A NEW GENERAL FRACTIONAL-ORDER DERIVATIVE WITH RABOTNOV FRACTIONAL-EXPONENTIAL KERNEL APPLIED TO MODEL THE ANOMALOUS HEAT TRANSFER

by

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In this paper, we consider a general fractional-order derivative of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function for the first time. A new general fractional-order derivative heat-transfer model is discussed in detail. The general fractional-order derivative formula is a new mathematical tool proposed to model the anomalous behaviors in complex and power-law phenomena.

Key words: general fractional-order derivative, Rabotnov fractional-exponential function, heat transfer, non-singular kernel, power law

Introduction

The general fractional-order derivatives, where the non-singular kernels are the special functions (for more details, see[1-3]), such as exponential, Mittag-Leffler-Gauss, Kohlrusch-Williams-Watts, Miller-Ross, Lorenzo-Hartley, Gorenflo-Mainardi, Bessel, Mittag-Leffler, Wiman, and Prabhakar, have been applied to investigate the mathematical models in mathematical physics. The general fractional-order diffusion was reported in [4]. The general-order chemical kinetics via Mittag-Leffler kernel was proposed in [5]. The general fractional-order relaxation via exponential kernel was discussed in [6]. The general fractional-order rheological model via Prabhakar kernel was considered in [7]. The general fractional-order Burgers via Mittag-Leffler was investigated in [8]. For more models via the special functions, we refer to the results for the relaxation and rheological arising in complex and power-law phenomena[1].

The Rabotnov fractional-exponential function, proposed in 1954 by Rabotnov [9], was used to describe the viscoelasticity [10,11]. However, up to now, the general fractional-order derivative with the non-singular kernel of the Rabotnov fractional-exponential function [11] has not

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been developed. Motivated by the new idea, the main target of the paper is to propose the general fractional-order derivative with the non-singular kernel of the Rabotnov fractional-exponential function in the sense of Liouville-Caputo type and to investigate the general fractional-order derivative heat-transfer model. The structure of the article is as follows. In Section 2, we propose a new general fractional-order derivative of the Liouville-Caputo type without the singular kernel of the Rabotnov fractional-exponential function. In Section 3, we investigate the general fractional-order derivative heat-transfer model. Finally, the conclusion is drawn in Section 4.

A new general fractional-order derivative with the non-singular kernel of the Rabotnov fractional-exponential function

Let \( \mathbb{C}, \mathbb{R}, \mathbb{R}_0^+, \mathbb{N} \) and \( \mathbb{N}_0 \) be the sets of complex numbers, real numbers, non-negative real numbers, positive integers and \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \), respectively.

The Rabotnov fractional-exponential function

Let \( \tau \in \mathbb{R}, \alpha \in \mathbb{R}_0^+, \lambda \in \mathbb{R}_0^+ \) and \( \kappa \in \mathbb{N}_0 \). The Rabotnov fractional-exponential function is defined as [1,9]:

\[
\Phi_\alpha(\lambda \tau^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k \Gamma(k+1)(\alpha+1)}{\Gamma((\kappa+1)(\alpha+1))} \tau^k
\]

and its Laplace transform is [1]

\[
L\left[\Phi_\alpha(\lambda \tau^\alpha)\right] = \frac{1}{s^{1+\alpha}} \frac{1}{1 - \lambda s^{1+\alpha}} \left(\frac{\lambda s^{1+\alpha}}{1+s^{1+\alpha}}\right)^{-1},
\]

where the Laplace transform of the function \( \phi(\tau) \) is given as [1-3]

\[
L[\phi(\tau)] := \phi(s) = \int_0^\infty e^{-s\tau} \phi(\tau) d\tau
\]

with \( s \in \mathbb{C} \).

A new general fractional-order derivative with Rabotnov fractional-exponential kernel

Let \( L(a,b) \) be the set of those Lebesgue measurable functions on a finite interval \( (a,b) \) \(( -\infty \leq a \leq b \leq +\infty \) \) (for more details, see[1]).

Let \( AC(a,b) \) be the space of the functions which are absolutely continuous on a finite interval \( (a,b) \) \(( -\infty \leq a \leq b \leq +\infty \) \) (for more details, see[1]).

Let \( AC^1(a,b) \) be the Kolmogorov-Fomin condition (for more details, see[1]).

Let \( \lambda \in \mathbb{R}_0^+ \). The general fractional-order integral operator via Rabotnov fractional-exponential kernel is defined as:

\[
\left( \int_a^T \Phi_\alpha(\Theta)(t) \right) = \int_a^T \Phi_\alpha(\Theta(\lambda(\tau-t)^\alpha)) \Theta(t) dt,
\]

which leads to
\[
\left( a\mathbb{D}_a^{(\alpha)}\Theta \right)(\tau) = \int_0^\tau \Phi_a (\tau - t)^\alpha \Theta(t) \, dt ,
\]
(5)

where \( a = 0 \) and \( \Theta \in L(a,b) \), and

\[
\left( \mathbb{D}_a^{(\alpha)}\Theta \right)(\tau) = \int_{-\infty}^\tau \Phi_a (\tau - t)^\alpha \Theta(t) \, dt ,
\]
(6)

where \( \Theta \in L(-\infty,b) \), and

\[
\left( \mathbb{D}_b^{(\alpha)}\Theta \right)(\tau) = \int_0^\infty \Phi_a (\tau - t)^\alpha \Theta(t) \, dt ,
\]
(7)

where \( \Theta \in L(0,\infty) \).

The left-sided general fractional-order derivative of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined as follows:

\[
\left( a\mathbb{D}_a^{(\alpha)}\Theta \right)(\tau) = a\mathbb{D}_a^{(\alpha)}\Theta(\tau) = \int_{-\infty}^\tau \Phi_a (\tau - t)^\alpha \Theta^{(\alpha)}(t) \, dt ,
\]
(8)

which can be written as follows:

\[
\left( \mathbb{D}_a^{(\alpha)}\Theta \right)(\tau) = \mathbb{D}_a^{(\alpha)}\Theta(\tau) = \int_{-\infty}^\tau \Phi_a (\tau - t)^\alpha \Theta^{(\alpha)}(t) \, dt ,
\]
(9)

where \( \Theta \in AC^1(a,b) \).

The right-sided general fractional-order derivative of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined as follows:

\[
\left( b\mathbb{D}_b^{(\alpha)}\Theta \right)(\tau) = b\mathbb{D}_b^{(\alpha)}\Theta(\tau) = - \int_\tau^\infty \Phi_a (t - \tau)^\alpha \Theta^{(\alpha)}(t) \, dt ,
\]
(10)

which can be written as follows:

\[
\left( \mathbb{D}_b^{(\alpha)}\Theta \right)(\tau) = \mathbb{D}_b^{(\alpha)}\Theta(\tau) = - \int_{-\infty}^\tau \Phi_a (t - \tau)^\alpha \Theta^{(\alpha)}(t) \, dt ,
\]
(11)

where \( \Theta \in AC^1(a,b) \).

The left-sided general fractional-order derivative of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined as follows:

\[
\left( a\mathbb{D}_a^{(\alpha)}\Theta \right)(\tau) = a\mathbb{D}_a^{(\alpha)}\Theta(\tau) = \int_{-\infty}^\tau \Phi_a (\tau - t)^\alpha \Theta^{(\alpha)}(t) \, dt ,
\]
(12)

which implies that
\[
\left( \mathcal{D}_a^{(\alpha)} \right) (\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{\tau}^{\infty} \left( -\lambda (\tau-t)^{\alpha} \right) \Theta(t) \, dt ,
\]
where \( \Theta \in AC^n(a,b) \) and \( n \in \mathbb{N} \).

The right-sided general fractional-order derivative of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined as follows:

\[
\left( \mathcal{D}_b^{(\alpha)} \right) (\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{\tau}^{b} \left( -\lambda (t-\tau)^{\alpha} \right) \Theta(t) \, dt ,
\]

which implies that

\[
\left( \mathcal{D}_b^{(\alpha)} \right) (\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{\tau}^{b} \left( -\lambda (t-\tau)^{\alpha} \right) \Theta(t) \, dt ,
\]

where \( n \in \mathbb{N} \).

The Laplace transforms of (5), (9) and (13) can be given as follows:

\[
\mathcal{L} \left[ \left( \mathcal{D}_a^{(\alpha)} \right) (\tau) \right] = \frac{1}{s^{\alpha+1}} \frac{1}{1+\lambda s^{-(\alpha+1)}} \Theta(s) ,
\]

(16)

\[
\mathcal{L} \left[ \left( \mathcal{D}_b^{(\alpha)} \right) (\tau) \right] = \frac{1}{s^{\alpha+1}} \frac{1}{1+\lambda s^{-(\alpha+1)}} \left( s \Theta(s) - \Theta(0) \right) ,
\]

(17)

and

\[
\mathcal{L} \left[ \left( \mathcal{D}_r^{(\alpha)} \right) (\tau) \right] = \frac{1}{s^{\alpha+1}} \frac{1}{1+\lambda s^{-(\alpha+1)}} \left( s^n \Theta(s) - \sum_{r=1}^{n} s^{n-r} \Theta^{(r)}(0) \right) ,
\]

(18)

with \( r \in \mathbb{N} \).

**General fractional-order integrals via special function**

The left-sided general fractional-order integral of \( \Omega(\tau) \) is defined as follows:

\[
\left( \mathcal{I}_a^{(\alpha)} \right) (\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{\tau} \lambda_a \left( -\lambda (\tau-t)^{\alpha} \right) \Omega(t) \, dt = \int_{a}^{\tau} \left( \tau-t \right)^{\alpha-1} E_{\alpha+1,\alpha-\alpha+1}^{-1} \left( -\lambda (\tau-t)^{\alpha-1} \right) \Omega(t) \, dt ,
\]

(19)

where \( \Lambda_a \left( -\lambda t^{\alpha} \right) = t^{\alpha-1} E_{\alpha+1,\alpha-\alpha+1}^{-1} \left( -\lambda t^{\alpha-1} \right) \) with the Prabhakar function, denoted as \([1]\)

\[
E_{\alpha,\beta}^\gamma (t) = \sum_{k=0}^{n} \frac{1}{\Gamma(\kappa \alpha + \beta)} \frac{\Gamma(\gamma + \kappa)}{\Gamma(\gamma + 1)} \frac{t^{\kappa \alpha}}{\Gamma(\kappa + 1)} .
\]

The right-sided general fractional-order integral of \( \Theta(\tau) \) is defined as follows:

\[
\left( \mathcal{I}_b^{(\alpha)} \right) (\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{\tau}^{b} \lambda_a \left( -\lambda (t-\tau)^{\alpha} \right) \Theta(t) \, dt .
\]

(20)

For \( a = 0 \), (19) can be written as follows:

\[
\left( \mathcal{I}_b^{(\alpha)} \right) (\tau) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\tau} \lambda_a \left( -\lambda (\tau-t)^{\alpha} \right) \Theta(t) \, dt ,
\]

(21)
where \( \Omega \in L(a,b) \).

The Laplace transform of (19) can be presented as follows:

\[
\mathcal{L}\left[\left(\mathbb{G}^{[\alpha]} \Omega\right)(\tau)\right] = s^{\alpha-1-a} \left(1 + \lambda s^{-\alpha+1}\right) \Omega(s).
\]  

(22)

**A new application in the heat-transfer process**

In this section, a new general fractional-order derivative heat-transfer model is presented.

We now consider the new general fractional-order derivative heat-transfer model

\[
\sigma \mathbb{D}^{(\alpha)}_x X(x) = \chi
\]

with the initial value condition

\[
X(x)|_{x=0} = X(0),
\]

(23)

(24)

where \( \sigma \) represents the thermal conductivity of the material and \( \chi \) is the heat flux density.

With the use of (17), we have

\[
\frac{1}{s^{\alpha+1}} \frac{\sigma}{1 + \lambda s^{-(\alpha+1)}} \left(sX(s) - X(0)\right) = \chi,
\]

which implies that

\[
X(s) = \frac{\chi}{\sigma} \left(1 + \lambda s^{-(\alpha+1)}\right) s^a + \frac{X(0)}{s}.
\]

(25)

(26)

Finally, we have the solution of the general fractional-order derivative heat-transfer model as follows:

\[
X(x) = \frac{\chi}{\sigma} x^{-(\alpha+1)} E_{\alpha+1,-a} \left(-\lambda x^{\alpha+1}\right) + X(0).
\]

(27)

**Conclusion**

In our work, we have addressed the new general fractional-order derivative of the Liouville-Caputo type without the singular kernel of the Rabotnov fractional-exponential function and its Laplace transform. As an potential application, the general fractional-order derivative heat-transfer model and its solution based on the general Prabhakar function have been investigated in detail. The general fractional-order derivative is accurate and efficient for description of the general fractional-order dynamics in complex and power-law phenomena.

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Nomenclature

- $\alpha$ - fractional order, [-]
- $x$ - space coordinate, [m]
- $\kappa$ - thermal conductivity, [Wm$^{-1}$K$^{-1}$]
- $\mathcal{L}\{\cdot\}$-Laplace transform, [-]
- $X(x)$-temperature distribution, [K]
- $\chi$ - heat flux density, [Wm$^{-2}$]

References


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