAL FRACTIONAL HEAT EQUATIONS

by

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Consider the nonlinear local fractional heat equation. The fractional complex transform method and the Adomian decomposition method are used to solve the equation. The approximate analytical solutions are obtained .

Key words: *local fractional heat equation, fractional complex transform, the Adomian decomposition method*

Introduction

In present investigation, we consider the following nonlinear local fractional heat equation:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}}-k^{2}u\left(\frac{\partial u}{\partial x}\right)^{2}+f(x,t) \tag{1}$$

with the conditions

$$u(x,0)=\varphi(x), \tag{2}$$

where k is a constant, $\frac{\partial^{\alpha}u}{\partial t^{\alpha}}$ and $\frac{\partial^{2\beta}u}{\partial x^{2\beta}}$ are the local fractional derivatives [11-15] ($0<\alpha\le 1, 0<\beta\le 1$), $\varphi(x)$ and $f(x,t)$ are given functions.

The classical heat equation is one of the most important partial differential equations to model problems in mathematical physics[1-10]. The nonlinear local fractional heat equation can be used to model the fractal electromagnetic radiation, the fractal seismology, the fractal acoustics and so on[11-15].

The linear heat equation involving local fractional derivative operators have been investigated over the last decade. In case of $k=0, \alpha=\beta,$ the Eq.(1) have been solved by applying the local fractional series expansion method and the local fractional variational iteration method[11-13]. The main objective of the present paper is to solve the problems (1)-(2) by means of the complex transform and Adomian decomposition method (ADM)[16-17]. The structure of the paper is as follows. In section 2, we give the some basic

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results on the local fractional derivative and the ADM. In section 3, the fractional complex transform and the ADM are applied to solve the problem (1)-(2). Finally, our conclusions are presented in section 4.

Preliminaries

Local fractional derivative

In this section, we give some definitions and properties of local fractional derivative, for more detail see [11-15].

Definition 1. For arbitrary $\varepsilon > 0$, assume that the relation below exists

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \tag{3}$$

with $|x - x_0| < \delta$. Then $f(x)$ is called local fractional continuous at x_0 which is denoted by $\lim_{x \to x_0} f(x) = f(x_0)$. If $f(x)$ is local fractional continuous on the interval (a,b), it is denoted by

$$f(x) \in C_{\alpha}(a,b).$$

Definition 2. Let $f(x) \in C_{\alpha}(a,b)$. In fractal space, the local fractional derivative of $f(x)$ of order α at the point $x = x_0$ is given by

$$D_x^{\alpha} f(x_0) = \frac{d^{\alpha}}{dx^{\alpha}} f(x)\bigg|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}}, \tag{4}$$

where $\Delta(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

Local fractional partial derivative of high order is defined in the form

$$\frac{\partial^{k\alpha} f(x,t)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k \ times} f(x,t). \tag{5}$$

The following formula on local fractional derivative hold true:

$$\frac{d^{\alpha} f(g(x))}{dx^{\alpha}} = f'(g(x)) g^{(\alpha)}(x), \tag{6}$$

where there exist $f'(g(x))$ and $g^{(\alpha)}(x)$.

Adomian decomposition method

To illustrate Adomian decomposition method[16], consider the following equation:

$$L(u)+N(u)=f(x), \tag{7}$$

where L is a linear operator, N is a non-linear operator and $f(x)$ is a given function.

We can solve the Eq.(7) by defining the unknown function as follows:

$$u(x)=\sum_{n=0}^{\infty}u_n(x), \tag{8}$$

where the components $u_n(x)$ are usually determined recurrently. The nonlinear operator $N(u)$ can be decomposed into the following result:

$$N(u)=\sum_{n=0}^{\infty}A_n\ , \tag{9}$$

where A_n are called Adomian's polynomials of $u_0,u_1,u_2...u_n$ defined by:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{n}\lambda^i u_i)]_{\lambda=0},\ n=0,1,2,3\cdots. \tag{10}$$

Substituting Eqs.(8) and (9) into (7), we get

$$\sum_{i=0}^{\infty}L(u_i)+\sum_{i=0}^{\infty}A_i=f(x) \tag{11}$$

Thus, the components $u_i(x,t)$ of the solution $u(x,t)$ can be computed by using the recursive relation

$$\begin{aligned} &Lu_0(x)=f(x),\\ &Lu_1(x)+A_0(u_0)=0,\\ &Lu_2(x)+A_1(u_0,u_1)=0,\\ &\vdots \end{aligned} \tag{12}$$

Finally, the k-term approximate solution of (7) is given by

$$u=u_0+u_1+\cdots+u_{k-1}.$$

Solution of the problem (1)-(2)

In this section, we consider the following initial value problem of nonlinear local fractional heat equation.

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}=\dfrac{\partial^{2\beta}u}{\partial x^{2\beta}}-k^2u\left(\dfrac{\partial u}{\partial x}\right)^2+f(x,t),\\ u(x,0)=\varphi(x). \end{cases} \tag{13}$$

Where we assume that the functions $f(x,t)$ and $\varphi(x)$ are local fractional continuous.

To solve this equation (13), we use the following fractional complex transform[17]:

$$X = \frac{x^{\beta}}{\Gamma(1+\beta)}, \quad T = \frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{14}$$

By (14), the problem (13) becomes

$$\begin{cases} \dfrac{\partial u}{\partial T} = \dfrac{\partial^2 u}{\partial X^2} - k^2 u\left(\dfrac{\partial u}{\partial X}\right)^2 + f(X,T), \\ u(X,0) = \varphi(X). \end{cases} \tag{15}$$

Next, we present the solutions of nonlinear fractional heat equations (17) by an application of the Adomian decomposition method.

For Eq. (13), we have

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}, \tag{16}$$

$$Nu = -k^2 u\left(\frac{\partial u}{\partial x}\right)^2. \tag{17}$$

Then, by Eq.(12), we obtain:

$$u_0(X,T) = \varphi(X),$$

$$\frac{\partial u_1}{\partial T} = \frac{\partial^2 u_0}{\partial X^2} - k^2 A_0 + f(X,T),$$

$$\frac{\partial u_n}{\partial T} = \frac{\partial^2 u_{n-1}}{\partial X^2} - k^2 A_{n-1}, \quad (n = 2,3,\cdots) \tag{18}$$

where

$$A_0 = u_0\left(\frac{\partial u_0}{\partial X}\right)^2,$$

$$A_1 = 2u_0\left(\frac{\partial u_0}{\partial X}\right)\left(\frac{\partial u_1}{\partial X}\right) + u_1\left(\frac{\partial u_0}{\partial X}\right)^2,$$

$$A_2 = 2u_0\left(\frac{\partial u_0}{\partial X}\right)\left(\frac{\partial u_2}{\partial X}\right) + u_0\left(\frac{\partial u_1}{\partial X}\right)^2 + u_2\left(\frac{\partial u_0}{\partial X}\right)^2 + 2u_1\left(\frac{\partial u_0}{\partial X}\right)\left(\frac{\partial u_1}{\partial X}\right),$$

and so on.

Thus the $n - term$ approximate solution of Eq.(15) is given by

$$u(X,T) = u_0(X,T) + u_1(X,T) + u_2(X,T) + \cdots u_n(X,T).$$

From (14), we get the solution of Eq.(15) as follows:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots u_n(x,t)+\cdots.$$

Example 1. Consider Eq. (1) in the form:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}=\dfrac{\partial^{2\beta}u}{\partial x^{2\beta}}-k^2u\left(\dfrac{\partial u}{\partial x}\right)^2, \\ u(x,0)=\exp\left(-\dfrac{x^{2\beta}}{\Gamma^2(1+\beta)}\right). \end{cases} \tag{19}$$

By the relations (18), we obtain:

$$u_0(X,T)=\exp(-X^2),$$

$$u_1(X,T)=\left(4X^2\exp(-X^2)-2\exp(-X^2)-4x^2k^2\exp(-3X^2)\right)T,$$

$$u_2(X,T)=\left(\left(8x^4-24x^2+6\right)\exp(-x^2)+k^2\left(28x^2-24x^4\right)\exp(-3x^2)+56k^2x^4\exp(-5x^2)\right)T^2,$$

$$\vdots$$

Thus, by (14), we obtain

$$u_0(x,t)=\exp\left(-\frac{x^{2\beta}}{\Gamma^2(1+\beta)}\right),$$

$$u_1(x,t)=\left(\frac{4x^{2\beta}}{\Gamma^2(1+\beta)}\exp(\frac{-x^{2\beta}}{\Gamma^2(1+\beta)})-2\exp(-\frac{x^{2\beta}}{\Gamma^2(1+\beta)})-\frac{4k^2x^{2\beta}}{\Gamma^2(1+\beta)}\exp(\frac{-3x^{2\beta}}{\Gamma^2(1+\beta)})\right)\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$u_2(x,t)=\left(\left(8\frac{x^{4\beta}}{\Gamma^4(1+\beta)}-24\frac{x^{2\beta}}{\Gamma^2(1+\beta)}+6\right)\exp(\frac{-x^{2\beta}}{\Gamma^2(1+\beta)})\right)\frac{t^{2\alpha}}{\Gamma^2(1+\alpha)}+$$

$$\left(k^2\left(\frac{28x^{2\beta}}{\Gamma^2(1+\beta)}-\frac{24x^{4\beta}}{\Gamma^4(1+\beta)}\right)\exp(\frac{-3x^{2\beta}}{\Gamma^2(1+\beta)})+k^2\frac{56x^{4\beta}}{\Gamma^4(1+\beta)}\exp(\frac{-5x^{2\beta}}{\Gamma^2(1+\beta)})\right)\frac{t^{2\alpha}}{\Gamma^2(1+\alpha)},$$

$$\vdots$$

Finally, the solution of (19) is given by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+\cdots.$$

When $k=0, \alpha=\beta=1,$ we have

$$u(x,t)=e^{-x^2}+2(2x^2-1)e^{-x^2}t+2(4x^4-12x^2+3)e^{-x^2}t^2+\cdots,$$

which is close to the exact solution[18]

$$u(x,t)=\frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}.$$

Conclusion

In this paper, we consider a nonlinear local fractional heat equation.The fractional complex transform and Adomian decomposition method are used to solve the equation. The approximate analytical solutions are obtained .We believe that for engineers and scientists the approximate analytical solutions would be quite useful to analyze the properties of the above nonlinear local fractional heat equation.

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