A NEW GENERAL FRACTIONAL-ORDER DERIVATIVE WITH RABOTNOV FRACTIONAL-EXPONENTIAL KERNEL APPLIED TO MODEL THE ANOMALOUS HEAT TRANSFER

by

Xiao-Jun YANG [a*], ***Mahmoud ABDEL-ATY*** [b,c,d], ***and Carlo CATTANI*** [e,f]

[a] State Key Laboratory for GeoMechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, China
[b] Center for Photonics and Smart Materials (CPSM), Zewail City of Science and Technology, Zewail, Egypt
[c] Mathematics Department, Faculty of Sciences, Sohag University, Sohag, Egypt
[d] Applied Science University, Sitra, Kingdom of Bahrain
[e] Engineering School, DEIM, University of Tuscia, Viterbo, Italy
[f] Ton Duc Thang University, HCMC, Ho Chi Minh City, Vietnam

Original scientific paper
https://doi.org/10.2298/TSCI180320239Y

In this paper, we consider a general fractional-order derivataive of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function for the first time. A new general fractional-order derivataive heat transfer model is discussed in detail. The general fractional-order derivataive formula is a new mathematical tool proposed to model the anomalous behaviors in complex and power-law phenomena.

Key words: power law Rabotnov fractional-exponential function, general fractional-order derivataive, heat transfer, non-singular kernel

Introduction

The general fractional-order derivatives, where the non-singular kernels are the special functions, for more details see [1-3], such as exponential, Mittag-Leffler-Gauss, Kohlrausch-Williams-Watts, Miller-Ross, Lorenzo-Hartley, Gorenflo-Mainardi, Bessel, Mittag-Leffler, Wiman, and Prabhakar, have been applied to investigate the mathematical models in mathematical physics. The general fractional-order diffusion was reported [4]. The general-order chemical kinetics via Mittag-Leffler kernel was proposed [5]. The general fractional-order relaxation via exponential kernal was discussed [6]. The general fractional-order rheologitcal model via Prabhakar kernel was considered [7]. The general fractional-order Burgers via Mittag-Leffler was investigated [8]. For more models via the special functions, we refer to the results for the relaxation and rheological arsising in complex and power-law phenomena [1].

The Rabotnov fractional-exponential function, proposed in 1954 by Rabotnov [9], was used to describe the viscoelasticity [10, 11]. However, up to now, the general fractional-order derivative with the non-singular kernel of the Rabotnov fractional-exponential function [11] has not been developed. Motivated by the new idea, the main target of the paper is to propose the general fractional-order derivative with the non-singular kernel of the Rabotnov fraction-

* Corresponding author, e-mail: dyangxiaojun@163.com

al-exponential function in the sense of Liouville-Caputo type and to investigate the general fractional-order derivataive heat transfer model.

A new general fractional-order derivataive of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}$, and $\mathbb{N}_0$ be the sets of complex numbers, real numbers, non-negative real numbers, positive integers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, respectively.

The Rabotnov fractional-exponential function

Let $\tau \in \mathbb{R}$, $\alpha \in \mathbb{R}_0^+$, $\lambda \in \mathbb{R}_0^+$, and $\kappa \in \mathbb{N}_0$. The Rabotnov fractional-exponential function is defined as [1, 9]:

$$\Phi_\alpha\left(\lambda\tau^\alpha\right) = \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa \tau^{(\kappa+1)(\alpha+1)-1}}{\Gamma\left[(\kappa+1)(\alpha+1)\right]} \tag{1}$$

and its Laplace transform is [1]:

$$L\left\{\Phi_\alpha\left(\lambda\tau^\alpha\right)\right\} = \frac{1}{s^{\alpha+1}} \frac{1}{1-\lambda s^{-(\alpha+1)}} \quad \left(\left|\lambda s^{-(\alpha+1)}\right| < 1\right) \tag{2}$$

where the Laplace transform of the function $\phi(\tau)$ is given as [1-3]:

$$\mathbb{L}\left[\phi(\tau)\right] := \phi(s) = \int_0^\infty \mathrm{e}^{-s\tau}\phi(\tau)\,\mathrm{d}\tau \tag{3}$$

with $s \in \mathbb{C}$.

A new general fractional-order derivataive with Rabotnov fractional-exponential kernel

Let $L(a, b)$ be the set of those Lebesgue measurable functions on a finite interval $(a, b)(-\infty \le a \le b \le +\infty)$, for more details, see [1].

Let $AC(a, b)$ be the space of the functions which are absolutely continuous on a finite interval $(a, b)(-\infty \le a \le b \le +\infty)$, for more details, see [1].

Let $AC^1(a, b)$ be the Kolmogorov-Fomin condition, for more details, see [1].

Let $\lambda \in \mathbb{R}_0^+$. The general fractional-order integral operator via Rabotnov fractional-exponential kernel is defined:

$$\left({}_a\mathbb{I}_\tau^{(\alpha)}\Theta\right)(\tau) = \int_a^\tau \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta(t)\,\mathrm{d}t \tag{4}$$

which leads

$$\left({}_0\mathbb{I}_\tau^{(\alpha)}\Theta\right)(\tau) = \int_0^\tau \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta(t)\,\mathrm{d}t \tag{5}$$

where $a = 0$ and $\Theta \in L(a, b)$

$$\left(\mathbb{I}_+^{(\alpha)}\Theta\right)(\tau) = \int_{-\infty}^\tau \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta(t)\,\mathrm{d}t \tag{6}$$

where $\Theta \in L(-\infty, b)$

$$\left(\mathbb{I}_-^{(\alpha)}\Theta\right)(\tau) = \int_0^{+\infty} \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta(t)\,\mathrm{d}t \tag{7}$$

where $\Theta \in L(-\infty, b)$.

The left-sided general fractional-order derivataive of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined:

$$\left({}_a\mathbb{D}_\tau^{(\alpha)}\Theta\right)(\tau)={}_a\mathbb{D}_\tau^{(\alpha)}\Theta(\tau)=\int_a^\tau \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta^{(1)}(t)\mathrm{d}t \tag{8}$$

which can be written

$$\left(\mathbb{D}_+^{(\alpha)}\Theta\right)(\tau)=\mathbb{D}_+^{(\alpha)}\Theta(\tau)=\int_{-\infty}^\tau \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta^{(1)}(t)\mathrm{d}t \tag{9}$$

where $\Theta \in AC^1(a, b)$.

The right-sided general fractional-order derivataive of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined:

$$\left({}_\tau\mathbb{D}_b^{(\alpha)}\Theta\right)(\tau)={}_\tau\mathbb{D}_b^{(\alpha)}\Theta(\tau)=-\int_\tau^b \Phi_\alpha\left[-\lambda(t-\tau)^\alpha\right]\Theta^{(1)}(t)\mathrm{d}t \tag{10}$$

which can be written:

$$\left(\mathbb{D}_-^{(\alpha)}\Theta\right)(\tau)=\mathbb{D}_-^{(\alpha)}\Theta(\tau)=-\int_\tau^{+\infty} \Phi_\alpha\left[-\lambda(t-\tau)^\alpha\right]\Theta^{(1)}(t)\mathrm{d}t \tag{11}$$

where $\Theta \in AC^1(a, b)$.

The left-sided general fractional-order derivataive of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined:

$$\left({}_a^n\mathbb{D}_+^{(\alpha)}\Theta\right)(\tau)={}_a^n\mathbb{D}_\tau^{(\alpha)}\Theta(\tau)=\int_a^\tau \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta^{(n)}(t)\mathrm{d}t \tag{12}$$

which implies that:

$$\left({}_\tau^n\mathbb{D}_+^{(\alpha)}\Theta\right)(\tau)={}_\tau^n\mathbb{D}_+^{(\alpha)}\Theta(\tau)=\int_{-\infty}^\tau \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta^{(n)}(t)\mathrm{d}t \tag{13}$$

where $\Theta \in AC^n(a, b)$ and $n \in \mathbb{N}$.

The right-sided general fractional-order derivataive of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined:

$$\left({}_\tau^n\mathbb{D}_b^{(\alpha)}\Theta\right)(\tau)={}_\tau^n\mathbb{D}_b^{(\alpha)}\Theta(\tau)=(-1)^n\int_\tau^b \Phi_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Theta^{(n)}(t)\mathrm{d}t \tag{14}$$

which implies that:

$$\left({}_\tau^n\mathbb{D}_-^{(\alpha)}\Theta\right)(\tau)={}_\tau^n\mathbb{D}_-^{(\alpha)}\Theta(\tau)=(-1)^n\int_\tau^{+\infty} \Phi_\alpha\left[-\lambda(t-\tau)^\alpha\right]\Theta^{(n)}(t)\mathrm{d}t \tag{15}$$

where $n \in \mathbb{N}$.

The Laplace transforms of (5), (9), and (13) can be given:

$$\mathbb{L}\left[\left({}_0\mathbb{I}_\tau^{(\alpha)}\Theta\right)(\tau)\right]=\frac{1}{s^{\alpha+1}}\frac{1}{1+\lambda s^{-(\alpha+1)}}\Theta(s) \tag{16}$$

$$\mathbb{L}\left[{}_0\mathbb{D}_\tau^{(\alpha)}\Theta(\tau)\right]=\frac{1}{s^{\alpha+1}}\frac{1}{1+\lambda s^{-(\alpha+1)}}\left[s\Theta(s)-\Theta(0)\right] \tag{17}$$

and

$$\mathbb{L}\left[{}_0^n\mathbb{D}_\tau^{(\alpha)}\Theta(\tau)\right]=\frac{1}{s^{\alpha+1}}\frac{1}{1+\lambda s^{-(\alpha+1)}}\left[s^n\Theta(s)-\sum_{r=1}^n s^{n-r}\Theta^{(r)}(0)\right] \tag{18}$$

with $r \in \mathbb{N}$.

General fractional-order integrals via special function

The left-sided general fractional-order integral of $\Omega(\tau)$ is defined:

$$\left({}_a\mathbb{I}_\tau^{(\alpha)}\Omega\right)(\tau)=\int_a^\tau \Lambda_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Omega(t)\mathrm{d}t=\int_a^\tau(\tau-t)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left[-\lambda(\tau-t)^{\alpha+1}\right]\Omega(t)\mathrm{d}t \tag{19}$$

where

$$\Lambda_\alpha\left(-\lambda\tau^\alpha\right)=\tau^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left(-\lambda\tau^{\alpha+1}\right)$$

with the Prabhakar function, denoted [1]:

$$E_{\alpha,\beta}^{\gamma}(\tau)=\sum_{\kappa=0}^{\infty}\frac{1}{\Gamma(\kappa\alpha+\beta)}\frac{\Gamma(\gamma+\kappa)}{\Gamma(\gamma)}\frac{\tau^\kappa}{\Gamma(\kappa+1)}$$

The right-sided general fractional-order integral of $\Omega(\tau)$ is defined:

$$\left({}_\tau\mathbb{I}_b^{(\alpha)}\Omega\right)(\tau)=(-1)^n\int_\tau^b \Lambda_\alpha\left[-\lambda(t-\tau)^\alpha\right]\Omega(t)\mathrm{d}t \tag{20}$$

For $a = 0$, eq. (19) can be written:

$$\left({}_0\mathbb{I}_\tau^{(\alpha)}\Omega\right)(\tau)=\int_0^\tau \Lambda_\alpha\left[-\lambda(\tau-t)^\alpha\right]\Omega(t)\mathrm{d}t \tag{21}$$

where $\Omega \in (a, b)$.

The Laplace transform of eq. (19) can be presented:

$$\mathbb{L}\left[\left({}_0\mathbb{I}_\tau^{(\alpha)}\Omega\right)(\tau)\right]=s^{\alpha+1-n}\left(1+\lambda s^{-(\alpha+1)}\right)\Omega(s) \tag{22}$$

A new application in the heat transfer process

In this section, a new general fractional-order derivataive heat transfer model is presented.

We now consider the new general fractional-order derivataive heat transfer model:

$$\sigma\,{}_0\mathbb{D}_x^{(\alpha)}\mathrm{X}(x)=\chi \tag{23}$$

with the initial value condition:

$$\mathrm{X}(x)|x=0=\mathrm{X}(0) \tag{24}$$

where σ represents the thermal conductivity of the material and χ – the heat flux density.

With the use of eq. (17), we have:

$$\frac{1}{s^{\alpha+1}}\frac{\sigma}{1+\lambda s^{-(\alpha+1)}}\left[s\mathrm{X}(s)-\mathrm{X}(0)\right]=\chi \tag{25}$$

which implies that:

$$\mathrm{X}(s)=\frac{\chi}{\sigma}\left(1+\lambda s^{-(\alpha+1)}\right)s^\alpha+\frac{\mathrm{X}(0)}{s} \tag{26}$$

Finally, we have the solution of the general fractional-order derivataive heat transfer model:

$$\mathrm{X}(x)=\frac{\chi}{\sigma}x^{-(\alpha+1)}E_{\alpha+1,-\alpha}^{-1}\left(-\lambda x^{\alpha+1}\right)+\mathrm{X}(0) \tag{27}$$

Conclusion

In our work, we have addressed the new general fractional-order derivataive of the Liouville-Caputo type without the singular kernel of the Rabotnov fractional-exponential function and its Laplace transform. As an potential application, the general fractional-order derivative heat transfer model and its solution based on the general Prabhakar function have been investigated in detail. The general fractional-order derivataive is accurate and efficient for description of the general fractional-order dynamics in complex and power-law phenomena.

Acknowledgment

This research has been supported by the financial support of the 333 Project of Jiangsu Province, People's Republic of China (Grant No. BRA2018320), the Yue-Qi Scholar of the China University of Mining and Technology (Grant No. 102504180004) and the State Key Research Development Program of the People's Republic of China (Grant No. 2016YFC0600705).

Nomenclature

$X(x)$ – temperature distribution, [K]
x – space co-ordinate, [m]
$\mathbb{L}[\bullet]$ – Laplace transform, [–]

Greek symbols

α – fractional order, [–]
κ – thermal conductivity, [$Wm^{-1}K^{-1}$]
χ – heat flux density, [Wm^{-2}]

References

[1] Yang, X. J., *General Fractional Derivatives: Theory, Methods and Applications*, CRC Press, New York, USA, 2019
[2] Samko, S. G., *et al.*, *Fractional Integrals and Derivatives*, Gordon and Breach Science, Yverdon, Switzerland, 1993
[3] Kilbas, A. A., *et al.*, Generalized Mittag-Leffler Function and Generalized Fractional Calculus Operators, *Integral Transforms and Special Functions, 15* (2004), 1, pp. 31-49
[4] Yang, X. J., *et al.*, Anomalous Diffusion Models with General Fractional Derivatives within the Kernels of the Extended Mittag-Leffler Type Functions, *Romanian Reports in Physics*, *69* (2017), 4, pp. 1-19
[5] Singh, J., *et al.*, On the Analysis of Chemical Kinetics System Pertaining to a Fractional Derivative with Mittag-Leffler Type Kernel, *Chaos*, *27* (2017), 10, ID 103113
[6] Caputo, M., *et al.*, A New Definition of Fractional Derivative without Singular Kernel, *Progress in Fractional Differentiation and Applications*, *1* (2015), 2, pp. 73-85
[7] Yang, X. J., New General Fractional-Order Rheological Models with Kernels of Mittag-Leffler Functions, *Romanian Reports in Physics*, *69* (2017), 4, ID 118
[8] Saad, K. M., *et al.*, New Fractional Derivatives with Non-Singular Kernel Applied to the Burgers Equation, *Chaos*, *28* (2018), 6, ID 063109
[9] Rabotnov, Y., Equilibrium of an Elastic Medium with After-Effect, Prikladnaya Matematika i Mekhanika, *12* (1948), 1, pp. 53-62 (in Russian), Reprinted: *Fractional Calculus and Applied Analysis*, *17* (2014), 3, pp. 684-696
[10] Rossikhin, Y. A., *et al.*, The Fractional Derivative Kelvin-Voigt Model of Viscoelasticity with and without Volumetric Relaxation, *Journal of Physics: Conference Series*, *991* (2018), 1, ID 012069
[11] Meshkov, S. I., *et al.*, Internal Friction Described with the Aid of Fractionally-Exponential Kernels, *Journal of Applied Mechanics and Technical Physics*, *7* (1969), 3, pp. 63-65

Paper submitted: March 20, 2018
Paper revised: June 15, 2018
Paper accepted: July 13, 2018